Markowitz Portfolio Analysis: The Demonstration Portfolio Problem

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Abstract

When introducing students to modern portfolio analysis, the most impressive example of diversification benefits comes when the portfolio standard deviation and coefficient of variation are lower than those of either of the two equally weighted assets contained in the portfolio. Randomly selected values for portfolio inputs will not always result in the most impressive portfolio outcome. This article derives the correlation coefficient necessary to produce an equally weighted two-asset portfolio that dominates either of the two individual assets in terms of standard deviation and coefficient of variation. Spreadsheet models based on these derivations are provided to assist finance educators in setting the necessary portfolio input values. The information in this article is most useful for educators who generate their own homework/quiz/test problems.

Introduction

In the early 1950's, Harry Markowitz, a Noble prizewinner, founded modern portfolio analysis when he demonstrated the nature of portfolio risk and how that risk can be minimized (Markowitz, 1952). Modern portfolio analysis is now a mainstream finance topic that is included in virtually every entry-level finance text and course. When introducing students to portfolio diversification, authors and instructors typically take a two-step approach. The first step is to calculate the expected return and standard deviation for two individual assets. The second step is to demonstrate that combining the two equally weighted assets into a portfolio, results in a portfolio standard deviation and coefficient of variation that is lower than the standard deviation and coefficient of variation for either of the two assets held in isolation. See for example (Weston, Besley, & Brigham, 1996; Gallagher & Andrew, 1997; Brigham & Gapenski, 1999). I refer to this result as "demonstration portfolio benefits." Unfortunately, this demonstration doesn't always work!

The Demonstration Portfolio Problem

The key factors in calculating portfolio standard deviation are the weights, standard deviations, and correlation coefficients of the assets held in the portfolio. Depending on the correlation between the assets, combining two equally weighted assets into a portfolio does *not* always result in a portfolio standard deviation that is lower than either of the individual assets (i.e., no "demonstration portfolio benefits"). Depending on the asset weights and the difference in the standard deviations between the two assets, even a correlation coefficient of -1.0 won't necessarily cause portfolio standard deviation and coefficient of variation to be below the standard deviation and coefficient of variation for the lower risk asset held in isolation.

I generate my own homework/quiz/test problems for my students. To encourage "independent efforts," I change the setup numbers each quarter. For portfolio problems, I have the students calculate the returns, standard deviations, and coefficients of variation for two individual assets and then the returns, standard deviations, and coefficients of variation for a portfolio containing those two individual assets. Once completed, I ask them to choose, based on the risk/return tradeoff, asset 1, asset 2, or the portfolio. I want the problem inputs to result in the students selecting the portfolio over either of the two individual assets. The following discussion develops a spreadsheet model that simplifies the process of setting up the portfolio problem.

Background Equations

The solution to the "demonstration portfolio problem" involves the use of several standard equations for calculating returns, standard deviations, and correlation coefficients. For convenience, these equations are presented below. In these equations; x and y represent individual assets, k indicates returns, w indicates weighting, i identifies an individual observation, and n is the total number of observations.

Individual asset average return over n periods:

$$\bar{k}_x = \frac{\sum\limits_{i=1}^{n} k_{x_i}}{n}$$

Standard deviation for the individual asset return over n periods:

$$\acute{o}_x = \left[\frac{\sum\limits_{i=1}^{n} (k_{x_i} - \bar{k}_x)^2}{n} \right]^{.5}$$

The covariance between the returns of two assets over n periods:

$$cov_{x,y} = \frac{\sum\limits_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{n}$$

The correlation coefficient for the returns of two assets:

$$\tilde{n}_{x,y} = \frac{cov_{x,y}}{\acute{o}_x \acute{o}_y}$$

The return for a two asset portfolio over n periods:

$$\hat{k}_p = \frac{\sum\limits_{i=1}^{n} w_x k_{x_i} + w_y k_{y_i}}{n}$$

Portfolio Standard Deviation Solution

Assume a two-asset portfolio, comprising assets 1 and 2. Given the asset weights and standard deviations and assuming that asset 1 has the lower standard deviation of the two assets in the portfolio (i.e., $\sigma_{1<}\sigma_2$) it's possible to derive the correlation coefficient necessary to cause the portfolio standard deviation to equal the standard deviation of asset 1. Setting the correlation coefficient to a value less than the solution value will give a portfolio standard deviation that is lower than the standard deviation of either of the individual assets. To find the benchmark correlation, set the portfolio standard deviation equal to that of asset 1 and solve for the correlation coefficient.

Start from the equation for the standard deviation of a two-asset portfolio:

$$\sigma_{portfolio} = (w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_{2\_1,2}\sigma_1\sigma_2)^{.5}$$

where:

$w_{1,2}$ = asset weights

σ^2_1, σ^2_2 = asset variations

$\rho_{1,2}$ = correlation between returns for assets 1 and 2

Below is a brief derivation of the benchmark correlation coefficient required to make the portfolio variance equal to the variance of asset 1 (the asset with lower variance.)

$$\sigma_1^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1 w_2 \rho_{1,2}\sigma_1\sigma_2$$

$$\sigma_1^2 - w_1^2\sigma_1^2 - w_2^2\sigma_2^2 = 2w_1 w_2 \rho_{1,2}\sigma_1\sigma_2$$

$$\frac{\sigma_1^2 - w_1^2\sigma_1^2 - w_2^2\sigma_2^2}{2w_1 w_2 \sigma_1\sigma_2} = \rho_{1,2} \qquad\qquad (1)$$

In equation (1), $\rho_{1,2}$ is the correlation coefficient that will result in a portfolio variance equal to the lower variance of the two assets in the portfolio. Setting the correlation coefficient to a value lower than the solution value will result in "demonstration portfolio benefits" where the standard deviation of the portfolio is lower than the standard deviation of either of the two assets contained in the portfolio.

Consider the following numerical example.

Return$_1$ = 11.39%	σ_1 = 1.7000%
Return$_2$ = 14.91%	σ_2 = 2.4500%
Correlation$_{1,2}$ = .4000	
Weight$_1$ = .Weight$_2$ = .5000	

The portfolio standard deviation resulting from these inputs is equal to 1.7482%, which is greater than the standard deviation of asset 1 held in isolation. Solving for the benchmark correlation coefficient gives a value of .3202. Using this value for Correlation$_{1,2}$ will result in a portfolio standard deviation of 1.7000%. Setting the correlation to any value less than .3202 will give the desired result, a portfolio standard deviation less than 1.7000%. Setting the correlation to any value greater than .3202 will generate a portfolio standard deviation greater than that of asset 1. Under the current scenario, portfolio return is the simple average of the returns of assets 1 and 2, or 13.15%.

The Excel spreadsheet model shown in figure 1 does the necessary calculations. In cell C2 of the spreadsheet the necessary formula is:

(B1^2-B4^2*B1^2-B5^2*B3^2)/(2*B4*B5*B1*B2)

Setting the correlation coefficient to a value less than .3202, say .2500, results in a portfolio standard deviation of 1.6564%, which is lower than either of the two individual assets.

Portfolio Coefficient of Variation Solution

The coefficient of variation (CV), defined as the standard deviation divided by the expected return, may provide a more complete assessment of the risk/return aspects of individual assets or of portfolios of assets. The coefficient of variation is often referred to as a "relative" measure; it represents the variability in returns relative to the expected return. This characteristic is especially beneficial if the numerical values for assets being compared are significantly different in size. Further, students are easily able to grasp the concept of "units of risk per units of return."

Consider the following numerical example.

Expected return$_1$ = 5.0%   Expected return$_2$ = 15.0%
σ_1 = 10.0% σ_2 = 25.0%
CV_1 = 2.0 CV_2 = 1.67

In this example, asset 2 clearly has greater total risk as measured by its standard deviation of expected return. This may lead a risk-averse investor (or introductory student) to select asset 1 which has a much lower standard deviation of expected return. However, evaluating the coefficients of variation for the two assets indicates that asset 1, with 2.0 units of risk for each unit of return is clearly inferior, on a risk/return basis, to asset 2 with only 1.67 units of risk for each unit of return.

Continuing with the first numerical example and including the coefficients of variation we have:

Return$_1$ = 11.39% σ_1 = 1.7000%

Return$_2$ = 14.91% σ_2 = 2.4500%

Return$_{portfolio}$ = 13.15% $\sigma_{portfolio}$ = 1.7482%

Correlation$_{1,2}$ = .4000
Weight$_1$ = Weight$_2$ = .5000

CV_1 = .1493
CV_2 = .1643
$CV_{portfolio}$ = .1329

The risk/return tradeoff for the portfolio (as measured by its CV) is superior to that of either of the individual assets, however, the standard deviation of asset 1 (1.7000%) is lower than the standard deviation of the portfolio (1.7482%). This situation is less than optimal for the demonstration of portfolio benefits to the first time finance student.

Consider the extreme situation in which the correlation coefficient for assets 1 and 2 is equal to +1.0000. In this case, the coefficient of variation for the portfolio would be .1578, which is greater than the coefficient of variation for asset 1 held in isolation. In such a case, an investor would be better off to put 100% of his or her funds into asset 1. This is not the desired result in an introductory lecture or homework assignment.

Following the same approach that was used to find the greatest correlation coefficient that would result in a portfolio standard deviation less than or equal to that of the lower of the two assets, it's also possible to find a benchmark correlation coefficient that will give a portfolio coefficient of variation that is lower than either of the individual assets. Again assuming a two-asset portfolio comprising assets 1 and 2 and assuming the coefficient of variation for asset 1 is the lower of the two assets, we can find the benchmark correlation coefficient by setting the portfolio coefficient of variation equal to the coefficient of variation for asset 1 and then solving the equation for the correlation coefficient.

$$\frac{(w^2_1\sigma^2_1 + w^2_2\sigma^2_2 + 2w_1w_2\sigma_1\sigma_2\tilde{n}_{1,2})^{.5}}{k_{portfolio}} = \frac{\sigma_1}{k_1}$$

$$(w^2_1\sigma^2_1 + w^2_2\sigma^2_2 + 2w_1w_2\sigma_1\sigma_2\tilde{n}_{1,2})^{.5} = \left(\frac{\sigma_1}{k_1}\right)\left(k_{portfolio}\right)$$

$$\tilde{n}_{1,2} = \frac{\left[\left(\frac{\sigma_1}{k_1}\right)\left(k_{portfolio}\right)\right]^2 - w^2_1\sigma^2_1 - w^2_2\sigma^2_2}{2w_1w_2\sigma_1\sigma_2} \qquad (2)$$

In equation (2), $\rho_{1,2}$ is the correlation coefficient that will result in a portfolio coefficient of variation equal to that of the lower of the two assets in the portfolio.

Consider the earlier numerical example where:

$$Return_1 = 11.39\% \qquad \sigma_1 = 1.7000\%$$
$$Return_2 = 14.91\% \qquad \sigma_2 = 2.4500\%$$
$$Weight_1 = Weight_2 = .5000$$

Solving the equation (2) indicates that the portfolio coefficient of variation will equal the coefficient of variation for asset 1 when the correlation between assets 1 and 2 is set to a value of .7822. The recommendation for any correlation coefficient greater than .7822 would be to invest 100% of available funds into asset 1.

The Excel spreadsheet model shown in figure 2 does the necessary calculations. In cell 2 of the spreadsheet the necessary formula is:

$$((((B1/B6)*B8\wedge2)-(B4\wedge2*B1\wedge2)-(B5\wedge2*B3\wedge2))/(2*B4*B5*B1*B3)$$

The Impact of Portfolio Weights

There is no quick and easy solution to the demonstration portfolio problem if the unknown component is the required weights. Typically, this isn't a problem because most introductory examples use equal weights. Although finding optimal portfolio weights is almost certainly beyond the scope of an introductory finance lecture, it may be useful to demonstrate the performance impact resulting from changing individual asset weights in the portfolio.

Continuing with the first numerical example, but changing the asset weights we have:

$$Return_1 = 11.39\% \qquad \sigma_1 = 1.7000\%$$
$$Return_2 = 14.91\% \qquad \sigma_2 = 2.4500\%$$
$$CV_1 = .1493 \qquad CV_2 = .1643$$

$$Correlation_{1,2} = .4000$$
$$Weight_1 = .6000, \; Weight_2 = .4000$$

$$Return_{portfolio} = 12.80\% \qquad \sigma_{portfolio} = 1.6735\%$$
$$CV_{portfolio} = .1308$$

Changing asset weights has resulted in the desired "demonstration portfolio" outcome; a portfolio return that is a weighted average of the returns of the individual assets in the portfolio, but a portfolio standard deviation and coefficient of variation that are each lower than either of the individual assets in the portfolio.

Summary

When introducing students to modern portfolio analysis, the most impressive example of diversification benefits comes when the portfolio standard deviation and coefficient of variation are lower than those of either of the two equally weighted assets contained in the portfolio. I refer to this result as "demonstration portfolio benefits." The "demonstration portfolio problem" is that randomly selecting values for portfolio inputs does not always produce a portfolio that dominates both of the individual assets in terms of standard deviation and coefficient of variation. Sub-optimal portfolio outcomes potentially lead to a sub-optimal learning experience for introductory students.

The principal objective of this paper is to provide solutions for the "demonstration portfolio problem." To reach that objective, this paper provides numerical examples of the "demonstration portfolio problem," derivations leading to solutions for the problem, and also presents simple Excel spreadsheet models which implement these solutions. Using the models provided in this paper, it is an easy matter to select input values for class demonstrations, quiz, test, or homework problems that will provide the desired end result - a portfolio risk/return

tradeoff that is superior to that of either of the two individual assets held in isolation. Ultimately, use of the models developed in this paper allow the finance educator to provide the most effective teaching examples of modern portfolio theory and to do so with a reasonable time commitment.

References

Brigham, Eugene F. and Gapenski, Louis C., *Financial Management Theory and Practice, 9th Ed.*, The Dryden Press, Harcourt Brace College Publishers, 1999

Gallagher, Timothy J. and Andrew, Joseph D, Jr., Financial Management Principles and Practice, Prentice-Hall Inc., New Jersey, 1997

Markowits, Harry M., "Portfolio Selection," *Journal of Finance*, March 1952

Weston, J. Fred, Besley, Scott and Brigham, Eugene F., *Essentials of Managerial Finance, 12th Ed.*, The Dryden Press, Harcourt Brace College Publishers, 1996

Figures

	A	B	C
1	Std Dev Asset 1	1.7000%	Benchmark Correlation
2	Correlation 1,2	0.4000	.3202
3	Std Dev Asset 2	2.4500%	
4	Weight Asset 1	0.5000	
5	Weight Asset 2	0.5000	
6	Return for Asset 1	11.39%	
7	Return for Asset 2	14.91%	

Figure 1, Excel spreadsheet for calculating benchmark correlation necessary for portfolio standard deviation

	A	B	C
1	Std Dev Asset 1	1.7000%	Benchmark Correlation
2	Correlation 1,2	0.4000	.7822
3	Std Dev Asset 2	2.4500%	
4	Weight Asset 1	0.5000	
5	Weight Asset 2	0.5000	
6	Return for Asset 1	11.39%	
7	Return for Asset 2	14.91%	
8	Portfolio Return	13.15%	

Figure 2, Excel spreadsheet for calculating benchmark correlation necessary for portfolio coefficient of variation

Period	Returns		
	Asset #1	Asset #2	Portfolio
1	13.26	16.31	14.78
2	10.42	11.92	11.17
3	9.26	17.17	13.22
4	10.47	11.95	11.21
5	13.54	17.19	15.37

Figure 3, Data set used in the calculation of returns, standard deviations, and correlation coefficient used in the preceding discussion. The data in this figure is rounded to two decimal places.