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Computing the Performance Parameters of Fuzzy Markovian Queueing System FM/FM/1 In Transient Regime by Flexible Alpha-Cuts Method

Jean Alonge w'Omatete [1], Rostin Mabela Makengo Matendo[2], Daniel Lama Okenge[3], Claude Mfumu Mbikudi[4], Bopatriciat Boluma Mangata[5*]

[1]*Department of Mathematics and Physics, Higher Pedagogical Institute of Wembo Nyama , D.R.Congo*
[2,5] *Departement of Mathematics and Computer Science, Faculty of Science and Technology, University of Kinshasa, Kinshasa, D.R.Congo*
[3] *Department of Mathematics and Physics, Pedagogical Institute of Kindu, D.R.Congo*
[4]*Kimwenza High School in Kinshasa, D.R.Congo*

*Corresponding author: * bopatriciat.boluma@unikin.ac.cd*
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HIGHLIGHTS

- Solving the fundamental problem to calculate the performance parameters of the FM/FM/1 queuing system in transient state.
- Resolving the problem of the performance parameters of the FM/FM/1 queuing system in transient state by the fuzzy arithmetic of $\propto -$ cuts and intervals using in transient state (TS).
- Using the analytical method with its different stages.
- Representation of the graphs of the performance parameters of the FM/FM/1 queuing system in the three-dimensional space in the transient regime in a fuzzy environment.

ABSTRACT

This scientific article sets out to solve the following fundamental problem to calculate the performance parameters of the FM/FM/1 queuing system in transient state. To resolve this problem the fuzzy arithmetic of $\propto -$ cuts and intervals was used in transient state (TS). To achieve the expected results of our work, we used the analytical method with its different stages which are : selecting textual or visual documents (books, articles, conferences, seminars,... with or without internet); read and analyze the content of these documents; record selected documents related to our research topic; interpret during the reading and the classification, the data obtained. This method helped us to identify the appropriate mathematical approach to evolve the performance of the queuing system under study. This is precisely the relaxed a-cuts and intervals. To illustrate this approach, a numerical example is given in the sixth section. The membership function enabled us to represent the graphs of the performance parameters of the FM/FM/1 queuing system in the three-dimensional space in the transient regime in a fuzzy environment.

Keywords: *Performance parameters, fuzzy markovian queuing system, flexible α-cuts method, Zadeh's extension principle, α-cuts intervals arithmetic.*

INTRODUCTION

Nowadays, queues play an important role in several areas, whether in production systems, computer systems, communication systems, sanitary systems or any other system in the daily life. The world today is in search of the best quality of service and the best system performance. Regarding to the system performance, Baynat. B points out that ‘’it becomes inconceivable to build any system whatsoever without first having done a performance analysis. This approach must go through a stage of modeling and performance analysis (Dervaraj, 2015).

Thus, facing to such a position and in view of the fact that none of the mathematical models introduced so far in fuzzy queuing theory to evaluate the performance parameters, has been devoted to fuzzy Markovian queueing systems in transient regime. This scientific article sets out to solve the following fundamental problem : “Would the method of relaxed α-cuts be able to calculate the performance parameters of the fuzzy Markovian FM/FM/1 queuing system in the transient regime? ". Thus the main hypothesis of our work is formulated in these terms: “it would be possible to calculate the performance parameters of the queuing system under study via of the α-cuts arithmetic and intervals”. The literature reviewed in fuzzy mathematics points out that many operational researchers such as (Jeeva, 2015, Noora, 2017, Kalayanaraman, 2015, Gani, 2016, Bede, 2015, Mukeba, 2015 & Palpandi, 2016), used the relaxed alpha-cuts method based on the arithmetic of α-cuts and intervals to calculate the performance parameters of the FM/FM/1 queuing system in steady state but not, in transient regime (TR). The question still remains a concern of researchers today. Thus, as for the calculation of the performance parameters of the FM/FM/1 fuzzy Markovian holding system in transient regime by the flexible α-cuts method, we dare affirm with regard to the literature, to bring a not insignificant contribution in the fuzzy queuing theory.

In this work, we limit ourselves to calculate the performance parameters such as the average number of the customers and the average time of the stay of customers in the system at an instant $t(t \geq 0)$of the queueing model in study in transient state To achieve this, our approach is presented as follows: the second section will recall the notions of fuzzy sets and fuzzy numbers. The third section will present Zadeh's extension principle and the two kinds of fuzzy arithmetic. The fourth section will define the fuzzy functions. The fifth section will give the calculation procedure of the relaxed α-cut method. The sixth section will treat a numerical example which will give all the steps of the resolution of the method. Finally the seventh section will give the conclusion of the work.

Fuzzy sets and numbers

Fuzzy sets

Definition 1. (Bede, 2015; Mukeba, 2016). Let E be a classical or a universe set. A fuzzy set Ã (or a fuzzy subset Ã) of E is defined by the function $n_{\tilde{A}}$, called membership function of Ã, ranging from E in the unit interval [0,1]. Thus, $n_{\tilde{A}}(x)$is called the membership degree of x into $E, \forall x \in \tilde{A}$. For each $x \in E$ such that $n_{\tilde{A}}(x) = 1, x$ is said mean value, modal value or mode of Ã.

Definition 2. (Gani, 2017; Mukeba, 2016). Let Ã be a fuzzy subset on E. The alpha-cut $\tilde{A}_{\alpha}$, the support supp (Ã), the height $h(\tilde{A})$and the core core (Ã)of Ã, are classical sets defined respectively as follows, for every $\propto \in [0,1]$:

- $\tilde{A}_{\propto} = \{x \in E / n_{\tilde{A}}(x) \geq \propto\}$ (1)
- $\text{supp}\ (\tilde{A}) = \{x \in E / n_{\tilde{A}}(x) > 0\}$ (2)
- $h\ (\tilde{A}) = \max\{n_{\tilde{A}}\ (x) / x \in E\}$ (3)
- $core\ (\tilde{A}) = \{x \in E / n_{\tilde{A}}\ (x) = 1\}$ (4)

The membership function of a fuzzy set Ã can be expressed in terms of characteristic functions of these α-cuts according to the formula (Dervaraj, 2015):

$$n_{\tilde{A}}(x) = \sup_{\propto \in [0,1]} \min\{\propto, n_{\tilde{A}_\propto}(x)\} \quad (5)$$

where $n_{\tilde{A}_\propto}(x) = \begin{cases} 1, if\ x \in \tilde{A} \propto \\ 0, otherwise \end{cases}$

Definition 3. (Zadeh, 2015; Mukeba, 2016). A fuzzy set Ã is said to be normal $iff\ h(\tilde{A}) = 1$ and convex $iff\ n_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq \min\{n_{\tilde{A}}(x), n_{\tilde{A}}(y)\}, \forall \lambda \in [0,1], \forall x, y \in E$.

Fuzzy numbers

Definition 4. (Buckley and, 2015; Mukeba, 2015). A fuzzy number Ã is a fuzzy subset defined on the universe ℝ such that:
$(i) core\ (\tilde{A}) \neq \emptyset$
$(ii) \tilde{A}_\propto$ is a closed and bounded intervals of ℝ, $for\ every\ \propto \in [0,1]$:
$(iii)\ supp\ (\tilde{A})$ is bounded.

Definition 5. (Barak, 2017). A fuzzy number Ã is said to be strictly positive iff $\forall x < 0, n_{\tilde{A}}(x) = 0$ and negative iff $\forall x > 0, n_{\tilde{A}}(x) = 0$

Definition 6. (Mukeba and al. 2015). Let Ã and $\hat{B}$ be two fuzzy numbers. $\tilde{A} < \hat{B}$ if $\forall x \in supp\ (\tilde{A}), \forall y \in supp\ (\hat{B}), x < y$. In other words, $\tilde{A} < \hat{B}\ iff\ sup\{supp\ (\tilde{A})\} < \inf\{supp\ (\hat{B})\}$. (6)

Definition 7.
A fuzzy number Ã is said a triangular fuzzy number if there exists three real numbers $a < b < c$ such that:

$$n_{\tilde{A}}(x) = \begin{cases} \frac{x-a}{b-a}.if\ a \leq x \leq b \\ \frac{c-x}{c-b}.if\ b < x \leq c \\ o \quad otherwise \end{cases}$$

Note 1.

a. Such triangular fuzzy number is often denoted $\tilde{A} = (a, b, c)$or $\tilde{A}\ (a/b/C)$and b is called the modal value of Ã.
b. If]a , c[is the support of Ã with modal value b , then the cut of Ã of level α is the closed interval:
$\tilde{A}_\propto = [(b-a) \propto +a, (b-c) \propto +c]$ (7)

Zadeh 's Extension Principle and Fuzzy Arithmetic

Zadeh extension principle

Introduced by (Zadeh, 2015) and later developed by (Mary, 2018; Vincent, 2017; Gani, 2016), the extension principle is one of the powerful ideas in the fuzzy set theory. It makes it possible to exploit classical knowledge in the case of fuzzy data (Wang, 2016).

Definition 8. (Bede, 2015). Let $E = E_1 \times E_2 \times \ldots \times E_n$ *and* F *be* two universes and let $\nexists$ an application from E in F. The principle of extension_is an application $\hat{\nexists}$ *, from* $\hat{P}(E)$ into $\hat{P}(F)$, such that $\forall\, \tilde{A} \in \hat{P}(E), \exists\ \hat{B} \in \hat{P}(F)$: $\hat{\nexists}(\tilde{A}) = \hat{B}$, *which is defined, for all* $y \in F$: *by*

$$n_{\hat{B}}(y) = \begin{cases} \sup\left\{\min\left\{n_{\tilde{A}_1}(x_1), n_{\tilde{A}_2}(x_2), \ldots\ldots n_{\tilde{A}_n}(x_n)\right\}\right\} \ if\ \nexists^{-1}_{(y)} \neq \emptyset \\ x \in E/\nexists(x) = y \\ o,\ otherwise \end{cases}$$

where $\hat{P}(E), \hat{P}(F)$ and $\nexists^{-1}$ are respectively the family of fuzzy subsets of E, the family of fuzzy subsets of F and the inverse map of $\nexists$.

Fuzzy arithmetic

Fuzzy arithmetic based on Zadeh 's extension principle

One of the techniques which makes the strength of the principle of extension of Zadeh is the fuzzy arithmetic based on this principle. Using this technique, we have succeeded to extend a classical operation $*$defined in $\mathbb{R}$ to a fuzzy binary operation $\circledast$ in F ($\mathbb{R}$), such that $\forall x, y \in \mathbb{R}$ *and* $\forall \widehat{M}, \widehat{N} \in F(\mathbb{R})$,

$$n_{\widehat{M} \circledast \widehat{N}}(z) = sup\ \{\min\{n_{\widehat{M}}(x), n_{\widehat{N}}(y)\}/x, y \in \mathbb{R}\ and\ x * y = z\}. \quad (8)$$

Fuzzy arithmetic based on α-cuts and intervals arithmetic

Arithmetic of intervals

Definition 9. (Buckley, 2016). Let $[a_1, b_1]$ and $[a_2, b_2]$ be two closed and bounded intervals of $\mathbb{R}$. If $* \in \{+, -, X, \div\}$, then the operation $*$ is defined by:

$$[a_1, b_1] * [a_2, b_2] = [\propto, \beta] \text{or } [\propto, \beta] = \{a * b / a_1 \leq a \leq b_1, a_2 \leq b \leq b_2\} \quad (9)$$

Where the division only has sense if $0 \notin [a_2, b_2]$. With the fundamental operations, the expression (9)is developed as follows:

- $[a_1, b_1] + [a_2, b_2] = [a_1 + a_2, b_1 + b_2]$ (10)
- $[a_1, b_1] - [a_2, b_2] = [a_1 - b_2, b_1 - a_2]$ (11)
- $[a_1, b_1].[a_2, b_2] =$ $[\min\{a_1 a_2, a_1 b_2, b_1 a_2, b_1 b_2\}, \max\{a_1, a_2, a_1 b_1, b_1 a_2, b_1 b_2,\}]$ (12)
- $\frac{[a_1, b_1]}{[a_2, b_2]} = \left[\max\left\{\frac{a_1}{a_2}, \frac{a_1}{b_2}, \frac{b_1}{a_2}, \frac{b_1}{b_2}\right\}, \max\{\frac{a_1}{a_2}, \frac{a_1}{b_2}, \frac{b_1}{a_2}, \frac{b_1}{b_2}\}\right]$ (13)

Arithmetic of α-cuts

Let $\tilde{A}$ and $\hat{B}$ *be* two fuzzy numbers. The fuzzy addition $\oplus$, subtraction $\ominus$, multiplication $\odot$ and division $\oslash$ of $\tilde{A}$ and $\hat{B}$ are defined through their α-cuts($0 \leq \propto \leq 1$) which are closed and bounded intervals of $\mathbb{R}$. If $\tilde{A}_\alpha = [\tilde{A}^L(\propto), \tilde{A}^U(\propto)]$ *and* $\hat{B}_\propto = [\hat{B}^L(\propto), \hat{B}^U(\propto)]$ represent respectively the α-cuts of $\tilde{A}$ and the α-cuts of $\hat{B}$, then (Bede, 2015; Buckley, 2016) give the fundamental operations relative to the α-cuts of $\tilde{A}$ and $\hat{B}$as follows:

- $[\tilde{A} \oplus \hat{B}]_\propto = \tilde{A}_\propto + \hat{B}_\propto = [\tilde{A}^L(\propto), \tilde{A}^U(\propto)] + [\hat{B}^L(\propto), \hat{B}^U(\propto)]$ (14)
- $[\tilde{A} \ominus \hat{B}]_\propto = \tilde{A}_\propto - \hat{B}_\propto = [\tilde{A}^L(\propto), \tilde{A}^U(\propto)] - [\hat{B}^L(\propto), \hat{B}^U(\propto)]$ (15)
- $[\tilde{A} \odot \hat{B}]_\propto = \tilde{A}_\propto . \hat{B}_\propto = [\tilde{A}^L(\propto), \tilde{A}^U(\propto)].[\hat{B}^L(\propto), \hat{B}^U(\propto)]$ (16)

- $\left[\tilde{A} \odot \hat{B}\right]_{\propto} = \frac{\tilde{A}_{\propto}}{\hat{B}_{\propto}} = \frac{[\tilde{A}^{L}_{(\propto)}, \tilde{A}^{U}_{(\propto)}]}{[\hat{B}^{L}_{(\propto)}, \hat{B}^{U}_{(\propto)}]}$ (17)

The α-cuts in (14), (15), 16 *and* (17) *are calculated using the* intervals arithmetic formulas in (10), (11), (12) and (13).

Arithmetic of α-cuts and intervals

Definition 10. To perform the fuzzy arithmetic operations by "the arithmetic of α-cuts and intervals" we must use the following relations :

- (14), (15), (16), (17) for defuzzification,
- (10), (11), (12) *and* (13) for classical arithmetic on closed intervals of $\mathbb{R}$,
- (5) for fuzzification.

These three operations work together to characterize the α-cuts approach.

Fuzzy function

Definition 11. (Allahviranlov, 2015)
Let $\Im(\mathbb{R})$ be the set of fuzzy numbers. We say that $\hat{f}(x)$ is a fuzzy function (or fuzzy-valued function) if it's defined as follows:

$$\hat{f} : \mathbb{R} \longrightarrow \Im(\mathbb{R}) \qquad x \longrightarrow \hat{f}(x) \tag{18}$$

Definition 12. The α-cut representation of a fuzzy function $\hat{f}$ is given by:

$$\hat{f}_{\propto}(x) = [\hat{f}^{L}_{\propto}(x), \hat{f}^{U}_{\propto}(x)] = [\hat{f}^{L}(x,\propto), \hat{f}^{U}(x,\propto)],\ 0 \leq \propto \leq 1 \tag{19}$$

Definition 13. The support of a fuzzy function $\hat{f}$ is defined by:

$$supp\ (\hat{f}(x))_{\propto=0} =]\hat{f}^{L}(x,0), \hat{f}^{U}(x,0)[\tag{20}$$

Definition 14. The kernel (or mode) of $\hat{f}$ is defined by:

$$core\ \left(\hat{f}(x)\right) = \hat{f}^{L}(x,1) = \hat{f}^{U}(x,1), \qquad \propto = 1 \tag{21}$$

Definition 15. The membership function $n_{\hat{f}(x)}$ of the fuzzy function $\hat{f}$ is given by:

$$n_{\hat{f}(x)}(x, s_X) = \begin{cases} (\hat{f}^{L})^{-1}(x, s_X), if\ \hat{f}^{L}(x,0) \leq s_X \leq \hat{f}^{L}(x,1) \\ (\hat{f}^{U})^{-1}(x, s_X), if\ \hat{f}^{U}(x,1) < s_X \leq \hat{f}^{U}(x,0) \\ o, \quad otherwise \end{cases} \tag{22}$$

Where $\hat{f}^{L}(x,o)$ and $\hat{f}^{U}(x,0)$ are positive and continuous real functions

$$n_{\hat{f}(x)}(x, s_x) = \begin{cases} \frac{s_X - \hat{f}^{L}(X,0)}{\hat{f}^{L}(x,1) - \hat{f}^{L}(x,0)}, if\ \hat{f}(x,0) \leq s_x \leq \hat{f}^{L}(x,1) \\ \frac{\hat{f}^{U}(x,0) - s_X}{\hat{f}^{U}(X,0) - \hat{f}^{U}(x,1)}\ if\ \hat{f}^{U}(x,1) \leq s_x \leq \hat{f}^{U}(x,0) \\ 0, \quad othervise \end{cases} \tag{23}$$

where s_x is a real function of x.

Definition 16. (Alonge, 2019). Let $\tilde{p}_x(t)(t \geq 0)$ the fuzzy sate probability of queue FM/FM/1. Then :

$$\tilde{p}_x(t) = \left(\frac{\tilde{\lambda}}{\tilde{\mu}}\right)^x \left(1 - \frac{\tilde{\lambda}}{\tilde{\mu}}\right)\left(1 - e^{-(\tilde{\mu}-\tilde{\lambda})t}\right). \qquad (24)$$

Definition 17. (Alonge, 2019). The fuzzy flow $\tilde{d}\,(t)$ of the system at a date t is defined by:

$$\tilde{d}(t) = [1 - \tilde{p}_0(t)]\,\tilde{\mu}$$
$$= \tilde{\mu}\left[\tilde{\rho} + (1-\tilde{\rho})e^{-(\tilde{\mu}-\tilde{\lambda})t}\right] \qquad (25)$$

$with\ \tilde{\rho} = \frac{\tilde{\lambda}}{\tilde{\mu}}$ is called traffic intensity

Theorem 1. Let $\tilde{N}_S(t)$ be the fuzzy mean number of customers in the system at time $t\ (t \geq 0). So$:

$$\tilde{N}_S(t) = \frac{\tilde{\rho}}{1-\tilde{\rho}}\left(1 - e^{-(\tilde{\mu}-\tilde{\lambda})t}\right) \qquad (26)$$

Demonstration

$$\tilde{N}_S(t) = \tilde{\mathbb{E}}(X_t) = \sum_{x=0}^{\infty} x\, \tilde{p}_x(t)$$
$$= \sum_{x=0}^{\infty} x \left(\frac{\tilde{\lambda}}{\tilde{\mu}}\right)^x \left(1 - \frac{\tilde{\lambda}}{\tilde{\mu}}\right)\left(1 - e^{-(\tilde{\mu}-\tilde{\lambda})t}\right)$$
$$= \left(1 - \frac{\tilde{\lambda}}{\tilde{\mu}}\right)\left(1 - e^{\breve{\upsilon}}\right)\left(\sum_{x=0}^{\infty} x \left(\frac{\tilde{\lambda}}{\tilde{\mu}}\right)^x \qquad with\ \ \breve{\upsilon} = -(\tilde{\mu} - \tilde{\lambda})t$$
$$= \tilde{\rho}(1-\tilde{\rho})\left(1 - e^{\breve{\upsilon}}\right)\left(\sum_{x=0}^{\infty} x\, \tilde{\rho}^{x-1}\right)\ with\ \tilde{\rho} = \frac{\tilde{\lambda}}{\tilde{\mu}}$$
$$= \ \tilde{\rho}(1-\tilde{\varrho})\left(1 - e^{\breve{\upsilon}}\right)\left(\sum_{x=0}^{\infty} \tilde{\rho}^{x}\right)'$$
$$= \tilde{\rho}(1-\tilde{\rho})\left(1 - e^{\breve{\upsilon}}\right)\left(\frac{1}{1-\tilde{\rho}}\right)',\ \left(with\ \sum_{x=0}^{\infty} x = \frac{1}{1-\tilde{\rho}}\right)$$
$$= \tilde{\rho}(1-\tilde{\rho})\left(1 - e^{\breve{\upsilon}}\right)\left[\frac{1}{(1-\tilde{\rho})^2}\right]$$
$$= \frac{\tilde{\rho}\left(1 - e^{\breve{\upsilon}}\right)}{1-\tilde{\rho}}$$

Finally:

$$\tilde{N}_s\,(t) = \frac{\tilde{\rho}(1-e^{-(\tilde{\mu}-\tilde{\lambda})t})}{1-\tilde{\rho}}, \tilde{\rho} = \frac{\tilde{\lambda}}{\tilde{\mu}}\ and\ t \geq 0$$

Theorem 2. Let $\tilde{T}_s\,(t)$ be the average fuzzy residence time of customers in the system at a date $t \geq 0$. So:

$$\tilde{T}_s\,(t) = \tilde{N}_s\,(t)\,[\tilde{d}(\mathrm{t})]^{-1}$$

Demonstration

$$\tilde{T}_s\,(t) = \tilde{N}_s\,(t)\,[\tilde{d}(\mathrm{t})]^{-1}$$
$$= \left[\frac{\tilde{\rho}(1-e^{\tilde{\upsilon}})}{1-\tilde{\rho}}\right]\left[\frac{1}{\tilde{\mu}[\tilde{\rho}+(1-\tilde{\rho})e^{\tilde{\upsilon}}}\right], \tilde{\rho} = \frac{\tilde{\lambda}}{\tilde{\mu}}, \tilde{\upsilon} = -(\tilde{\mu} - \tilde{\lambda})t$$

From where:

$$\tilde{T}_s\,(t) = \frac{\tilde{\rho}(1-e^{-(\tilde{\mu}-\tilde{\lambda})t})}{\tilde{\mu}(1-\tilde{\rho})[\tilde{\rho}+(1-\tilde{\rho})e^{-(\tilde{\mu}-\tilde{\lambda})t}]}, \tilde{\rho} = \frac{\tilde{\lambda}}{\tilde{\mu}}, t \geq 0$$

Computation procedure of the flexible α - cuts method

Consider a fuzzy queue whose rates are fuzzy numbers $\tilde{\xi}_1, \tilde{\xi}_2, \ldots, \widetilde{\xi_n}$; and the performance measure that we want to compute is a denoted $\tilde{\Psi}(t) in$ transient regime (TR) fuzzy function.

In the classical model, this measure and these rates are denoted respectively by Ψ(t) and $\xi_1, \xi_2, \dots \xi_n$. The formula for Ψ(t) in the classical model is often given by :

$$\Psi\,(t)= \not{f}(t, \xi_1, \xi_2, \dots, \xi_n) \tag{27}$$

Where t is a time variable and $\not{f}$ a function with $(n+1)$real variables defined using the fundamental operations " $+, -, X\ and\ \div$ "in $\mathbb{R}$; while in fuzzy model, this same formula is written:

$$\widehat{\Psi}(t) = \hat{\not{f}}(t, \widehat{\xi_1}, \widehat{\xi_2} \dots, \widehat{\xi_n}) \tag{28}$$

Where $\hat{\not{f}}$ is a fuzzy function with $(n+1)$variables one of which is classical (i.e. the time t) and others fuzzy (i.e. $\widehat{\xi_\iota}, i = 1, 2, \dots, n$) ; using the fuzzy fundamental operations " $\oplus, \ominus, \odot$ and $\oslash$ " in $\mathcal{F}(\mathbb{R})$.
To determine the performance parameter $\widehat{\Psi}$ of a fuzzy Markovian queue in transient regime by the method of $\propto -cuts$ relaxations, it suffices to use the arithmetic of $\propto -cuts$ and intervals according to the procedure below.

1. First determine the $\propto -cuts$ of all fuzzy rates $\widehat{\xi_1}, \widehat{\xi_2} \dots, \widehat{\xi_n}$ using (1). We get the closed intervals of $\mathbb{R}$, $\widehat{\xi_{1\propto}} = [\varphi_1(\propto), \omega_1(\propto)]$, $\widehat{\xi_{2\propto}} = [\varphi_2(\propto), \omega_2(\propto)] \dots, \widehat{\xi_{n\propto}} = [\varphi_n(\propto), \omega_n(\propto)]$,where $\varphi_i(\propto)$and $\omega_i(\propto)$are real functions of$\propto$ $(1 \le i \le n)$.
2. Applying to (28) the appropriate expressions of (14), (15), (16) and (17), we obtain:
$$\widehat{\Psi}_\propto(t) = \hat{\not{f}}\,(t, \widehat{\xi_{1\propto}}, \widehat{\xi_{2\propto}} \dots, \widehat{\xi_{n\propto}}) = \hat{\not{f}}(t, [\varphi_1(\propto), \omega_1(\propto)], \dots, [\varphi_2(\propto), \omega_2(\propto)], \dots, [\varphi_n(\propto), \omega_n(\propto)]) \tag{29}$$
3. Applying (10), (11), (12) and (13) to (29), we obtain the following interval formed of $\mathbb{R}$:
$$\widehat{\Psi}_\propto(t) = [\widehat{\Psi}^{\llcorner}(t, \propto), \widehat{\Psi}^{\cup}(t, \propto)] \tag{30}$$

where $\widehat{\Psi}^{\llcorner}(t, \propto)\ et\ \widehat{\Psi}^{\cup}(t, \propto)$ are of real functions of two real variables $t\ and\ \propto$ whose reciprocals define the membership function $of\ \ \widehat{\Psi}\,(t)$as follows:

$$n_{\widehat{\Psi}_{(t)}}(t, X_t) = \begin{cases} (\widehat{\Psi}^{\llcorner})^{-1}(t, x_t), if\ \widehat{\Psi}^{\llcorner}(t,0) \le x_t \le \widehat{\Psi}^{\llcorner}(t,1) & , t \ge 0 \\ \left(\widehat{\Psi}^{\cup}\right)^{-1}(t, x_t)\,, if\ \widehat{\Psi}^{\cup}(t,1) < x_t \le \widehat{\Psi}^{\cup}(t,0) & , t \ge 0 \\ 0 \qquad\qquad\qquad , othervise \end{cases}$$

4. The real $functions$ $\widehat{\Psi}^{\llcorner}(t,0)$and $\widehat{\Psi}^{\cup}(t,0)$obtained in (30) for $\propto = 0$,determine the support bounded of $\widehat{\Psi}(t)$and indicate that the fuzzy function $\widehat{\Psi}(t)$is between $\widehat{\Psi}^{\llcorner}(t,0)$ and $\widehat{\Psi}^{\cup}(t,0)$. Therefore $\widehat{\Psi}(t)$can not go below $\widehat{\Psi}^{\llcorner}(t,0)$or exceed the upper bound $\widehat{\Psi}^{\cup}(t,0)$. The modal function $\widehat{\Psi}^{\llcorner}(t,1)$= $\widehat{\Psi}^{\cup}(t,1)$ is the most possible function of $\widehat{\Psi}(t)$

In order to explain the steps of the method, we show below how it applies to a concrete numerical example.

Numerical example

States

In a reference hospital, an ophthalmologist consults patients on odd-numbered days each week from 10:00 a.m. to 1:30 p.m. The patients arrive there according to a Poisson law of parameter $\tilde{\lambda}$ and the consultation of the doctor according to an exponential negative law of parameter $\tilde{\mu}$. Fuzzy Parameters $\tilde{\lambda}\ et\ \tilde{\mu}$ are such that $\frac{\tilde{\lambda}}{\tilde{\mu}}$ is approximately 0.4. Note that $\tilde{\rho} = {}^{\tilde{\lambda}}/_{\tilde{\mu}}$ is the fuzzy trafic intensity. We also warn that this traffic intensity is a triangular fuzzy number and is noted by $\tilde{\rho} = (0,3/0{,}4/0{,}5)$.

Questions

Determine in transient regime (RT), the following performance measures:

a) The average number of patients in the system;
b) The average length of stay of patients in the system

Give the graphical representation of these fuzzy performance parameters and interpret the results.

Solution

Careful reading of the given example reveals that it is a fuzzy Markovian waiting system denoted FM/FM/1 with a single server and infinite capacity. The traffic intensity being about 0.4 implies that the fuzzy rates $\tilde{\lambda}$ and $\tilde{\mu}$ are respectively about 2 and 5. By assumption, the fuzzy traffic intensity $\tilde{\rho}$ being a triangular fuzzy number, the rates $\tilde{\lambda}$ and $\tilde{\mu}$ are also fuzzy numbers triangular and can be written (cfr. Remark 2.1. point a.) $\tilde{\lambda} = (1/2/3)$ and $\tilde{\mu} = (4/5/6)$., $\tilde{\lambda} < \tilde{\mu}$.

Let us suppose that $\lambda$and $u$are arrival and service rates in the classical model and let us also suppose that $\widetilde{N}s\ (t) et\, \widetilde{T}s\ (t)$ are respectively the average number of patients and the average length of stay of patients in the system at a time t. In queuing theory, it is known that a queuing system is stable if $\frac{\lambda}{\mu} < 1$, this condition makes it possible to calculate these performance parameters in RT using the following formulas (Alonge, 2019):

- $N_s\ (\mathrm{t}) = \frac{\lambda\,(1-e^{-(\mu-\lambda)t})}{\mu-\lambda}$ (31)
- $T_s(t) = \frac{\lambda\,(1-e^{-(\mu-\lambda)t})}{(\mu-\lambda\,)[\lambda+(\mu-\lambda\,)e^{-(\mu-\lambda\,)t}]}$ (32)

By Zadeh's extension principle, the performance parameters in (31) and (32) can be extended to fuzzy performance parameters in transient regime (TR) $\widetilde{N}s\ (t) et\, \widetilde{T}s\ (t)$ whose mathematical formulas are:

- $\widetilde{N}s\ (t) = \frac{\tilde{\lambda}(1-e^{-(\tilde{\mu}-\tilde{\lambda})t})}{\tilde{\mu}-\tilde{\lambda}}$ (33)
- $\widetilde{T}s\ (t) = \frac{\tilde{\lambda}(1-e^{-(\tilde{\mu}-\tilde{\lambda})t})}{(\tilde{\mu}-\tilde{\lambda})[\tilde{\lambda}+(\tilde{\mu}-\tilde{\lambda})e^{-(\tilde{\mu}-\tilde{\lambda})t}]}$ (34)

where rates $\tilde{\lambda}\ and\ \tilde{\mu}$ are fuzzy variables.

Note 2. The $\propto -cuts$ of $\tilde{\lambda}\ and\ \tilde{\mu}$ are (cfr. Note 1, b):

$\tilde{\lambda}_\propto = [\propto +1, -\propto +3]$ (35)
$\tilde{\mu}_\propto = [\propto +4, -\propto +6]$ (36)

Average number of patients in the system at a time t ($t \geq 0$) $\widetilde{N}s\ (t)$

To obtain $\widetilde{N}s\ (t)$, we must first determine the $\propto -cuts[\widetilde{N}s\ (t)]_\propto$ as shown in (30) in the following way:

$[\widetilde{N}s\ (t)]_{\propto} = \left[\frac{\tilde{\lambda}(1-e^{-(\tilde{\mu}\ominus\tilde{\lambda})t}}{\tilde{\mu}\ominus\tilde{\lambda}}\right]_{\propto}$

using (14) *into* (17), *we obtain*

$= \frac{\tilde{\lambda}_{\propto}(1-e^{-(\tilde{u}\ominus\tilde{\lambda})t})}{\tilde{\mu}_{\propto}-\tilde{\lambda}_{\propto}} \Downarrow$ *Replacing* $\tilde{\lambda}_{\propto}$ *and* $\tilde{\mu}_{\propto}$ *by their values from* (35) *and* (36) *we obtain*

$= \frac{[\propto+1,-\propto+3]-[\propto+1-\propto+3]e^{-(2\propto+1,-2\propto+5)t}}{[2\propto+1,-2\propto+5]}$, (*with* $\tilde{\mu} - \tilde{\lambda} = [2 \propto +1 - 2 \propto +5]$ *according to* (11))

using (11)*to subtract in numerator, we get* :

$= \frac{[\propto+1,-(\propto+3)e^{x},-\propto+3-(\propto+1)e^{x}]}{[2\propto+1,-2\propto+5]}$, *with* $x = -[2 \propto +1, \quad -2 \propto +5]t$

Using (13) for the division, we get

$[\widetilde{N}s\ (t)]_{\propto} = [min\ A(t, \propto), \max A(t, \propto)]$

Where $\min A(t, \propto)$ *and* $\max A(t, \propto)$are obtained by solving the following two parametric nonlinear programs (PNLP):

$$\begin{cases} \min A(t,\propto) = \quad ? \\ \min A(t,\propto) = \min\{a_1(t,\propto), a_2(t,\propto), a_3(t,\propto), a_4(t,\propto)\} \\ \quad a_1(t,\propto) = \frac{\propto+1-(-\propto+3)e^{X}}{2\propto+1} \\ \quad a_2(t,\propto) = \frac{\propto+1-(-\propto+3)e^{X}}{-2\propto+5} \\ \quad a_3(t,\propto) = \frac{-\propto+3-(\propto+1)e^{X}}{2\propto+1} \\ \quad a_4(t,\propto) = \frac{-\propto+3-(+\propto+1)e^{X}}{-2\propto+5} \\ 0 \le\propto\le 1 \quad , \ t \ge 0 \end{cases}$$

$$\begin{cases} \max A(t,\propto) = \quad ? \\ \max A(t,\propto) = \max\{a_1(\propto,t), a_2(t,\propto), a_3(t,\propto), a_4(t,\propto)\} \\ \quad a_1(t,\propto) = \frac{\propto+1-(-\propto+3)e^{X}}{2\propto+1} \\ \quad a_2(t,\propto) = \frac{\propto+1-(-\propto+3)e^{X}}{-2\propto+5} \\ \quad a_3(t,\propto) = \frac{-\propto+3-(\propto+1)e^{X}}{2\propto+1} \\ \quad a_4(t,\propto) = \frac{-\propto+3-(+\propto+1)e^{X}}{-2\propto+5} \\ 0 \le\propto\le 1 \quad , \ t \ge 0 \end{cases}$$

Solving these two PNLPs gives:

$$[\min A(t,\propto), \max A(t,\propto)] = [a_2(t,\propto), a_3(t,\propto)]$$
$$= \left[\frac{\propto +1 - (-\propto +3)e^{X}}{-2 \propto +5}, \frac{-\propto +3 - (-\propto +1)e^{X}}{2 \propto +1}\right]$$

Whose $[\widetilde{N}s\ (t)]_{\propto}$is written:

$$[\widetilde{N}s\ (t)]_{\propto} = \left[\frac{\propto+1-(-\propto+3)e^{X}}{-2\propto+5}, \frac{-\propto+3-(\propto+1)e^{X}}{2\propto+1}\right] \quad (37)$$
$$\text{with } x = -[2 \propto +1, -2 \propto +5]t \qquad , t \ge 0$$

Average length of stay of patients in the system at a date t$(t \ge 0)$

To obtain $\widetilde{T}s\ (t)$, we must first determine the $\propto -cuts\ [\widetilde{T}s\ (t)]_{\propto}$as indicated in (30) and we have progressively:

$$[\widetilde{T}s\ (t)]_{\propto} = \left[\frac{\tilde{\lambda} - \tilde{\lambda} e^{-(\tilde{\mu}-\tilde{\lambda})t}}{(\tilde{\mu}-\tilde{\lambda})[\,\tilde{\lambda} + (\tilde{\mu}-\tilde{\lambda})e^{-(\tilde{\mu}-\tilde{\lambda})t}}\right]_{\propto}$$

Using formula from (14) *at* (17), *we obtain*

$$\left[\widetilde{T}s\ (t)\right]_{\propto} = \frac{\tilde{\lambda}_{\propto} - \widetilde{\tilde{\lambda}_{\propto}}\ e^{-(\tilde{\mu}\propto - \tilde{\lambda}\propto)t}}{(\tilde{\mu}_{\propto} - \tilde{\lambda}_{\propto})\left[\tilde{\lambda}_{\propto} + (\mu - \tilde{\lambda}_{\propto})e^{-(\tilde{\mu}\propto - \tilde{\lambda}\propto)t}\right]}$$

Replacing $\tilde{\lambda}_{\propto}$ *and* $\tilde{u}_{\propto}$ *by their values from* (35) *and* (36), *we get*

$$[\widetilde{T}s\ (t)]_{\propto} = \frac{[\propto+1,-\propto+3]-[\propto+1,-\propto+3]e^{-[2\propto+1,-2\propto+5]t}}{[2\propto+1,-2\propto+5]\ \{[\propto+1,-\propto+3]+[2\propto+1,-2\propto+5]e^{-[2\propto+1,-2\propto+5]}\}}$$, namely that

$$\tilde{\mu}_{\propto} - \tilde{\lambda}_{\propto} = [\propto +4, -\propto +6] - [\propto +1, -\propto +3] = [2 \propto +1, -2 \propto +5].$$

using (11) *to substract in numerator and denominator, we obtain*

$$[\widetilde{T}s\ (t)]_{\propto} = \frac{[\propto +1 - (-\propto +3)e^{x}, -\propto +3 - (\propto +1)e^{x}]}{[2 \propto +1, -2 \propto +5].[\propto +1 + (2 \propto +1)e^{x}, -\propto +3 + (-2 \propto +5)e^{x}]}\ avec$$
$$x = -[2 \propto +1, -2 \propto +5]t, t \geq 0$$

Using (12)*to denominator, we obtain*

$$[\widetilde{T}s\ (t)]_{\propto} = \frac{[\propto +1 - (-\propto +3)e^{x}, -\propto +3 - (\propto +1)e^{x}]}{[\min B(t,\propto), \max B(t,\propto)]}$$

where $\min B(t,\propto)$and max B (t,∝) are solutions of the following parametric non linear programs $(PNLP)$:

$$\begin{cases} \min B(t,\propto) = \qquad ? \\ \min B(t,\propto) = \min\{b_1(t,\propto), b_2(t,\propto), b_3(t,\propto), b_4(t,\propto)\} \\ \quad b_1(t,\propto) = (2 \propto +1)[\propto +1 + (2 \propto +1)e^{x}] \\ \quad b_2(t,\propto) = (2 \propto +1)[-\propto +3 + (-2 \propto +5)e^{x}] \\ \quad b_3(t,\propto) = (-2 \propto +5)[\propto +1 + (2 \propto +1)e^{x}] \\ \quad b_4(t,\propto) = (-2 \propto +5)[-\propto +3 + (-2 \propto +5)e^{x}] \\ 0 \leq \propto \leq 1 \quad , \quad t \geq 0 \end{cases}$$

$$\begin{cases} \max B(t,\propto) = \qquad ? \\ \max B(t,\propto) = \max\{b_1(t,\propto), b_2(t,\propto), b_3(t,\propto), b_4(t,\propto)\} \\ \quad b_1(t,\propto) = (2 \propto +1)[\propto +1 + (2 \propto +1)e^{x}] \\ \quad b_2(t,\propto) = (2 \propto +1)[-\propto +3 + (-2 \propto +5)e^{x}] \\ \quad b_3(t,\propto) = (-2 \propto +5)[\propto +1 + (2 \propto +1)e^{x}] \\ \quad b_4(t,\propto) = (-2 \propto +5)[-\propto +3 + (-2 \propto +5)e^{x}] \\ 0 \leq \propto \leq 1 \quad , \quad t \geq 0 \end{cases}$$

Solving these two PNLPs gives:

$$\begin{aligned}[\min B(t,\propto), \max B(t,\propto)] &= [b_1(t,\propto), b_4(t,\propto)] \\ &= [(2 \propto +1)[\propto +1 + (2 \propto +1)e^{x}, \\ &\qquad (-2 \propto +5)[-\propto +3 + (-2 \propto +5)e^{x}],\end{aligned}$$

and we have:

$$[\tilde{T}s\ (t)]_{\propto} = \frac{[\propto +1 - (-\propto +3)e^{x}, -\propto +3 - (\propto +1)e^{x}]}{[(2 \propto +1)[\propto +1 + (2 \propto +1)e^{x}, (-2 \propto +5)[-\propto +3 + (-2 \propto +5)e^{x}]]}$$

$$\Downarrow\ Appling\ (13), to\ divide\ the\ two\quad factors, we\ obtain$$

$$[\widetilde{T}s\ (t)]_{\propto} = [\min C(t,\propto), \max C\ (t,\propto)]$$

where $\min C(t,\propto)$ *and* $\max C\ (t,\propto)$are solutions of the following two PNLPS:

$$\begin{cases} \min C(t,\propto) = \quad ? \\ \min C(t,\propto) = \min\{c_1(t,\propto), c_2(t,\propto), c_3(t,\propto), c_4(t,\propto)\} \\ c_1(t,\propto) = \dfrac{\propto +1 - (-\propto +3)e^x}{(2\propto +1)[\propto +1 + (2\propto +1)e^x]} \\ c_2(t,\propto) = \dfrac{\propto +1 - (-\propto +3)e^x}{(-2\propto +1)[-\propto +3 + (2\propto +1)e^x]} \\ c_3(t,\propto) = \dfrac{-\propto +3 - (\propto +1)e^X}{(2\propto +1)[\propto +1 + (2\propto +1)e^x]} \\ c_4(t,\propto) = \dfrac{-\propto +3 - (\propto +1)e^x}{(-2\propto +5)[-\propto +3 + (-2\propto +5)e^x]} \\ 0 \le \propto \le 1 \ , \quad t \ge ,0 \end{cases}$$

$$\begin{cases} \max C(t,\propto) = \quad ? \\ \max C(t,\propto) = \max\{c_1(t,\propto), c_2(t,\propto), c_3(t,\propto), c_4(t,\propto)\} \\ c_1(t,\propto) = \dfrac{\propto +1 - (-\propto +3)e^X}{(2\propto +1)[\propto +1 + (2\propto +1)e^x]} \\ c_2(t,\propto) = \dfrac{\propto +1 - (-\propto +3)e^X}{(-2\propto +1)[-\propto +3 + (2\propto +1)e^x]} \\ c_3(t,\propto) = \dfrac{-\propto +3 - (\propto +1)e^X}{(2\propto +1)[\propto +1 + (2\propto +1)e^x]} \\ c_4(t,\propto) = \dfrac{-\propto +3 - (\propto +1)e^x}{(-2\propto +5)[-\propto +3 + (-2\propto +5)e^x]} \\ 0 \le \propto \le 1 \ , \quad t \ge ,0 \end{cases}$$

Solving these two PNLPs gives:

$$[\min C(t,\propto), \max C(t,\propto)] = [c_2(t,\propto), c_3(t,\propto)] = \left[\frac{\propto +1 - (-\propto +3)e^x}{(-2\propto +5)[-\propto +3 + (-2\propto +5)e^x]} \ , \ \frac{-\propto +3 - (\propto +1)e^x}{(2\propto +1)[\propto +1 + (2\propto +1)e^x]}\right]$$

Finally, we have:

$$[\tilde{T}s\ (t)]_\propto = \left[\frac{\propto+1-(-\propto+3)e^x}{(-2\propto+5)[-\propto+3+(-2\propto+5)e^x]} \ , \ \frac{-\propto+3-(\propto+1)e^x}{(2\propto+1)[\propto+1+(2\propto+1)e^x]}\right] \tag{38}$$

with $x = -[2\propto +1, -2\propto +5].t\ , \quad t \ge 0$

Note 3.

For $\propto = 0$

$i) The\ interval\ x = -[1,5]t\ \ have\ to\ center - valued\ 3\ \ and\ we\ write\ x = -3t, (t \ge 0).$

$$ii) \left[\tilde{N}s\ (t)\right]_{\propto=0} = supp(\tilde{N}s\ (t)) =]0{,}2 - 0{,}6\, e^{-3t}, 3 - e^{-3t}[\tag{39}$$

is the support of $\tilde{N}s\ (t)$.

$$iii)\ [\tilde{T}s\ (t)]_{\propto=0} = supp\left(\tilde{T}s\ (t)\right) = \left]\frac{1-3e^{-3t}}{15+25e^{-3t}}, \frac{3-e^{-3t}}{1+e^{-3t}}\right[\tag{40}$$

Is the support of $\tilde{T}s\ (t)$.

For$\propto = 1$:
$i)\ The\ interval\ x = -[3,3]t\ have\ to\ center - valued\ 3\ t\ and\ we\ writ : X = -3t\ (t \geq 0)$
$ii)\ [\widetilde{N}s\ (t)]_{\propto=1} = Noy\ \left(\widetilde{N}s\ (t)\right) = \{0{,}7 - 0{,}7e^{-3t}\}\ , t \geq 0$ (41)
is the kernel (or mode) of $\widetilde{N}s\ (t)$.
$iii)[\tilde{T}s\ (t)]_{\propto=1} = Noy\ \left(\tilde{T}s\ (t)\right) = \left\{\frac{2-2e^{-3t}}{6+9e^{-3t}}\right\}, t \geq 0$ (42)
is the kernel (or mode) of $\tilde{T}s\ (t)$.

Graphics

If $\propto$ varies from 0 to 1, the closed intervals in (37) and (38) describe the graphs of the membership functions of the fuzzy performance parameters $\widetilde{N}s\ (t)\ and\ \tilde{T}s\ (t)$ of fuzzy markovian queuing system FM/FM/1 in transient regime. These graphs are represented on Figures 1 and 2 as follows:

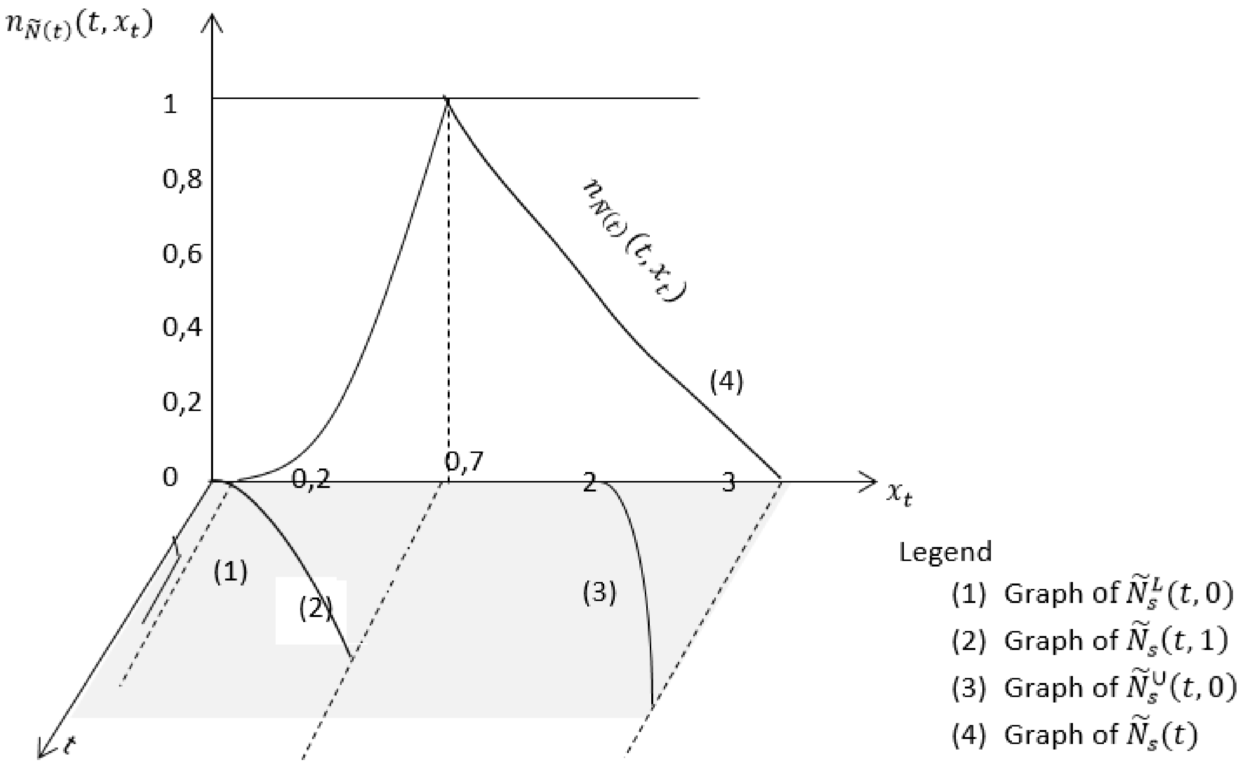

Figure 1: Graph of the membership function $n_{\widetilde{N}s(t)}$of the fuzzy function $\widetilde{N}_s(t)$

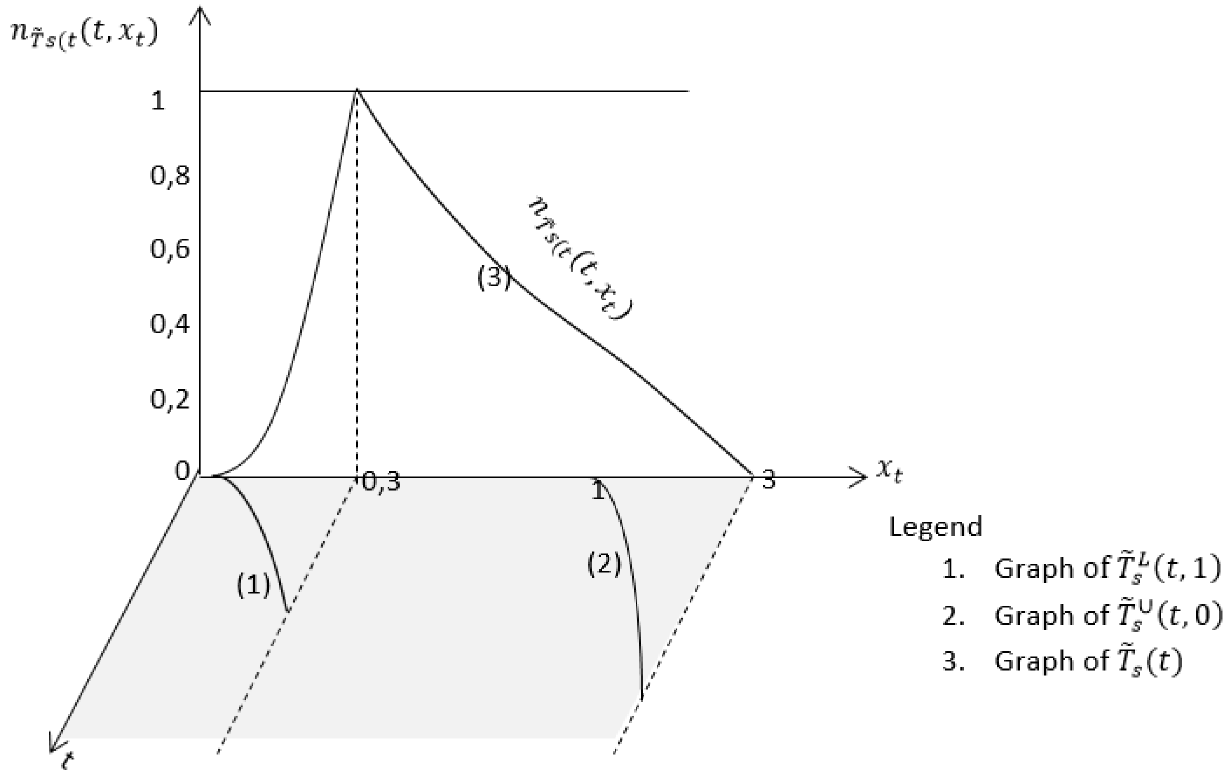

Figure 2 : Graph of the membership function $n_{\tilde{N}s(t)}$ of the fuzzy function $\tilde{T}_s(t)$.

Results and discussion

The system stability condition $\tilde{\lambda} < \tilde{\mu}$ being satisfied, the standby system in this case operates normally and we can evaluate the performance parameters of system under study in transient regime. Thus, if we pose successively $\propto = 0\ et\ \ \propto = 1\ in\ the\ expressions$ $(37) and$ (38),we obtain respectively $\widetilde{N_s}(t,0) = [0{,}2 - 0{,}6e^{-3t};\ 3 - e^{-3t}]$, $\widetilde{Ts}(t,0) = \left[\frac{1-3e^{-3t}}{15+25e^{-3t}}, \frac{3-e^{-3t}}{1+e^{-3t}}\right]$ and $\widetilde{Ns}(t,1) = \{0{,}7 - 0{,}7e^{-3t}\}$, $\widetilde{Ts}(t,1) = \left\{\frac{2-2e^{-3t}}{6+9e^{-3t}}\right\}$. Figure 1 shows that the average number of patients in the waiting system $\widetilde{Ns}(t)$,is a fuzzy function of time whose the graph is included between the curves of equation $\widetilde{N_s}^L(t,0) = 0{,}2 - 0{,}6e^{-3t} and$ $\widetilde{N_s}^U(t,0) = 3 - e^{-3t}$.The curve $\widetilde{Ns}(t,1) = 0{,}7 - 0{,}7e^{-3t}$is its most possible modal curve. For $t \to +\infty$,the average fuzzy number $\widetilde{Ns}$ is approximately between 0 and 3 patients in the system.

Figure 2 in turn indicates that the mean time of stay of patients in the system $\widetilde{Ts}(t)$, is a fuzzy function of time whose the graph is included between the curves of equations $\widetilde{T_s}^L(t,0) = \frac{1-3e^{-3t}}{15+25e^{-3t}}\ and\ \widetilde{T_s}^U(t,0) = \frac{3-e^{-3t}}{1+e^{-3t}}$.The curve of equation $\widetilde{Ts}(t,1) = \frac{2-2e^{-et}}{6+9e^{-3t}}$is the curve of its most modal function possible. For $t \to +\infty$,the average waiting time $\widetilde{Ts}$of patients in the system varies approximately between 0.1 and 3 hours, or between 6 and 180 minutes. Its maximum possible value is 0.3 hours, or 18 minutes.

Conclusion

At the end of our scientific article which focused on "the computing of performance parameters of fuzzy Markovian system FM/FM/1 in transient regime by flexible $\propto -cuts$ method", it was a question of calculating these.

Performance of the holding system in transient regime using flexible $\propto -cuts\ method$ based on the fuzzy arithmetic of $\propto -cuts$ and intervals.

In practice, to facilitate the calculations, a procedure of flexible $\propto -cuts$ was developed in transient regime and enabled us to find the $\propto -cuts$parameters of performance, their supports and modes as well as their membership functions which made it possible to graph in three-dimensional space. This underlines the originality and the contribution of this research in the theory of queuing. We can also examine in transient regime, the same problem when the number of servers increases and becomes greater than or equal to two for a fuzzy Markovian queue.

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