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On the graded identities of the Grassmann algebra*

Review Article

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Abstract: We survey the results concerning the graded identities of the infinite dimensional Grassmann algebra.

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1. Introduction

All algebras we refer to are assumed to be associative with unit and all fields are assumed to be infinite unless explicitely written. Moreover, every group is an abelian group unless explicitely written.

The Grassmann algebra is the algebra of the wedge product, also called an alternating algebra or exterior algebra. It fits in several places in mathematics and, in general, in sciences as well. We recall that if F is a field and $X = \{x_1, x_2, \ldots\}$ is a countable infinite set of variables, let $F\langle X\rangle$ be the free associative algebra freely generated by X over F. We shall refer to the elements of $F\langle X\rangle$ as *polynomials* in the set of variables X. If A is an F-algebra, we say that $f(x_1, \ldots, x_n) \in F\langle X\rangle$ is a *polynomial identity of* A if $f(a_1, \ldots, a_n) = 0$ for any $a_1, \ldots, a_n \in A$. If A has a non-trivial polynomial identity we say that A is a *polynomial identity algebra* or *PI-algebra* and we denote by $T(A)$ the set of all polynomial identities satisfied by A. It is well known that $T(A)$ is an ideal of $F\langle X\rangle$ invariant under all endomorphisms of $F\langle X\rangle$, i.e., it is a T-ideal called the T*-ideal of* A.

Concerning the mathematical aspects of the Grassmann algebra and, in particular, the algebraic ones, the Grassmann algebra E generated by an infinite dimensional vector space and its identities, play an important role in the structure theory of Kemer on varieties of associative algebras with polynomial identities [21, 22]. More precisely, Kemer proved that any associative PI-algebra over a field F of characteristic zero satisfies the same identities (is *PI-equivalent*) of the Grassmann envelope of a finite dimensional associative superalgebra, i.e., they have the same T-ideal. Moreover for any associative algebra A, $T(A)$ is finitely generated as a T-ideal.

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One of the main goals of the theory of PI-algebras is finding a complete set of generators of the T-ideal of a given algebra. For example, it is well known that if A is a commutative unitary algebra, then $T(A)$ is generated by the Lie commutator $[x, y] := xy - yx$. The problem turns out to be very hard even for finite dimensional algebras. In fact, if we consider $M_n(F)$ the matrix algebra over a field F of characteristic 0, we know that its T-ideal is well known only for the case $n = 2$ (see [31] and [15]), provided that the case $n = 1$ is trivial. If F has positive characteristic $p \neq 2$ we have a description of a finite basis of identities for $M_2(F)$ (see [24]). We note that some further partial results for $M_2(F)$ in the case of characteristic 2 were obtained in [16] and [23] but it is still unknown if the T-ideal of $M_2(F)$ is finitely generated or not in this case.

Let us consider a generalization of the definition of polynomial identity for a G-graded algebra $A = \bigoplus_{g\in G} A^g$, where G is any group. If we specialize the variable from X with a G-degree, to say $\|\cdot\|$, we obtain a set $F\langle X|G\rangle$ of "graded polynomials". Of course we may generalize the notion of polynomial identity with the notion of G-graded polynomial identity in a natural way. We say that $f(x_1, \dots, x_n) \in F\langle X|G\rangle$ is a *G-graded polynomial identity of A* if $f(a_1, \dots, a_n) = 0$ for all $a_1, \dots, a_n \in \bigcup_{g\in G} A^g$ such that $a_i \in A^{\|x_i\|}$. As well as in the ungraded (or ordinary) case we denote by $T_G(A)$ the set of all G-graded polynomial identities satisfied by A. It is well known that $T_G(A)$ is an ideal of $F\langle X|G\rangle$ invariant under all the G-graded endomorphisms of $F\langle X|G\rangle$, i.e., it is a T_G-ideal called the *T_G-ideal of A*. In [38] Vasilovsky gives a complete description of the $T_{\mathbb{Z}_n}$-ideal of $M_n(F)$ for a particular $\mathbb{Z}_n$-grading and for all $n \geq 2$ in characteristic 0 whereas Azevedo in [3] obtained the same results without any restriction on the ground field. We recall that the works by Vasilovsky and by Azevedo are a generalization of the work by Di Vincenzo for 2 by 2 matrices (see [11]). As well as in the ordinary case, if A is an associative algebra graded by a finite group G, then $T_G(A)$ is finitely generated as a T_G-ideal (see [1]).

Coming back to the Grassmann algebra E, we know that in the ordinary case $T(E)$ is generated by the triple commutator $[x, y, z] := [[x, y], z]$ as shown in the papers by Latyshev [27] and [26] by Krakovski and Regev. For this purpose, we want to point out that the latter two papers deal with characteristic 0 only even if the argument used in [27] is still valid in positive characteristic as the argument used in [17] Theorem 5.1.2. For this purpose, see also the paper by Giambruno and Koshlukov [18]. In light of what we said until now and with the intent of a future use, in this paper we want to collect the results concerning the graded identities (and other related topics) of the Grassmann algebra trying to be as exaustive as possible. We recall that in [36] the author collected the results related to $\mathbb{Z}_2$-gradings of E.

2. Graded PI-algebras

We introduce the terminology for the study of graded polynomial identities. We start off with the following definition. In the sequel every algebra is associative with unit and every field is infinite unless explicitely written.

Definition 2.1. *Let G be a group and A be an algebra over a field F. We say that the algebra A is G-*graded *if there exist subspaces A^g, $g \in G$ such that $A = \bigoplus_{g\in G} A^g$ as a vector space and for all g, $h \in G$, one has $A^g A^h \subseteq A^{gh}$.*

It is easy to note that if a is any element of A it can be uniquely written as a finite sum $a = \sum_{g\in G} a_g$, where $a_g \in A^g$. We shall call the subspaces A^g the *G-homogeneous components of A*. Accordingly, an element $a \in A$ is called *G-homogeneous* if exists $g \in G$ such that $a \in A^g$. If $B \subseteq A$ is a subspace of A, B is G-graded if and only if $B = \bigoplus_{g\in G}(B \cap A^g)$. Analogously one can define G-graded *algebras, subalgebras, ideals, etc.* We say that a G-grading on A is *homogeneous* if there exists a linear basis $\mathcal{B}$ of A such that every element of $\mathcal{B}$ is a homogeneous element of A.

Let $\{X^g \mid g \in G\}$ be a family of disjoint countable sets of indeterminates. Set $X = \bigcup_{g\in G} X^g$ and denote by $F\langle X|G\rangle$ the free associative algebra freely generated by X over F. An indeterminate $x \in X$ is said to be of *homogeneous G-degree* g, written $\|x\| = g$, if $x \in X^g$. We always write x^g if $x \in X^g$. The homogeneous G-degree of a monomial $m = x_{i_1} x_{i_2} \cdots x_{i_k}$ is defined to be $\|m\| = \|x_{i_1}\| \cdot \|x_{i_2}\| \cdot \cdots \cdot \|x_{i_k}\|$. For every $g \in G$, denote by $F\langle X|G\rangle^g$ the subspace of $F\langle X|G\rangle$ spanned by all monomials having homogeneous G-degree g. Notice that $F\langle X|G\rangle^g F\langle X|G\rangle^{g'} \subseteq F\langle X|G\rangle^{gg'}$ for all $g, g' \in G$. Thus

$$F\langle X|G\rangle = \bigoplus_{g \in G} F\langle X|G\rangle^g$$

is a G-graded algebra. The elements of the G-graded algebra $F\langle X|G\rangle$ are referred to as *G-graded polynomials* or, simply, *graded polynomials*. In order to simplify the notation we shall sometimes use y_i's to denote variables of homogeneous degree 1_G, z_i's to denote variables of homogenous degree different than 1_G and x_i's for any variables without distinguish their homogenous degree.

Definition 2.2. *If A is a G-graded algebra, a G-graded polynomial*

$$f(x_1, \ldots, x_n)$$

is said to be a graded polynomial identity *of A if*

$$f(a_1, a_2, \ldots, a_n) = 0$$

for all $a_1, a_2, \ldots, a_n \in \bigcup_{g \in G} A^g$ such that $a_k \in A^{\|x_k\|}$, $k = 1, \ldots, n$. We shall write $f \equiv 0$ in order to say that f is a graded polynomial identity for A. If A satisfies a non-trivial graded identity we say that A is a PI G-graded algebra. When the algebra A is graded by the trivial group (in fact it has no grading) we refer to polynomial identities of A and T-ideal of A.

Given an algebra A graded by a group G, we define

$$T_G(A) := \{f \in F\langle X|G\rangle | f \equiv 0 \text{ on } A\},$$

the set of G-graded polynomial identities of A.

Definition 2.3. *An ideal I of $F\langle X|G\rangle$ is said to be a T_G-ideal if it is invariant under all F-endomorphisms $\varphi : F\langle X|G\rangle \to F\langle X|G\rangle$ such that $\varphi(F\langle X|G\rangle^g) \subseteq F\langle X|G\rangle^g$ for all $g \in G$.*

Hence $T_G(A)$ is a T_G-ideal of $F\langle X|G\rangle$. On the other hand, it is easy to check that all T_G-ideals of $F\langle X|G\rangle$ are of this type. If $S \subseteq F\langle X|G\rangle$, we shall denote by $\langle S\rangle^{T_G}$ the *T_G-ideal generated by the set S*, i.e., the smallest T_G-ideal containing S. Moreover, given a set of polynomials $\mathcal{S} \subseteq F\langle X|G\rangle$, we say that I is the T_G-ideal generated by $\mathcal{S}$, if I is the smallest T_G-ideal containing $\mathcal{S}$. In this case we say that $\mathcal{S}$ is a basis for I or that the elements of I *follow from* or *are consequences of* the elements of $\mathcal{S}$. If $\mathcal{S}$ is a finite set generating the T-ideal I we say I *is finitely based*. Notice that being a basis for a T-ideal does not imply being a minimal basis.

The theory of PI G-graded algebras in characteristic zero passes through the representation theory of the symmetric group. We consider the following S_n-modules.

Definition 2.4. *Let*

$$P_n^G = span\langle x_{\sigma(1)}^{g_1} x_{\sigma(2)}^{g_2} \cdots x_{\sigma(n)}^{g_n} | g_i \in G, \sigma \in S_n\rangle,$$

then the elements in P_n^G are called multilinear polynomials *of degree n of $F\langle X|G\rangle$.*

It turns out that P_n^G is a left S_n-module under the natural left action of the symmetric group S_n; we denote the S_n-character of the factor module $P_n^G/(P_n^G \cap T_G(A))$ by $\chi_n^G(A)$, and by $c_n^G(A)$ its dimension over F. We say that

$$\left(\chi_n^G(A)\right)_{n \in \mathbb{N}} \text{ is the } G\text{-graded cocharacter sequence of } A$$

$$\left(c_n^G(A)\right)_{n \in \mathbb{N}} \text{ is the } G\text{-graded codimension sequence of } A$$

Now, for $l_{g_1}, \ldots, l_{g_r} \in \mathbb{N}$ let us consider the blended components of the multilinear polynomials in the indeterminates labeled as follows: $x_1^{g_1}, \ldots, x_{l_{g_1}}^{g_1}$, then $x_{l_{g_1}+1}^{g_2}, \ldots, x_{l_{g_1}+l_{g_2}}^{g_2}$ and so on. We denote this linear space by $P_{l_{g_1},\ldots,l_{g_r}}^G$. Of course, this is a left $S_{l_{g_1}} \times \cdots \times S_{l_{g_r}}$-module. We shall denote by $\chi_{l_{g_1},\ldots,l_{g_r}}^G(A)$ the character of the module $P_{l_{g_1},\ldots,l_{g_r}}^G(A)/(P_{l_{g_1},\ldots,l_{g_r}}^G(A) \cap T_G(A))$ and by $c_{l_{g_1},\ldots,l_{g_r}}^G(A)$ its dimension. When the algebra A is graded by the trivial group (in fact it has no grading) we refer to *cocharacter sequence of A* and *codimension sequence of A*.

For a more detailed account on PI-algebras, see Chapters 1 and 3 of [20] or [17].

Since the ground field F is infinite, a standard *Vandermonde-argument* yields that a polynomial f is a G-graded polynomial identity for A if and only if its multihomogeneous components are identities as well. Moreover, since $\mathrm{char}(F) = 0$, the well known multilinearization process shows that the T_G-ideal of a G-graded algebra A is determined by its multilinear polynomials, i.e. by the various $P_{l_{g_1},\ldots,l_{g_r}}^G(A)$. We remark that, given the cocharacter $\chi_{l_{g_1},\ldots,l_{g_r}}^G(A)$, the graded cocharacter $\chi_n^G(A)$ is known as well. More precisely, the following is due to Di Vincenzo (see [12], Theorem 2).

Proposition 2.5. *Let A be a G-graded algebra with graded cocharacter sequences $\chi_{l_{g_1},\ldots,l_{g_r}}^G(A)$. Then*

$$\chi_n^G(A) = \sum_{\substack{(l_{g_1},\ldots,l_{g_r}) \\ l_{g_1}+\ldots+l_{g_r}=n}} \chi_{l_{g_1},\ldots,l_{g_r}}^G(A)^{\uparrow S_n}.$$

Moreover

$$c_n^G(A) = \sum_{\substack{(l_{g_1},\ldots,l_{g_r}) \\ l_{g_1}+\ldots+l_{g_r}=n}} \binom{n}{l_{g_1},\ldots,l_{g_r}} c_{l_{g_1},\ldots,l_{g_r}}^G(A).$$

Actually, if A is a G-graded PI-algebra, it is more convenient studying $P_n^G(A)$ than the whole $T_G(A) \cap P_n^G(A)$. In fact, the latter grows factorially while a graded generalization of a well celebrated work by Regev (see [32]) says that $P_n^G(A)$ grows at most exponentially.

3. The Grassmann algebra

In this section we define the Grassmann algebra and we list some results about its ordinary polynomial identities and the polynomial identities of some related algebras.

Definition 3.1. *Let $X = \{x_1, x_2, \ldots\}$ and let us consider $F\langle X\rangle$. If I is the two-sided ideal of $F\langle X\rangle$ generated by the set of polynomials $\{x_ix_j + x_jx_i | i,j \geq 1\}$, we set $E := F\langle X\rangle/I$. Then we say that E is the infinite dimensional Grassmann algebra. Indeed if the set X is finite we define analogously the finite dimensional Grassmann algebra. We denote by L the vector space spanned by the $e_i := x_i + I$'s and we call it underlying vector space of E and we write $E = E(L)$. Moreover, if $w = e_{i_1} \cdots e_{i_t}$ is a monomial in the e_i's we say t is the length of w and we write $l(w) = t$. The set of different e_i's appearing in w is called support of w.*

We observe that E has the following presentation:

$$E = \langle 1, e_1, e_2, \ldots | e_ie_j = -e_je_i, \text{for all } i,j \geq 1\rangle.$$

Remark 3.2. *Of course over a field of characteristic 2, the Grassmann algebra turns out to be commutative. Hence in the sequel every field is supposed to have characteristic $p \neq 2$.*

The set

$$B = \{1, e_{i_1} \cdots e_{i_k} | 1 \leq i_1 < \cdots < i_k\}$$

is a basis of E over F.

It is convenient to write E in the form $E = E^0 \oplus E^1$ where

$$E^0 := \text{span}\{1, e_{i_1} \cdots e_{i_{2k}} | 1 \leq i_1 < \cdots < i_{2k}, k \geq 0\},$$

$$E^1 := \text{span}\{1, e_{i_1} \cdots e_{i_{2k+1}} | 1 \leq i_1 < \cdots < i_{2k+1}, k \geq 0\}.$$

It is easily checked that $E^0E^0 + E^1E^1 \subseteq E^0$ and $E^0E^1 + E^1E^0 \subseteq E^1$. This says that the decomposition $E = E^0 \oplus E^1$ is a $\mathbb{Z}_2$-grading of E, called *natural* or *canonical* $\mathbb{Z}_2$-grading of E. Notice that E^0 coincides with the center of E whereas it is not true if L has finite dimension. For example, if L has dimension d, d odd, then $e_1 \cdots e_d$ annihilates any element of E, then it is central but it does not fit in E^0. The next is a well known fact.

Proposition 3.3. *E satisfies the identity $[[x, y], z] \equiv 0$.*

Let us suppose E over a field of characteristic zero, then the triple commutator is the only generator of $T(E)$. In fact we have the following.

Theorem 3.4. *(Latyshev [27], Krakowski and Regev [26]) The T-ideal of E is generated by the polynomial*

$$[x_1, x_2, x_3].$$

We list some results concerning the polynomial identities of some algebras related to E. Unless otherwise stated the base field is supposed to be of characteristic 0. In what follows we shall denote by $UT_n(R)$ the F-algebra of $n \times n$ upper triangular matrices with entries of the F-algebra R.

Theorem 3.5. *(Berele and Regev [6]) The T-ideal of $UT_n(E)$ is generated by the polynomial*

$$[x_1, x_2, x_3] \cdots [x_{3n-2}, x_{3n-1}, x_{3n}].$$

More precisely, in [6] Theorem 2.8 the authors give a more general version of the previous result. We also recall that in [28] Latyshev proved that $T(UT_n(E))$ is finitely based.

Theorem 3.6. *(Popov [30]) The T-ideal of $E \otimes E$ is generated by the polynomials*

$$[x_1, x_2, [x_3, x_4], x_5], \ [[x_1, x_2], x_2^2].$$

We also want to point out that in [30] the author described the structure of the relatively free algebra of $E \otimes E$.

If we consider the finite dimensional Grassmann algebra, we have the next result by Di Vincenzo.

Theorem 3.7. *(Di Vincenzo [10]) Let E be the Grassmann algebra generated by the k dimensional underlying vector space L_k. Then the T-ideal of E is generated by the polynomials*

$$[x_1, x_2, x_3], \ [x_1, x_2] \cdots [x_{2t-1}, x_{2t}],$$

where $t = [k/2] + 1$ and $[a]$ is the integer part of a.

Even if the characteristic zero is the most investigated case, we also have several works in positive characteristic. Here we cite the analog of Theorem 3.4 for any infinite field.

Theorem 3.8. *The T-ideal of E is generated by the polynomial*

$$[x_1, x_2, x_3].$$

Notice that the T-ideal of E does not depend on the characteristic of the field. This is not the case of the Grassmann algebra over a finite field or, as we shall see later, of the $(\mathbb{Z}_2)$-graded case. We may cite the work by Regev [34] which gave a lot of information about the T-ideal of E, introducing the so called *class identities* whose definition goes far from the intent of this survey. By the way it will be interesting to note the following.

Proposition 3.9. *Let E be the infinite dimensional Grassmann algebra over a field of characteristic p and let us consider $E^* = E - \{1\}$, then E^* satisfies the identity x^p.*

We close this section by citing a famous result by Olsson and Regev about the cocharacters and the codimensions of the infinite dimensional Grassmann algebra in characteristic 0. We recall that a partition of the non-negative integer n is a sequence of integers $\lambda = (\lambda_1, \ldots, \lambda_r)$ such that

$$\lambda_1 \geq \cdots \geq \lambda_r > 0 \text{ and } \lambda_1 + \cdots + \lambda_r = n.$$

In this case we shall write

$$\lambda \vdash n.$$

We assume two partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_s)$ to be equal if $r = s$ and

$$\lambda_1 = \mu_1, \ldots, \lambda_r = \mu_r.$$

When $\lambda = (\lambda_1, \ldots, \lambda_{k_1+\cdots+k_p})$ and

$$\lambda_1 = \cdots = \lambda_{k_1} = \mu_1, \ldots, \lambda_{k_1+\cdots+k_{p-1}+1} = \cdots = \lambda_{k_1+\cdots+k_p} = \mu_p,$$

we accept the notation

$$\lambda = (\mu_1^{k_1}, \ldots, \mu_p^{k_p}).$$

Definition 3.10. *Given a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$, we associate to λ its Young diagram $[\lambda]$ having r rows such that its i-th row contains λ_i squares. Moreover we denote by λ' the partition associated to the transpose diagram of $[\lambda]$.*

Theorem 3.11. *(Olsson and Regev [29] for cocharacters and Krakowsky and Regev [26] for codimensions) The cocharacter sequence of the Grassmann algebra is the following:*

$$\chi_n(E) = \sum_{k=1}^{n} (k, 1^{n-k}),$$

where $n \geq 1$. Moreover its codimension sequence is such that for each $n \geq 1$ we have $c_n(E) = 2^{n-1}$.

4. $\mathbb{Z}_2$-graded identities

In this section we collect the results concerning the $\mathbb{Z}_2$-graded identities of E based on the works by the author [7] and Di Vincenzo and da Silva [14]. For the sake of completeness we want to cite the papers [37] and [2] by Anisimov in which the author computes the sequence of involutive codimensions of Grassmann algebra for some special involutions (see [37]), then generalized in [2]. In the latter paper the author gives also an explicit form of the sequence of involutive codimensions of the Grassmann algebra for arbitrary involution (exept one case) and for some other groups. The work by Anisimov has been completed by da Silva in [35] for the remaining case.

It is also interesting to say that the non-homogeneous G-gradings on E are unknown as well as their corresponding ideals of graded identities.

In the sequel for G-grading we mean homogeneous G-grading. In order to simplify the notation we shall use the symbols y's for variables of $\mathbb{Z}_2$-degree 0 and the symbols z's for variables of $\mathbb{Z}_2$-degree 1 as declared in Section 2.

Let $E = E(L)$ be the infinite dimensional Grassmann algebra with underlying vector space L and let G be an abelian group. If G is finite and $B_L = \{e_1, e_2, \ldots\}$ is a basis of L, let

$$\varphi : B_L \to G$$

be any map. Then φ induces a homogeneous G-grading on E and viceversa. In this section we consider homogeneous $\mathbb{Z}_2$-gradings only.

Let us consider the map $\varphi : L \to \mathbb{Z}_2$ such that $e_i \mapsto 1$. The map φ gives out the natural grading over E. In this case, let E^0 be the homogeneous component of $\mathbb{Z}_2$-degree 0 and let E^1 be the component of degree 1. It is easy to see that E^0 is the center of E and $ab + ba = 0$ for all $a, b \in E^1$. This means that E satisfies the following graded polynomial identities: $[y_1, y_2]$, $[y_1, z_1]$, $z_1z_2 + z_2z_1$. Moreover, the latter generates the whole $T_{\mathbb{Z}_2}$-graded ideal of E endowed with its natural $\mathbb{Z}_2$-grading in the case of characteristic 0. In fact we have the following.

Theorem 4.1. *(Giambruno, Mischenko, Zaicev [19]) The $T_{\mathbb{Z}_2}$-graded ideal of E endowed with its natural $\mathbb{Z}_2$-grading is generated by the polynomials*

$$[y_1, y_2],\ [y_1, z_1],\ z_1z_2 + z_2z_1,$$

if the characteritic of the ground field is 0.

Now, let us consider the $\mathbb{Z}_2$-gradings over E induced by the maps $\deg_{k*}$, $\deg_\infty$, and $\deg_k$, defined respectively by

$$\deg_{k*}(e_i) = \begin{cases} 1 \text{ for } i = 1, \ldots, k \\ 0 \text{ otherwise,} \end{cases}$$

$$\deg_\infty(e_i) = \begin{cases} 1 \text{ for } i \text{ odd} \\ 0 \text{ otherwise,} \end{cases}$$

$$\deg_k(e_i) = \begin{cases} 0 \text{ for } i = 1, \ldots, k \\ 1 \text{ otherwise.} \end{cases}$$

We shall denote by E_{k^*}, E_∞, E_k the Grassmann algebra endowed with the $\mathbb{Z}_2$-grading induced by the maps $\deg_{k*}$, $\deg_\infty$, and $\deg_k$. We denote by E_d any of the superalgebras E_{k^*}, E_∞, E_k without distinguish them.

Let $f = z_{i_1}^{r_{i_1}} \cdots z_{i_s}^{r_{i_s}} [z_{j_1}, z_{j_2}] \cdots [z_{j_{t-1}}, z_{j_t}]$ and consider the set

$$S := \{\text{different homogeneous variables appearing in } f\}.$$

If $h = |S|$, then $S = \{z_{i_1}, \ldots, z_{i_h}\}$. We consider now

$$T = \{j_1, \ldots, j_t\} \subseteq S$$

and let us denote the previous polynomial by

$$f_T(z_{i_1}, \cdots, z_{i_h}).$$

For $m \geq 2$ let

$$g_m(z_{i_1}, \ldots, z_{i_h}) = \sum_{\substack{T \\ |T| \text{ even}}} (-2)^{-\frac{|T|}{2}} f_T,$$

moreover we set $g_1(z_1) = z_1$.

Let F be an infinite field of characteristic $p > 2$, then we have the next results (see [7]).

Theorem 4.2. *Let $k \in \mathbb{N}$. If $p > k$, then all $\mathbb{Z}_2$-graded polynomial identities of E_{k^*} are consequences of the graded identities:*

$$[x_1, x_2, x_3], \ z_1 \cdots z_{k+1}.$$

On the other side, if $p \leq k$, all $\mathbb{Z}_2$-graded polynomial identities of E_{k^*} are consequences of the graded identities:

$$[x_1, x_2, x_3], \ z_1 \cdots z_{k+1}, \ z^p.$$

Theorem 4.3. *All the $\mathbb{Z}_2$-graded polynomial identities of E_∞ are consequences of the graded identities:*

$$[x_1, x_2, x_3], \ z^p.$$

Theorem 4.4. *Let $k \in \mathbb{N}$ and set $X = Y \cup Z$. Then if $p > k$ all the $\mathbb{Z}_2$-graded identities of E_k are consequences of the graded identities:*

- $[x_1, x_2, x_3]$,
- $[y_1, y_2] \cdots [y_{k-1}, y_k][y_{k+1}, x]$ (if k is even)
- $[y_1, y_2] \cdots [y_k, y_{k+1}]$ (if k is odd)
- $g_{k-l+2}(z_1, \ldots, z_{k-l+2})[y_1, y_2] \cdots [y_{l-1}, y_l]$ (if $l \leq k$)
- $[g_{k-l+2}(z_1, \ldots, z_{k-l+2}), y_1][y_2, y_3] \cdots [y_{l-1}, y_l]$ (if $l \leq k$, l is odd)
- $g_{k-l+2}(z_1, \ldots, z_{k-l+2})[z, y_1][y_2, y_3] \cdots [y_{l-1}, y_l]$ (if $l \leq k$, l is odd)

If $p \leq k$ we have to add to the list above the identity

- z^p

From Theorem 4.4 it turns out that a minimal basis of the $\mathbb{Z}_2$-graded identities of E either in positive characteristic or in characteristic zero is generated by the polynomials $[y_1, y_2], \ [y_1, z_1], \ z_1 z_2 + z_2 z_1$.

The case of characteristic zero was the first case which has been considerated and it was completely solved by Di Vincenzo and da Silva in [14]. Their generators in the three cases are the ones above without the polynomial z^p. We observe that the identity z^p comes from the fact that the E^1 component of the Grassmann algebra lies in E^*, then we use Proposition 3.9.

In [14] the authors described the sequence of $\mathbb{Z}_2$-graded cocharacters and codimensions in the case of characteristic 0. We collect below their results. We shall adopt the following notation. Let $\lambda_s = (l - s, 1^s) \vdash l$, $\mu_t = (1 + t, 1^{m-t-1}) \vdash m$ be the hook partition of l and m with leg s and arm t respectively.

Theorem 4.5. *Let $k \in \mathbb{N}$, then for each $n \in \mathbb{N}$ the n-th $\mathbb{Z}_2$-graded codimension of E_{k^*} is*

$$c_n^{\mathbb{Z}_2}(E_{k^*}) = 2^{n-1} \sum_{t=0}^{k} \binom{n}{t}.$$

Let $k \in \mathbb{N}$, then the $\mathbb{Z}_2$-graded cocharacter sequence of E_{k^*} is given by

$$\chi_{l,0}(E_{k^*}) = \sum_{s=0}^{l-1} \lambda_s \otimes \emptyset \text{ if } l \geq 1;$$

$$\chi_{0,m}(E_{k^*}) = \sum_{t=0}^{m-1} \emptyset \otimes \mu_t \text{ if } m \geq 1;$$

$$\chi_{l,m}(E_{k^*}) = \sum_{s=0}^{l-1} \sum_{t=0}^{m-1} 2(\lambda_s \otimes \mu_t) \text{ if } l \geq 1,\ 1 \leq m \leq k;$$

$$\chi_{l,m}(E_{k^*}) = 0 \text{ if } l \geq 0,\ m \geq k+1.$$

Theorem 4.6. *For each $n \in \mathbb{N}$ the n-th $\mathbb{Z}_2$-graded codimension of E_∞ is*

$$c_n^{\mathbb{Z}_2}(E_\infty) = 4^{n-\frac{1}{2}}.$$

The $\mathbb{Z}_2$-graded sequence of E_∞ is given by

$$\chi_{l,0}(E_\infty) = \sum_{s=0}^{l-1} \lambda_s \otimes \emptyset \text{ if } l \geq 1;$$

$$\chi_{0,m}(E_{k^*}) = \sum_{t=0}^{m-1} \emptyset \otimes \mu_t \text{ if } m \geq 1;$$

$$\chi_{l,m}(E_{k^*}) = \sum_{s=0}^{l-1} \sum_{t=0}^{m-1} 2(\lambda_s \otimes \mu_t) \text{ if } m, l \geq 1;$$

Theorem 4.7. *Let $k \in \mathbb{N}$, then for each $n \in \mathbb{N}$ we have*

$$c_{n-m,m}(E_k) = \sum_{t=0}^{k} \binom{n-1}{t} \text{ if } m \geq 1;$$

$$c_{n,0}(E_k) = \sum_{t=0}^{e(k)} \binom{n-1}{t},$$

where

$$e(k) = \begin{cases} k \text{ if } k \text{ is even} \\ k-1 \text{ if } k \text{ is odd.} \end{cases}$$

Let $k \in \mathbb{N}$, then the $\mathbb{Z}_2$-graded cocharacter sequence of E_k is given by

$$\chi_{l,0}(E_k) = \sum_{s=0}^{l-1} \lambda_s \otimes \emptyset \text{ if } l \leq k;$$

$$\chi_{l,0}(E_k) = \sum_{s=0}^{e(k)} \lambda_s \otimes \emptyset \text{ if } l \geq k+1;$$

$$\chi_{0,m}(E_k) = \sum_{t=0}^{m-1} \emptyset \otimes \mu_t \text{ if } t \leq k;$$

$$\chi_{0,m}(E_k) = \sum_{t=0}^{e(k)} \emptyset \otimes \mu_t \text{ if } t \geq k+1;$$

$$\chi_{l,m}(E_k) = \sum_{s=0}^{l-1} \sum_{t=0}^{m-1} m_{s,t}(\lambda_s \otimes \mu_t) \text{ if } l, m \geq 1,$$

where

$$m_{s,t} = \begin{cases} 2 \text{ if } s+t \leq k-1 \\ 1 \text{ if } s+t = k \\ 0 \text{ otherwise.} \end{cases}$$

5. G-graded identities of E

Now we consider the more general case of a homogeneous G-grading of E, where G is a finite abelian group. We show that in order to study the G-graded identities of E we may reduce to G'-gradings, where G' is a group having a smaller number of elements than G. We give the proofs of the main results. The complete contents related to this section may be found in [8]. We also recall that from now on every field is supposed to be of characteristic 0. Finally, if $H \triangleleft G$ we shall denote by gH or simply by $\overline{g}$ the coset of g modulo H, where $g \in G$.

Let us consider the following homomorphism between free graded algebras

$$\pi : F\langle X|G\rangle \to F\langle Y|G/H\rangle,$$

such that for every $g \in G$ and for every $i \in \mathbb{N}$, $\pi(x_i^g) = y_i^{gH}$, where H is a subgroup of G.

Definition 5.1. *Let G be a finite abelian group and suppose E is G-graded. We say that the subgroup H of G has the property $\mathcal{P}$ when for any $h \in H$, E^h has infinitely many elements of even length with pairwise disjoint support.*

The importance of this property is given by the following proposition.

Proposition 5.2. *Let $H < G$ having the property $\mathcal{P}$ and let $f \in F\langle X|G\rangle$ be a multilinear polynomial. Then $f \in T_G(E)$ if and only if $\pi(f) \in T_{G/H}(E)$.*

Proof. We have to prove just the only if part. Let

$$f = f(x_1^{g_1}, x_2^{g_1}, \ldots, x_{l_{g_1}}^{g_1}, \ldots, x_{\sum_{i=1}^{r-1} l_{g_i}+1}^{g_r}, \ldots, x_{\sum_{i=1}^{r} l_{g_i}}^{g_r}) \in T_G(E)$$

and let $F = \pi(f)$. Let φ be any G/H-graded substitution, hence $\varphi(y_j^{gH}) = \sum_{h\in H} a_j^{gh}$, and by the multilinearity of f, we can consider only substitutions φ such that $y_j^{gH} \mapsto a_j^{gh}$, for some $h \in H$ and for any j. Now, we observe that every homogeneous component E^h has infinitely many elements of even length with pairwise disjoint supports because H satisfies the property $\mathcal{P}$, then for every j and for every $h \in H$ exists $b_j^{h^{-1}}$ of even length such that $\|b_j^{h^{-1}}\| = h^{-1}$. For every $h \in H$, $w_j^g = a_j^{gh} b_j^{h^{-1}}$ is a homogeneous element of degree g in the G-grading of E. Let us consider a new substitution ψ such that $x_j^g \mapsto w_j^g$. This is a G-graded substitution. Now, since $f \in T_G(E)$, $0 = f(w_1^{g_1}, \ldots, w_{l_{g_r}}^{g_r}) = \prod_{h\in H,j} b_j^{h^{-1}} \cdot F(a_j^{gh})$ because the $b_j^{h^{-1}}$'s are in $Z(E)$ and this implies $F(a_j^{gh}) = 0$. $\square$

We observe that if L^g is infinite dimensional and $|H| = n$ is odd, then $H = \langle g \rangle$ has the property $\mathcal{P}$. Moreover even if G is a finite abelian group and

$$H = \langle g | \dim_F L^g = \infty \text{ and } o(g) \text{ is odd}\rangle,$$

then H has the property $\mathcal{P}$. We shall adopt the following notation: if H is a subgroup of G, we shall denote by $\overline{g}$ the translate $gH \in G/H$. We have the following.

Theorem 5.3. *Let G be a finite abelian group of odd order and let*

$$H = \langle g | \dim_F L^g = \infty\rangle.$$

Then the following properties hold:

1. *for any multilinear polynomial $f(x_1, \ldots, x_n) \in F\langle X\rangle$ one has that*

$$f \in T_G(E) \text{ if and only if } \pi(f) \in T_{G/H}(E).$$

2. *In the quotient grading of E, $L^{\overline{g}}$ is infinite dimensional if and only if $\overline{g} = 1_{G/H}$.*

Now let us consider the following subsets of G :

$$\mathcal{I} = \{g \in G | \dim_F L^g = \infty\},$$

$$\mathcal{I}_1 = \{g \in \mathcal{I} | o(g) \text{ is odd}\},$$

$$\mathcal{I}_2 = \mathcal{I} - \mathcal{I}_1 \text{ and}$$

$$\mathcal{I}_3 = \{g^2 | g \in \mathcal{I}_2\} - \mathcal{I}_1.$$

We have the following.

Theorem 5.4. *Let G be a finite abelian group and let $H = \langle g | g \in \mathcal{I}_1 \cup \mathcal{I}_3\rangle$. Then the following properties hold:*

1. *for any multilinear polynomial $f = f(x_1, \ldots, x_n) \in F\langle X\rangle$ one has*

$$f \in T_G(E) \text{ if and only if } \pi(f) \in T_{G/H}(E).$$

2. *In the quotient grading of E, if $L^{\overline{g}}$ is infinite dimensional, then $\overline{g}^2 = 1_{G/H}$.*

Proof. (1). Let $h \in H$, then there exist $a_1, \ldots, a_r \in \mathcal{I}_1$, $b_1, \ldots, b_s \in \mathcal{I}_3$ and positive integers such that $h = a_1^{m_1} \cdots a_r^{m_r} b_1^{m_{r+1}} \cdots b_s^{m_{r+s}}$. Let $a_{r+1}, \ldots, a_{r+s} \in \mathcal{I}_2$ such that $b_i = a_{r+i}^2$, then $\dim_F L^{a_i} = \infty$ for any $i = 1, \ldots, r+s$. Let us denote by E_i the Grassmann algebra generated by the subspace $L^{a_i^{m_i}}$. For any $i = 1, \ldots, r+s$, E_i contains infinite elements

$$w_1^i, w_2^i, \ldots, w_m^i, \ldots$$

of even length with pairwise disjoint supports. Moreover, for all $m \geq 1$ we have that $\|w_m^i\| = a_i^{m_i}$ if $i = 1, \ldots, r$ and $\|w_m^i\| = b_{i-r}^{m_i}$ for $i = r+1, \ldots, r+s$. We consider in E^h the elements $u_m = w_m^1 \cdots w_m^{r+s}$, $m \geq 1$; clearly the elements $\{u_m | m \geq 1\}$ have pairwise disjoint supports and they have even length. Now H has the property $\mathcal{P}$ and the assertion comes by Proposition 5.2.

(2). Let $\overline{g} = gH \in G/H$ be such that $L^{\overline{g}} = \bigoplus_{h \in H} L^{gh}$ is infinite dimensional. Since G is finite there exists $g' \in gH$ such that $L^{g'}$ is infinite dimensional. If $o(g')$ is odd, then $g' \in H$ and so $gH = g'H = 1_{G/H}$. If $o(g')$ is even, then $g'^2 \in H$ and so $(gH)^2 = (g'H)^2 = 1_{G/H}$. □

In light of Theorems 5.3 and 5.4, we list the results about the G-graded identities of E in the case $\dim_F L^{1_G} = \infty$.

Theorem 5.5. *Let $G = \{g_1, \ldots, g_r\}$ be a finite abelian group with $g_1 = 1_G$. Suppose that L^{g_1} has infinite dimension. Let*

$$l_{g_1}, l_{g_2}, \ldots, l_{g_r} \in \mathbb{N}$$

such that

$$l_{g_1} + l_{g_2} + \ldots + l_{g_r} = m.$$

Then $P_{l_{g_1}, \ldots, l_{g_r}} \subseteq T_G(E)$ or for any $f \in P_{l_{g_1}, l_{g_2}, \ldots, l_{g_r}}$ one has

$$f(x_1^{g_1}, \ldots, x_{l_{g_1}}^{g_1}, \ldots, x^{g_r}_{\sum_{i=1}^{r-1} l_{g_i}+1}, \ldots, x^{g_r}_{\sum_{i=1}^{r} l_{g_i}}) \in T_G(E) \text{ if and only if } f(x_1, \ldots, x_m) \in T(E).$$

Theorem 5.6. *Let $G = \{g_1, \ldots, g_r\}$ be a finite abelian group with $g_1 = 1_G$. Let L be a G-homogeneous vector space over L such that $\dim_F L^{g_1} = \infty$ and $\dim_F L^{g_i} = k_i < \infty$, if $i \neq 1$. If $E = E(L)$ is the Grassmann algebra generated by L, then $T_G(E)$ is generated as a T_G-ideal by the following polynomials:*

1. *$[u_1, u_2, u_3]$ for any choice of the G-degree of the variables u_1, u_2, u_3.*
2. *monomials of $P_{0,t_2,\ldots,t_r}$ such that $\sum_{i=2}^r t_i = 1 + \sum_{i=2}^r k_i$*
3. *monomials of $P_{0,t_2,\ldots,t_r}$ such that $\sum_{i=2}^r t_i < 1 + \sum_{i=2}^r k_i$ and $P_{0,t_2,\ldots,t_r} \subseteq T_G(E)$.*

As a consequence of the previous results, we have, up to combinatorics, the following description of the G-graded cocharacters in the case L^{1_G} is infinite dimensional.

Corollary 5.7. *Let $G = \{g_1, \ldots, g_r\}$ be a finite abelian group with $g_1 = 1_G$. If L^{g_1} has infinite dimension and $l_{g_1}, l_{g_2}, \ldots, l_{g_r} \in \mathbb{N}$ such that $l_{g_1} + l_{g_2} + \ldots + l_{g_r} = m$, then*

$$c_{l_{g_1}, \ldots, l_{g_r}}(E) = 0 \text{ or}$$

$$c_{l_{g_1}, \ldots, l_{g_r}}(E) = 2^{m-1}$$

and in this last case, $P_{l_{g_1}, \ldots, l_{g_r}}(E)$ and $P_m(E)$ are isomorphic $S_{l_{g_1}} \times \cdots \times S_{l_{g_r}}$-modules.

Let us consider now the set

$$S(\varphi) = \left\{(l_{g_1}, l_{g_2}, \ldots, l_{g_r}) \in \mathbb{N}^r | P_{l_{g_1}, l_{g_2}, \ldots, l_{g_r}} \subseteq T_G(E)\right\}.$$

We note that if L^{1_G} is the only homogeneous subspace of L such that $\dim_F L^{1_G} = \infty$, then $S(\varphi) \neq \emptyset$.

$S(\varphi)$ allows us to give the complete description of the sequence of the graded cocharacters and codimensions of E. In fact, we have the following proposition.

Theorem 5.8. *Let $G = \{g_1, \ldots, g_r\}$ be a finite abelian group and L be a G-homogeneous vector space with linear basis $\{e_1, e_2, \ldots\}$. Let $\varphi : B_L \to G$ be a map such that $|\varphi^{-1}(1_G)| = \infty$ and consider E, the G-graded Grassmann algebra obtained by φ. Then*

$$\chi^G_{l_{g_1}, \ldots, l_{g_r}}(E) = 2^{|G|-1} \sum_{a_1=0}^{l_{g_1}-1} \sum_{a_2=0}^{l_{g_2}-1} \cdots \sum_{a_r=0}^{l_{g_r}-1} \lambda_{a_1} \otimes \lambda_{a_2} \otimes \cdots \otimes \lambda_{a_r}$$

if $(l_{g_1}, \ldots, l_{g_r}) \notin S(\varphi)$, where λ_{a_i} is the hook partition of leg a_i and arm $l_{g_i} - a_i + 1$.

Moreover

$$c^G_n(E) = 2^{n-1} \sum_{\substack{(l_{g_1}, \ldots, l_{g_r}) \notin S(\varphi) \\ l_{g_1} + \ldots + l_{g_r} = n}} \binom{n}{l_{g_1}, \ldots, l_{g_r}}.$$

6. Other results

In what follows we recall some results about the (graded) polynomial identities of structures related to the Grassmann algebra.

We shall consider a G-graded algebra A and the canonical $\mathbb{Z}_2$-grading of the Grassmann algebra $E = E^0 \oplus E^1$ over a field of characteristic 0, and compare the G-graded identities of A with the $G \times \mathbb{Z}_2$-graded identities of the $G \times \mathbb{Z}_2$-graded algebra $A \otimes E$ with homogeneous components given by $(A \otimes E)^{(g,i)} := A^g \otimes E^i$.

Notice that the free algebra $F\langle X | G \times \mathbb{Z}_2\rangle$ is both a G-graded algebra and a $\mathbb{Z}_2$-graded algebra. Referring to the $\mathbb{Z}_2$-grading of $F\langle X | G \times \mathbb{Z}_2\rangle$ one defines the map ζ as follows. Let m be a multilinear monomial in $F\langle X | G \times \mathbb{Z}_2\rangle$ and let $i_1 < \cdots < i_k$ be the indexes of the variables with odd $\mathbb{Z}_2$-degree occurring in m. Then, for some σ in the symmetric group $S_k(\{i_1, \ldots, i_k\})$, we may write $m = m_0 z_{\sigma(i_1)} m_1 z_{\sigma(i_2)} \cdots m_{k-1} z_{\sigma(i_k)} m_k$, where $m_0, \ldots, m_k$ are multilinear monomials in even variables only and z_{i_j} are odd variables. Then, as in Kemer [22], Di Vincenzo and Nardozza [13] define $\zeta(m) := (-1)^\sigma m$. Note that $\zeta(\zeta(m)) = m$. We define a similar map from the free G-graded algebra to the free $G \times \mathbb{Z}_2$-graded algebra.

Definition 6.1. *Let $J \subseteq \mathbb{N}$. Let $\varphi_J : F\langle X\rangle \to F\langle X | G \times \mathbb{Z}_2\rangle$ be the unique G-homomorphism defined by the map*

$$\varphi_J(x^g) = \begin{cases} x^{(g,0)} & \text{if } i \notin J \\ x^{(g,1)} & \text{if } i \in J. \end{cases}$$

Also, for a multilinear monomial $m \in P^G_n$, define $\zeta_J(m) := \zeta(\varphi_J(m))$. The map φ_J depends on J, of course. We may extend the map ζ_J by linearity to the space of all G-graded multilinear polynomials P^G_n . If $f \in P^G_n$, then $\zeta_J(f)$ is a multilinear element of $F\langle X | G \times \mathbb{Z}_2\rangle$.

We have the following result (see Theorem 11 of [13]).

Theorem 6.2. *Let S be a system of multilinear generators for $T_G(A)$. Then the system*

$$\{\zeta_J(f) \in F\langle X\rangle \,|\, f \in S, J \subseteq \mathbb{N}\}$$

is a set of multilinear generators for $T_{G \times \mathbb{Z}_2}(A \otimes E)$.

If we consider the case of positive characteristic, the previous result is verified for the algebra $UT_2(E)$ of upper triangular 2×2 matrices.

Theorem 6.3. *(C. and da Silva [9]) Let F be a field of characteristic $p > 2$ and E be graded with its natural $\mathbb{Z}_2$-grading. Let us set $S := \{[y_1, y_2],\ [y_1, z_1],\ z_1z_2 \circ z_2z_1\}$, then*

$$T_{\mathbb{Z}_2}(UT_2(E)) = \{\zeta_J(f) | f \in S\}.$$

We observe that the previous result does not depend on the characteristic of the field. Moreover $T_{\mathbb{Z}_2}(UT_2(E)) = T_{\mathbb{Z}_2}(E)T_{\mathbb{Z}_2}(E)$ as in the ordinary case.

We have the next related result about the $G \times \mathbb{Z}_2$-cocharacters of $A \otimes E$.

Theorem 6.4. *(Di Vincenzo and Nardozza [13]) Let $n \in \mathbb{N}$ and $k_1, l_1, \ldots, k_r, l_r \in \mathbb{N}$ such that $\sum_{i=1}^{r} k_i + l_i = n$ and consider $H = S_{k_1} \times S_{l_1} \times \cdots S_{k_r} \times S_{l_r}$. If*

$$(\chi_n^G(A))_{\downarrow H} = \sum m_{\lambda_1,\mu_1,\ldots,\lambda_r,\mu_r} \lambda_1 \otimes \mu_1 \otimes \cdots \otimes \lambda_r \otimes \mu_r,$$

then

$$\chi_n^{G\times\mathbb{Z}_2}(A \otimes E) = \sum_{\sum_i (k_i + l_i) = n} \ \sum_{\substack{\lambda_i \vdash k_i \\ \mu_i \vdash l_i}} m_{\lambda_1,\mu_1,\ldots,\lambda_r,\mu_r} \lambda_1 \otimes \mu_1' \otimes \cdots \otimes \lambda_r \otimes \mu_r'.$$

We give a short account of the structure theory of T-ideals developed by Kemer [22].

Definition 6.5. *The T-ideal S of $F\langle X\rangle$ is called T-semiprime or verbally semiprime if any T-ideal U such that $U^k \subseteq S$ for some k, lies in S, i.e. $U \subseteq S$. The T-ideal P is T-prime or verbally prime if the inclusion $U_1U_2 \subseteq P$ for some T-ideals U_1 and U_2 implies $U_1 \subseteq P$ or $U_2 \subseteq P$.*

Let $E = E^0 \oplus E^1$ be endowed with its canonical $\mathbb{Z}_2$-grading, then the vector subspace of $M_{a+b}(E)$,

$$M_{a,b}(E) := \left\{ \begin{pmatrix} r & s \\ t & u \end{pmatrix} | r \in M_a(E^{(0)}), s \in M_{a\times b}(E^{(1)}), u \in M_b(E^{(0)}) \right\}$$

is an algebra. The building blocks in the theory of Kemer are the polynomial identities of the matrix algebras over the field and over the Grassmann algebra and the algebras $M_{a,b}(E)$. In fact, we have the following theorem.

Theorem 6.6.
1. *For every T-ideal U of $F\langle X\rangle$ there exist a T-semiprime T-ideal S and a positive integer k such that*

$$S^k \subseteq U \subseteq S.$$

2. *Every T-semiprime T-ideal S is an intersection of a finite number of T-prime T-ideals $Q_1, \ldots, Q_m$,*

$$S = Q_1 \cap \cdots \cap Q_m.$$

3. *A T-ideal P is T-prime if and only if P coincides with one of the following T-ideals:*

$$T(M_n(F)),\ T(M_n(E)),\ T(M_{a,b}(E)),\ (0),\ F\langle X\rangle.$$

We recall that if two algebras A and B satisfy the same polynomial identities we say that A is PI-equivalent to B and denote by $A \sim B$. An important corollary to the structure theory of Kemer is the Tensor Product Theorem (TPT) which follows from the result by Kemer [22].

Theorem 6.7. *The tensor product of two verbally prime algebras is PI equivalent to a verbally prime algebra. More precisely, let* $a, b, c, d \in \mathbb{N}$ *such that* $a \geq b$ *and* $c \geq d$ *and* F *be a field of characteristic 0, then:*

1. $M_{a,b}(E) \otimes E \sim M_{a+b}(E)$;
2. $M_{a,b}(E) \otimes M_{c,d}(E) \sim M_{ac+bd,ad+bc}(E)$;
3. $M_{1,1}(E) \sim E \otimes E$.

The remaining PI equivalences follow from the isomorphism of the corresponding algebras.

An alternative proof of the TPT can be be found in the paper by Regev [33]. In [33] we also have the proof that the TPT is still valid for multilinear identities in the case E is the infinite dimensional Grassmann algebra over an infinite field of characteristic $p \neq 2$.

In the papers [4], [5] and [25] the authors deal with graded identities for certain gradings on some of the verbally prime algebras. In the paper [25] the authors constructed an appropriate model for the relatively free algebra in the variety of algebras determined by $E \otimes E$ when the field F has characteristic $p > 2$. This model is the generic algebra of $A = F \oplus M_{1,1}(E^*)$. It turned out that $E \otimes E$ and A satisfy the same graded and hence ordinary polynomial identities. In [4] the authors used the properties of A in order to show that that $T(M_{1,1}(E)) \subsetneq T(E \otimes E)$. Hence the TPT theorem fails in positive characteristic.

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