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Some new large sets of geometric designs of type $\mathrm{LS}[3][2,3,2^8]$

Research Article

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Abstract: Let V be an n-dimensional vector space over $\mathbb{F}_q$. By a *geometric* t-$[q^n, k, \lambda]$ design we mean a collection $\mathcal{D}$ of k-dimensional subspaces of V, called blocks, such that every t-dimensional subspace T of V appears in exactly λ blocks in $\mathcal{D}$. A *large set*, LS[N]$[t, k, q^n]$, of geometric designs, is a collection of N t-$[q^n, k, \lambda]$ designs which partitions the collection $\begin{bmatrix} V \\ k \end{bmatrix}$ of all k-dimensional subspaces of V. Prior to recent article [4] only large sets of geometric 1-designs were known to exist. However in [4] M. Braun, A. Kohnert, P. Östergard, and A. Wasserman constructed the world's first large set of geometric 2-designs, namely an LS[3][2,3,2^8], invariant under a Singer subgroup in $GL_8(2)$. In this work we construct an additional 9 distinct, large sets LS[3][2,3,2^8], with the help of lattice basis-reduction.

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1. Introduction

In this article we deal with *large sets* of *geometric* t-designs. By a *geometric* t-design we mean what earlier authors have called t-designs over a finite field, or designs on vector spaces. Geometric t-designs are the $\mathbb{F}_q$-analogs of ordinary t-(v, k, λ) designs. The earliest mention of t-$[q^n, k, \lambda]$ designs, although not using our terminology or notation, was by P.J. Cameron in 1974 [5, 6] and P. Delsarte in 1976 [7]. In 1987, S. Thomas [20] exhibited the first simple geometric 2-design, and in the 1990's H. Suzuki [19], M. Miyakawa et al. [17] , and T. Itoh [10] constructed new geometric 2-designs and families of such designs. In 1994, D.K. Ray-Chaudhuri and E.J. Schram [18] studied and constructed geometric t-designs from quadratic forms, allowing repeated blocks. For the first time, the latter authors also studied large sets of geometric t-designs.

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M. Braun, A. Kerber and R. Laue [3] constructed in 2005 the first simple geometric 3-design. In 2013, Braun et al. [2] constructed the first example of a q-Steiner system, that is a simple, geometric t-design with $\lambda = 1$, namely a 2-$[2^{13}, 3, 1]$ design.

In a short recent arXiv preprint [8], and based on a probabilistic existence theorem of G. Kuperberg, S. Lovett and R. Peled in preprint [14], A. Fazeli, S. Lovett, and A. Vardy, appear to have proved the remarkable theorem that simple geometric t-designs exist for all values of t. This would be a q-analog of the famous theorem of L. Tierlinck for ordinary t-designs. It should be noted however, that the result in [8] is purely existential and there is no known efficient algorithm which can produce t-$[q^n, k, \lambda]$ designs for $t > 3$. The authors present the following challenge:

Problem 1.1. *Design an efficient algorithm to produce simple, non-trivial* t-$[q^n, k, \lambda]$ *designs for large* t*, (say* $t \geq 4$*).*

Of course, finding large sets of geometric t-designs is even harder than just finding geometric t-designs. Prior to recent article [4] only large sets of geometric 1-designs were known to exist. However in [4] M. Braun, A. Kohnert, P. Östergard, and A. Wasserman constructed the world's first large set of geometric 2-designs, namely an LS[3][2,3,2^8], invariant under a Singer subgroup in $GL_8(2)$.

In this paper we construct 9 distinct large sets $LS[3][2, 3, 2^8]$, all different from the large set constructed in [4]. The computation involved our APL package *knuth* for group theoretic matters, and various LLL variants in the NTL library, augmented by certain optimization techniques for parallel lattice basis reduction.

It should be noted that some of the recent work on geometric t-designs has been motivated by present day *coding theoretic* applications as discussed in [9] and [11].

2. Preliminaries

Let V be an n-dimensional vector space over the field $\mathbb{F}_q$. If U is a j-dimensional subspace of V, we say that U is a j-subspace of V. If X is a set and $0 \leq s \leq |X|$, $\binom{X}{s}$ denotes the collection of all subsets of cardinality s of X.

A *geometric* t-$[q^n, k, \lambda]$ design is a pair $(V, \mathcal{B})$ where $\mathcal{B}$ is a multiset of k-subspaces of V, called *blocks*, such that any t-subspace T of V is contained in exactly λ blocks. $(V, \mathcal{B})$ is said to be *simple* if $\mathcal{B}$ is a set, i.e. if there are no repeated blocks.

In this paper we deal only with simple geometric designs, and the square brackets of the symbol t-$[q^n, k, \lambda]$ will imply "*geometric*" in contrast to the round parentheses for an ordinary t-(v, k, λ) design.

We denote the collection of all k-subspaces of V by $\begin{bmatrix} V \\ k \end{bmatrix}$ and note that $|\begin{bmatrix} V \\ k \end{bmatrix}| = \begin{bmatrix} n \\ k \end{bmatrix}_q$, where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the well known Gaussian binomial coefficient, given by:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} \tag{1}$$

where for positive integer r,

$$[r]_q! := [1]_q[2]_q \cdots [r]_q, \quad \text{and} \quad [j]_q := (1 + q + \cdots + q^{j-1}). \tag{2}$$

Analogously to the case of ordinary t-(v, k, λ) designs, a geometric t-$[q^n, k, \lambda]$ design $(V, \mathcal{B})$ is also an s-$[q^n, k, \lambda_i]$ design for every $0 \leq s \leq t$ with:

$$\lambda_s = \lambda \begin{bmatrix} n-s \\ t-s \end{bmatrix}_q / \begin{bmatrix} k-s \\ t-s \end{bmatrix}_q, \tag{3}$$

Thus, a necessary condition for the existence of a t-$[q^n, k, \lambda]$ design is that the λ_s given by the equations (3) must be integral for all $0 \leq s \leq t$.

By a *large set* $\mathrm{LS}[N][t, k, q^n]$ we mean a collection $\mathcal{L} = \{(V, \mathcal{B}_i)\}_{i=1}^{N}$ of simple t-$[q^n, k, \lambda]$ designs where $\{\mathcal{B}_i\}_{i=1}^{N}$ is a partition of $\begin{bmatrix} V \\ k \end{bmatrix}$. We can immediately see that for a given large set $\mathrm{LS}[N][t, k, q^n]$, N can be expressed in terms of the other parameters as :

$$N = \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q / \lambda \tag{4}$$

Two t-$[q^n, k, \lambda]$ designs $\mathcal{D} = (V, \mathcal{B})$ and $\mathcal{D}' = (V, \mathcal{B}')$ are said to be *isomorphic* if there exists $\alpha \in GL_n(q)$ such that $\mathcal{B}^\alpha = \mathcal{B}'$, that is, $B^\alpha \in \mathcal{B}'$ for all $B \in \mathcal{B}$, in which case we also write $\mathcal{D}^\alpha = \mathcal{D}'$. If $\mathcal{D}^\alpha = \mathcal{D}$, then α is said to be an *automorphism* of $\mathcal{D}$. The group of all automorphisms of $\mathcal{D}$ is denoted by $Aut(\mathcal{D})$.

Let $\mathbb{B} = \{\mathcal{B}_i\}_{i=1}^{N}$ be the collection of designs in a large set $\mathcal{L}$. A group $G \leq GL_n(q)$ is said to be an *automorphism group* of $\mathcal{L}$ if $\mathbb{B}^g = \mathbb{B}$ for all $g \in G$, that is, if $\mathcal{B}_i^g \in \mathbb{B}$ for all $\mathcal{B}_i \in \mathbb{B}$ and $g \in G$. Equivalently, we say that a large set with this property is G-invariant. The group of all automorphisms of $\mathcal{L}$ is denoted by $Aut(\mathcal{L})$. If the stronger condition holds, that $\mathcal{B}_i^g = \mathcal{B}_i$ for all $\mathcal{B}_i \in \mathbb{B}$ and $g \in G$, we say that the large set $\mathcal{L}$ is $[G]$-invariant.

In 1976, E.S. Kramer and D.M. Mesner [12] presented a theorem which provides necessary and sufficient conditions for the existence of an ordinary G-invariant t-(v, k, λ) design. Beginning with a given group action $G|X$, the authors define certain integer matrices, presently known as the *Kramer-Mesner* (KM) matrices. Roughly speaking such a matrix $A_{t,k}$ is the result of fusing under G the incidence matrix between $\binom{X}{t}$ and $\binom{X}{k}$ where incidence is set inclusion (fused R.Wilson matrix). These matrices extend naturally to the case of a group $G \leq GL_n(q)$ acting on $AG_n(q)$ or $PG_{n-1}(q)$, and provide necessary and sufficient conditions for the existence of geometric, G-invariant t-$[q^n, k, \lambda]$ designs. We proceed to define these matrices in the context of geometric t-designs, and state the analog of the Kramer-Mesner theorem.

Let V be an n-dimensional vector space over $\mathbb{F}_q$, and $G \leq GL_n(q)$. Suppose that t and k are integers, $0 \leq t < k \leq n$, and consider the actions of G on $\begin{bmatrix} V \\ t \end{bmatrix}$ and $\begin{bmatrix} V \\ k \end{bmatrix}$ respectively, with corresponding G-orbit decompositions:

$$\begin{bmatrix} V \\ t \end{bmatrix} = \Delta_1 + \Delta_2 + \cdots + \Delta_{\rho(t)}, \tag{5}$$

and

$$\begin{bmatrix} V \\ k \end{bmatrix} = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_{\rho(k)}. \tag{6}$$

where $\rho(s)$ denotes the number of G-orbits on $\begin{bmatrix} X \\ s \end{bmatrix}$. Just as in [12], it can be shown that for any fixed t-subspaces $T, T' \in \Delta_i$, we have that

$$|\{K \in \Gamma_j : T \leq K\}| = |\{K \in \Gamma_j : T' \leq K\}|, \tag{7}$$

that is, the number $a_{t,k}(i,j) = |\{K \in \Gamma_j : T \leq K\}|$ is independent of the choice of a fixed $T \in \Delta_i$. The Kramer-Mesner matrix $A_{t,k}$ is then defined as the $\rho(t) \times \rho(k)$ matrix :

$$A_{t,k} = (a_{t,k}(i,j)) \tag{8}$$

Dually, for K fixed in Γ_j, let $b_{t,k}(i,j) := |\{T \in \Delta_i : T \leq K\}|$, and define the dual KM matrix $B_{t,k}$ by:

$$B_{t,k} = (b_{t,k}(i,j)) \tag{9}$$

In the following Lemma we state without proof geometric analogs of some properties of the $A_{t,k}$ and $B_{t,k}$ as included for the ordinary t-design context in [13].

Lemma 2.1. *Let $A_{t,k}$ and $B_{t,k}$, Δ_i, Γ_j be as defined above.*

(i) *If* $t \le s \le k \le n$, *then* $\begin{bmatrix} k-t \\ k-s \end{bmatrix}_q A_{t,k} = A_{t,s} \cdot A_{s,k}$

(ii) $A_{t,k}$ *has constant row sums* $\begin{bmatrix} v-t \\ k-t \end{bmatrix}_q$

(iii) $|\Delta_i| \cdot A_{t,k}(i,j) = |\Gamma_j| \cdot B_{t,k}(i,j)$

Keeping in mind that we are only interested in *simple* geometric t-designs, we now state, without proof, the Kramer-Mesner theorem for geometric t-designs :

Theorem 2.2. *If* $G \le GL_n(q)$, *there is a* G*-invariant (simple)* t-$[q^n, k, \lambda]$ *design if and only if there is a* $\rho(k) \times 1$ *0-1 vector* $\mathbf{u}$ *which is solution of the matrix equation*

$$A_{t,k}\mathbf{u} = \lambda J \tag{10}$$

where J *is the* $\rho(t) \times 1$ *vector of all 1's.*

Here, the 1's in a solution $\mathbf{u}$ select the G-orbits of $\begin{bmatrix} V \\ k \end{bmatrix}$ whose union will constitute the design. The following corollary follows immediately:

Corollary 2.3. *There is a* $[G]$*-invariant large set* $LS[N][t, k, q^n]$ *of geometric designs if and only if there exist* N *distinct solutions,* $\mathbf{u}_1, \ldots, \mathbf{u}_N$, *to the matrix equation (10), whose sum is the* $\rho(k) \times 1$ *all 1's vector.*

3. Main result

It is well known that $GL_n(q)$ has a cyclic subgroup of order $q^n - 1$, called a *Singer subgroup*, acting regularly on the non-zero vectors of $V = \mathbb{F}_q^n$. It is also known that all Singer subgroups are conjugate in $GL_n(q)$. A Singer subgroup G of $\Gamma = GL_8(2)$ is the centralizer of a Sylow-17 subgroup of Γ and its normalizer N in Γ is a split extension of G by its Frobenius group Φ_8, thus $|N| = 2040$. In particular, for the rest of the paper we adopt the notation $V = \mathbb{F}_2^8$, $\Gamma = GL_8(2)$, and $G = \langle \alpha \rangle$, where α is the same Singer cycle as the one used in [4], that is :

$$\alpha = \begin{bmatrix} 0&0&0&0&0&0&0&1 \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&1 \\ 0&0&1&0&0&0&0&1 \\ 0&0&0&1&0&0&0&1 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \end{bmatrix}.$$

We will presently construct 9 distinct large sets of geometric $2-[2^8, 3, 21]$ designs which are $[G]$-invariant under Singer subgroup G of Γ. We have used the exact same Singer subgroup $G = \langle \alpha \rangle$ as in [4] so that it will be easy to check that our large sets are different from the one constructed in [4].

3.1. Computing and presenting $A_{2,3}$

Members of $\begin{bmatrix} V \\ 2 \end{bmatrix}$ are Klein 4-groups, and those of $\begin{bmatrix} V \\ 3 \end{bmatrix}$ elementary abelian groups of order 8. Viewed projectively, the 2- and 3-spaces can be seen as collinear triples and Fano planes respectively. There are in all 10795 2-spaces, and 97155 3-spaces.

We begin by computing the G-orbits on $\begin{bmatrix} V \\ 2 \end{bmatrix}$ and $\begin{bmatrix} V \\ 3 \end{bmatrix}$, where $G = \langle \alpha \rangle$. There are exactly 43 G-orbits on $\begin{bmatrix} V \\ 2 \end{bmatrix}$, all of which have length 255, except for one which has length 85. The short orbit is explained by the fact that the cyclic subgroup of order 3 in G fixes a collinear triple. There are 381 G-orbits on $\begin{bmatrix} V \\ 3 \end{bmatrix}$ all of length 255.

The vectors of $V = \mathbb{F}_2^8$ are represented by the radix-2 representation of integers in $\mathbb{Z}_{256}$. Orbits of 2- and 3-spaces are represented by the lexically smallest basis among all members of the orbit, but since G is transitive on the non-zero vectors, each such basis will consist of the vector $1 \leftrightarrow 00000001$, and one (or two) elements of $\mathbb{Z}_{256} - \{1\}$. Hence, to represent $\langle\alpha\rangle$-orbits of 2-spaces, it suffices to specify the second vector in the lexically minimal basis over all 2-spaces for that orbit. Thus, the $\langle\alpha\rangle$-orbits of 2-spaces are represented by the following 43 integers :

$$2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 24, 28, 30, 32, 34, 36, 38, 40, 42, 44, 50, 54, 56, 58$$

$$60, 62, 70, 74, 76, 78, 80, 86, 88, 96, 100, 106, 114, 128, 136, 146, 164, 210, 218.$$

Similarly, orbit representatives of the 381 orbits of 3-spaces are given by the pair of integers in $\mathbb{Z}_{256}$ which together with 1, form the lexically minimal basis among the members of the orbit of 3-spaces. The pairs $x, y \in \mathbb{Z}_{256}$ representing the G-orbits on 3-spaces will appear in our display of the KM-matrix $A_{2,3}$ below.

To compute $A_{2,3}$, we found it easier to first compute matrix $B_{2,3}$ and then compute the $A_{2,3}(i,j)$ entries, using Lemma 2.1, equation (iii) :

$$A_{2,3}(i,j) = \frac{|\Gamma_j|}{|\Delta_i|} B_{2,3}(i,j).$$

Almost all ratios $\frac{|\Gamma_j|}{|\Delta_i|}$ are 1, that is all, except for those involving the short orbit Δ_{43} of length 85, in which case the ratio is 3. For any particular Fano plane F in orbit Γ_j, it is easy to determine how the 7 lines of F are distributed among the orbits $\{\Delta_i\}$, thus computation of $B_{2,3}$ is straightforward.

In an effort to overcome the difficulty of presenting in this article the 43×381 matrix $A_{2,3}$, the next two pages display a coded version of $A_{2,3}$ from which, with a little effort, a user-friendly version of $A_{2,3}$ can be recovered. Each column of $A_{2,3}$ is a vector consisting of 43 elements from $\{0, 1, 3\}$. We adjoin two extra 0's at the top of the column and transpose, transforming the column to a row vector $\mathbf{v} \in \mathbb{Z}_4^{45}$. We then use the following alphabet of 64 characters, as *digits* with values from 0 to 63: 0123456789abcdefghijklmnopqrstuvwxyzABCDEGFHIJKLMNOPQRSTUVWXYZ+− .

The vector $\mathbf{v} \in \mathbb{Z}_4^{45}$ is separated into 15 triples, and each triple, belonging to $\mathbb{Z}_4^3$, is encoded as a symbol in the alphabet using radix-4 notation. For example,

100011010100010100000000000000000000000000000 $\Longrightarrow$
001 000 110 101 000 101 000 000 000 000 000 000 000 000 000 $\Longrightarrow$
1 0 k h 0 h 0 0 0 0 0 0 0 0 0 $\Longrightarrow$ 10kh0h000000000

The next page displays the first 192 columns, and the subsequent page the remaining columns of $A_{2,3}$. The 381 columns of $A_{2,3}$ correspond to 381 short rows in the display. For example the short row

5 2 20 10kh0h000000000

means the following: i) 5 is the column index for $A_{2,3}$, corresponding to the 5^{th} orbit of 3-spaces under G. (ii) Augmenting the pair 2 20 with 1, yields the basis of 3 vectors $\{1, 2, 20\}$ in $\mathbb{F}_2^8$ from which a Fano plane is constructed, and from which the complete 5^{th} G-orbit can be generated. (iii) By reversing the encoding process discussed earlier, the code "10kh0h000000000" yields the 5^{th} column of $A_{2,3}$, as the transpose of 100011010100010100000000000000000000000000000 .

Remark 3.1. *In passing, we present some properties of $A_{2,3}$ which may be used to establish still unknown features of designs and large sets related to $A_{2,3}$. We say that a vector with integer entries has type $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_m^{\lambda_m}$ if the value x_i appears λ_i times in the vector, for $1 \le i \le m$.*

(i) The row sums of $A_{2,3}$ are all 63, as expected,

(ii) The vector of column sums of $A_{2,3}$ is of type $7^{360} 9^{21}$,

(iii) The row vectors of the long orbits of 2-spaces are all of type $0^{320} 1^{60} 3^1$,

(iv) the row vector for the short orbit of 2-spaces is of type $0^{360} 3^{21}$,

(v) There are 4 column types for $A_{2,3}$ as follows:

1	2	4	3k0000000040004
2	2	8	1lg000100000100
3	2	12	1k5000000011000
4	2	16	111k0000g000040
5	2	20	10kh0h000000000
6	2	24	1140M0400000000
7	2	28	10h050000000013
8	2	32	100g1k00000040g
9	2	36	100541g000000g0
10	2	40	1101g05000000g0
11	2	44	110gg04g4000000
12	2	48	104gg004000g400
13	2	52	1045g0010004000
14	2	56	100g400gk00000g
15	2	60	100110001g10040
16	2	64	10144g001g00000
17	2	68	100004001l00100
18	2	72	10001h00gh00000
19	2	76	1g0000g040k1000
20	2	80	104400k01010000
21	2	84	1000001c000g010
22	2	88	1000000j000400g
23	2	92	1h40000000gg004
24	2	96	10400g0000g5040
25	2	100	10g004140000g10
26	2	104	14k004g00000400
27	2	108	1000010hg500000
28	2	112	11000000g400hg0
29	2	116	101g110g4000000
30	2	120	100040k05000g00
31	2	124	140004100g40004
32	2	128	100014k010000g0
33	2	132	151000004040004
34	2	136	10gg40000g00440
35	2	140	100004000c00gg0
36	2	144	100010g000g0k10
37	2	148	1000000101g4110
38	2	152	10004100g1g0040
39	2	156	1gh000000g4000g
40	2	160	10000g000015440
41	2	164	10040g00000504g
42	2	168	1000001400005h0
43	2	172	1104g040000gg00
44	2	176	10000g001105040
45	2	180	1000g01l0000010
46	2	188	10000010004h40g
47	2	192	100100k0100h000
48	2	196	1010000g0014013
49	2	200	1000010044g0gg0
50	2	204	10010011gg0000g
51	2	212	100000000gh0504
52	2	216	1000504g0010003
53	2	220	10000100g300100
54	2	224	10000004140ggg0
55	2	228	1g000400400010k
56	2	232	10110g000005040
57	2	236	10g40101000010g
58	2	240	1000400000hg0h0
59	2	244	10g51000g000g00
60	2	248	140000g0005g004
61	2	252	100g040k4400000
62	4	16	M0h00000104000
63	4	18	g0k00101110000
64	4	24	k00l0010000100
65	4	26	k00l0040001000
66	4	32	h000h040g00003
67	4	34	gg004g10000140
68	4	40	hg0004g00gg000
69	4	42	h040g500400000
70	4	48	gc00000h010000
71	4	50	g10ggg50000000
72	4	56	g540000h000010
73	4	58	g1001004g01010
74	4	64	ggh00001gg0000
75	4	66	g100001g410400
76	4	72	g04100g00k4000
77	4	74	ggg000001400k0
78	4	80	g0h10040410000
79	4	82	h04000000gM000
80	4	90	g040441000g010
81	4	96	g00g0000g11g0g
82	4	98	g0g101000044g0
83	4	104	k00l00g0400000
84	4	106	g0500000504400
85	4	112	g1040045000g00
86	4	114	g00g1g0400010g
87	4	120	k10l0000040000
88	4	122	g000400000ggk4
89	4	128	g000g004040gg4
90	4	130	k00l0100010000
91	4	136	g000c400000050
92	4	138	gg000hg0500000
93	4	144	g0041g10401000
94	4	146	h0000000ggg050
95	4	152	g004ggg0g00g00
96	4	160	g100g40000140g
97	4	162	g0100g001g4040
98	4	168	h00011g0000410
99	4	170	g00045000000hg
100	4	176	g00g10g0040g40
101	4	178	g000g01g0004h0
102	4	192	g44000g001gg00
103	4	194	g01000g0g10440
104	4	200	g01000411040g0
105	4	202	gg00000400030g
106	4	216	g000141g000440
107	4	218	g000h00g400007
108	4	224	g400000h000gg4
109	4	226	g0001040000ggk
110	4	232	g004g10400010g
111	4	234	g041001g001400
112	4	240	g400040h0g1000
113	4	242	gg004405000010
114	4	248	g0g00g40g44000
115	4	250	l0000040400007
116	6	16	40M00000001h00
117	6	18	c05000001000g0
118	6	32	4g00gg0g000g10
119	6	34	4l005100000000
120	6	40	5400040g000h00
121	6	42	410g11g0000040
122	6	48	5k004010001000
123	6	50	440014400g0040
124	6	56	5000g00gk00003
125	6	58	4h000005004010
126	6	64	400g0001k0004g
127	6	66	41011101g00000
128	6	74	401000401g0g0g
129	6	80	400g000001hg0g
130	6	82	40440401004g0
131	6	88	4000000400s004
132	6	96	40g40100001500
133	6	98	40g00g0000kk00
134	6	106	405000g0000k10
135	6	112	40g014g00g0g00
136	6	114	41g040g0001100
137	6	120	40544010040000
138	6	122	5000g000150g00
139	6	130	40010014g0g040
140	6	136	4001004140004g
141	6	138	4k00k004000100
142	6	144	44000g0g0h0010
143	6	146	4000g0140g00h0
144	6	154	40400004h400g0
145	6	160	44000104004440
146	6	162	4000000140k04g
147	6	168	4g00044g001400
148	6	170	50100050g000g0
149	6	176	400000g140005g
150	6	178	400gg01104g000
151	6	192	4001g0g001g100
152	6	194	401400g00h00g0
153	6	200	404000001100hg
154	6	202	50104004100400
155	6	216	40001c0g0g0000
156	6	218	4000104gg00007
157	6	224	4000g0014g004g
158	6	232	4040k0g0g0000g
159	6	234	40g0041000h100
160	6	240	4g0014110g0000
161	6	242	40000M0g000410
162	6	248	40000g0g404014
163	6	250	41g01100004004
164	8	20	10144010040004
165	8	22	10g10g400000gg
166	8	34	10045100014000
167	8	38	30010g0100000g
168	8	50	11g00054010000
169	8	52	14000gg10010g0
170	8	66	11004040500g00
171	8	70	11100400440100
172	8	80	1g00l000014000
173	8	82	1g44001000g040
174	8	98	10005000014k00
175	8	100	140g00g0100k00
176	8	102	1g00ghg0100000
177	8	112	1104g00g000h00
178	8	114	1040000g044h00
179	8	116	11041000040050
180	8	118	11005000015000
181	8	128	10g0001gg000k0
182	8	130	1g4100000g5000
183	8	144	10k40h01000000
184	8	146	100000041k1010
185	8	148	10000014k010g0
186	8	160	101000g4000c00
187	8	162	100101010041g0
188	8	166	10g0040g0k0040
189	8	182	10001000g01030
190	8	192	140g0010g0g0g0
191	8	194	101g0004g00410
192	8	196	10100104410400

193	8	198	10g000g10g000k	256	12	192	4h1h00000g000	319	18	226	400040gg0504
194	8	212	1000504005400 0	257	12	194	41400400g000k	320	18	230	4000hg00401g
195	8	224	1004g04100400g	258	12	196	500000g01g1g0	321	18	234	401g040010g4
196	8	226	10010041101004	259	12	198	44040g4g00400	322	18	236	4h00004g4100
197	8	228	1k04000000010k	260	12	214	4000000040hk3	323	20	38	101g500g000g
198	8	230	1041gg0000010g	261	12	224	40000g1010114	324	20	44	1100gkg0g000
199	8	240	14010g001g000g	262	12	226	44g0444100000	325	20	78	1k4110040000
200	8	242	111000g0001014	263	12	228	4h0l000000004	326	20	102	10010004h110
201	8	244	10g4000g0000c0	264	12	230	5004000h0100g	327	20	132	1g4g05000003
202	8	246	1g000010510004	265	12	240	4h0h0004g0000	328	20	140	14501004g000
203	10	16	gggg0000k00g0	266	12	242	4g10g00440010	329	20	162	1g0001404110
204	10	18	g5g1000400004	267	14	20	11g0g00040044	330	20	164	100040gh004g
205	10	50	g01004g003000	268	14	22	1gg400h0g0000	331	20	170	1000g10011k0
206	10	52	l040000h000g0	269	14	38	1100k000041g0	332	20	192	14010g01g00g
207	10	64	g00g001h50000	270	14	66	30000500g0g00	333	20	202	10010Q00000g
208	10	66	h0104g0004003	271	14	86	1000g0400k044	334	20	206	100g0410g00j
209	10	68	Mg00000g400g0	272	14	100	100g0110g0k00	335	20	230	10010g000g1k
210	10	70	g0000h04g400g	273	14	128	100400gk0g0g0	336	20	234	10g500401004
211	10	84	g00014040g440	274	14	130	10h41000040g0	337	24	38	g0g10k04100
212	10	100	g000hg0100g40	275	14	148	1000g141g0g00	338	24	106	g0001gg0504
213	10	102	h000hg010g000	276	14	150	10k00401g0400	339	24	110	g10g0100150
214	10	112	g110000000M10	277	14	160	10004000404 0M	340	24	128	k400g0h00g0
215	10	118	h0g1001040400	278	14	164	10104001004gg	341	24	130	h400g1g1000
216	10	130	g110040000g44	279	14	166	104000g00g4k0	342	24	162	g4g0g0k4000
217	10	132	g400444410000	280	14	178	140gh00000044	343	24	164	g4100011g0g
218	10	144	k000hh0100000	281	14	194	1300400001100	344	24	166	g0000111g4g
219	10	146	g040400014013	282	14	198	1000gg00100k4	345	24	168	g10g4400440
220	10	160	kg000000404gg	283	14	214	110g4g000g100	346	24	170	gM0100000h0
221	10	162	g04100040k400	284	14	224	1s00101000000	347	24	234	g1100g01014
222	10	164	h00040040g40g	285	14	226	1004g400h0100	348	24	236	k000g504100
223	10	180	g4g004g001004	286	14	240	1g000104g0h00	349	28	32	cg005000g00
224	10	192	g00000k00k00k	287	14	242	1k00000410014	350	28	74	41000h0h010
225	10	194	g1g0000g100k0	288	14	244	10gg00g00g0g4	351	28	96	40h0044h000
226	10	196	g0410000h00gg	289	16	38	g00h10g01g00	352	28	102	4010l004g00
227	10	198	g11g0g00g0g00	290	16	42	g1011k010000	353	28	226	40440500404
228	10	210	g050410001004	291	16	70	g0500040hg00	354	28	236	40000gg0514
229	10	212	g10000l010010	292	16	74	g00414110010	355	30	36	31004g0g000
230	10	224	g00404h005000	293	16	78	g0040014g01g	356	30	38	1kg0g010400
231	10	226	g0gg045000010	294	16	98	g40400g04h00	357	30	40	10411010404
232	10	228	g4g0004000144	295	16	108	gh00kg040000	358	30	64	1101110g100
233	10	240	g4g00000hg004	296	16	110	h0400110g003	359	30	78	10g40050140
234	10	244	g0g0g0g40g400	297	16	140	g04000gg0504	360	30	110	1k000100k04
235	12	16	4g4000000401k	298	16	164	gg0g0100g00j	361	30	162	105000040M0
236	12	18	4510000440010	299	16	166	k40400100050	362	30	174	14510000044
237	12	32	400g1k000g100	300	16	196	g00014430000	363	30	228	1041010000c
238	12	50	40000l0g0400g	301	16	200	gg04001040h0	364	32	78	g10015h000
239	12	54	4000154000g40	302	16	202	gg400410000j	365	32	94	k0gg10g100
240	12	66	501000044001g	303	16	226	g0g05000g0h0	366	32	156	g500k4000g
241	12	68	5010004h0g000	304	16	228	k0010hg00004	367	32	196	g400150044
242	12	70	5400100l00000	305	16	232	gggg00000414	368	32	216	gk40g10100
243	12	84	4h0h040004000	306	16	234	g40gg0401004	369	34	192	404100g150
244	12	98	4040040105400	307	18	32	44g4k4000000	370	34	200	40g14g10g0
245	12	112	4400010040k03	308	18	40	404k44000040	371	36	64	140104gk00
246	12	118	400100400kg04	309	18	44	4040kgg40000	372	36	78	1013040400
247	12	128	40440g00000gk	310	18	68	4440g0ggg000	373	36	196	1101050500
248	12	130	40100004k0050	311	18	96	400010001h43	374	36	198	14kg040100
249	12	132	4011104000140	312	18	100	4g1000011g0g	375	38	68	k00M04g00
250	12	144	4040k00k01000	313	18	132	40g0044k1000	376	40	68	4g0g54g00
251	12	148	4144g10040000	314	18	160	41k0000g4400	377	42	76	3000M0003
252	12	162	40g0500g04g00	315	18	164	414000000g5g	378	42	78	111140g04
253	12	166	40000100014k3	316	18	196	401g400101g0	379	44	64	M10010g4
254	12	176	400g504400040	317	18	200	4000000415g3	380	44	78	kg0c4000
255	12	182	4050000440110	318	18	206	403g000000gg	381	58	128	c0k10g0

a.) 320 columns of type $0^{36}1^{7}$

b.) 40 columns of type $0^{38}1^{4}3^{1}$

c.) 20 columns of type $0^{36}1^{6}3^{1}$

d.) a single column of type $0^{40}3^{3}$

(vi) Since all G-orbits on 3-spaces have length 255, each of the three constituent designs of any $[G]-$*invariant large set* $LS[3][2,3,2^8]$ *will be comprised of 127 G-orbits of 3-spaces.*

In particular, properties (v) d.) and (vi) imply that a large set $LS[3][2,3,2^8]$ whose automorphism group contains a Singer subgroup *as a normal subgroup*, can not have a group of automorphisms transitive on the 3 2-$[2^8,3,21]$ designs.

3.2. Constructing and presenting the designs and large sets

As the number of columns of $A_{2,3}$ is rather large, a backtrack, depth-first search or similar algorithm would be hopeless in finding solutions to equation (10). Instead, we use *lattice basis reduction* to seek solutions. This technique is nicely described in [15], pages 277-300. For each of the 9 large sets of 2-$[2^8, 3, 21]$ designs we proceed using the following non-deterministic procedure, which, in general, is not guaranteed to terminate.

Procedure 3.2.

(i) Determine a 0-1 solution $\mathbf{u_1}$ *to equation* $A_{2,3}\mathbf{u_1} = 21J$, *thus extracting a 2-*$[2^8, 3, 21]$ *design* $\mathcal{D}_1$ *as the union of 127 G-orbits of 3-spaces. If this step succeeds, proceed to step (ii), otherwise stop.*

(ii) Remove from $A_{2,3}$ *the 127 columns corresponding to design* $\mathcal{D}_1$ *to obtain a* 43×254 *matrix* $C_{2,3}$*, and find a 0-1 solution* $\mathbf{u_2}$ *to equation* $C_{2,3}\mathbf{u_2} = 21J$, *thus extracting a second design* $\mathcal{D}_2$ *consisting of 127 G-orbits among the orbits corresponding to the columns of* $C_{2,3}$*. If this step succeeds, proceed to step (iii), otherwise stop.*

(iii) Remove the 127 columns constituting $\mathcal{D}_2$ *from* $C_{2,3}$*. The remaining 127 columns of* $C_{2,3}$ *correspond to orbits whose union is a third design* $\mathcal{D}_3$*, and* $\mathcal{L} = \{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3\}$ *is a* $LS[3][2, 3, 2^8]$ *large set. If steps (i) and (ii) are successful, so is (iii) and we have a successful termination with output large set* $\mathcal{L}$*.*

Thus, the procedure of finding $\mathbf{u_1}$ and $\mathbf{u_2}$ becomes a matter of solving systems of integer equations through lattice basis reduction [15]. The following procedure describes briefly how the problems are set up so that lattice basis reduction can be used.

Procedure 3.3. *First we construct a matrix that will constitute a basis for an integral lattice* Λ_1 *by adjoining the identity matrix of order 381 above KM matrix* $A_{2,3}$*. To the right of the* 424×381 *matrix just formed we adjoin a* 424×1 *column vector which has zeros in the first 381 positions and* -21*'s in the remaining 43 positions. Let* M_1 *denote the* 424×382 *matrix just formed.*

$$M_1 = \left[\begin{array}{c|c} I & 0 \\ \hline A_{2,3} & -21J \end{array}\right], \qquad M_2 = \left[\begin{array}{c|c} I & 0 \\ \hline C_{2,3} & -21J \end{array}\right]$$

If basis reduction produces a short enough basis M_1' *for* Λ_1 *which contains a short vector* $\mathbf{v_1}$ *with 0's and 1's (or 0's and* -1*'s) in the first 381 positions and all 0's below, then the projection* $\mathbf{u_1}$ *of* $\mathbf{v_1}$ *(or* $-\mathbf{v_1}$*) to the first 381 coordinates is likely to be a solution to* $A_{2,3}\mathbf{u_1} = 21J$ *(see [15].) The weight of* $\mathbf{u_1}$ *will be 127, and the union of orbits of 3-spaces corresponding to the 1's in* $\mathbf{u_1}$ *will form a 2-*$[2^8, 3, 21]$ *design* $\mathcal{D}_1$*.*

If a solution $\mathbf{u_1}$ *is found, then replacing* $A_{2,3}$ *by* $C_{2,3}$ *yields a* 297×255 *matrix* M_2 *which spans a lattice* Λ_2*, and by the same process as above,* M_2 *can yield a solution to* $C_{2,3}\mathbf{u_2} = 21J$, *that is, a design* $\mathcal{D}_2$ *disjoint from* $\mathcal{D}_1$*.*

It is now clear that when the 127 columns corresponding to the orbits forming $\mathcal{D}_2$ are removed from $C_{2,3}$, the remaining 127 orbits will form a 2-$[2^8, 3, 21]$ design $\mathcal{D}_3$, and that $\{\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3\}$ will be a large set.

However, the above procedure is not guaranteed to find a solution at first try, so if the basis reduction algorithm was unable to find a column in reduced basis M_1' that met the conditions in Procedure 3.2.2 , we would repeat the process, twiking the order of the columns of M_1, and the same later for M_2. The above procedure was repeated a number of times and we successfully constructed 9 distinct large sets $\{\mathcal{L}_1, \ldots, \mathcal{L}_9\}$ which we exhibit below.

3.3. Reconstruction of the large sets

We briefly describe the display, to enable the reader to reconstruct the large sets and related designs. The first column is the index of the G-orbits on 3-spaces. There are 9 additional columns, each corresponding to one of the large sets. Each column has 127 1's, 127 2's and 127 3's in it, which select the orbits contained in $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ respectively, for each large set. Since the orbits can be computed from the representative bases in the presentation of $A_{2,3}$, the reader can readily reconstruct the 9 large sets and the designs involved.

Direct computation shows that indeed the 9 large sets are different from each other and different from the large set $\mathcal{L}_0$ constructed in [4]. However, a peculiar visual symmetry is observed in the structure of our 9 large sets

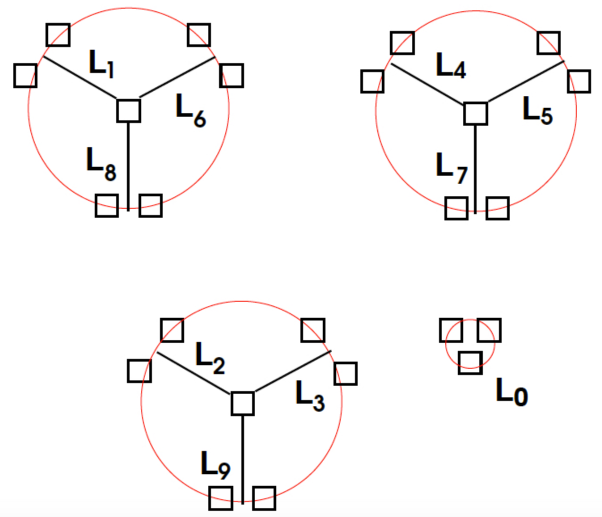

Figure 1. The 10 large sets $\{\mathcal{L}_0, \ldots, \mathcal{L}_9\}$

which is perhaps only related to our search method. The 9 large sets can be divided into 3 clusters of 3 large sets per cluster. The large sets of each cluster share a common 2-$[2^8, 3, 21]$ design forming a triad of large sets, with a central design and three peripheral pairs of designs as illustrated in Figure 1. The three centers are different from each other, and the 18 peripheral designs are also different from each other and the 3 centers. Actually, there are no elements of Γ permuting non-trivially the 3 clusters of large sets, nor elements of order 3 permuting the 3 designs of any one of the large sets.

Checking the list of maximal subgroups of $\Gamma = GL_8(2)$ shows that $N = N_\Gamma(G)$ is not maximal in Γ. Let $\Phi_8 = \langle\zeta\rangle \leq N$, $\zeta : \alpha \to \alpha^2$ be the Frobenius subgroup normalizing G. We have checked that Φ_8 does not fix any of the 9 large sets, and does not move any one of the 9 large sets to any other.

Let $\mathcal{L}_0$ be the $LS[3][2, 3, 2^8]$ discovered by the authors of [4], and let $S = \{\mathcal{L}_i : 0 \leq i \leq 9\}$. We already know that $G \leq Aut(\mathcal{L}_i)$ for each $i \in \{0, \ldots, 9\}$. It is conceivable that the automorphism groups of the 10 large sets in S are not all identical, but we think this is very unlikely and we conjecture that in fact $Aut(\mathcal{L}_i)$ are all identical, and equal to the Singer subgroup G.

$$\zeta = \begin{bmatrix} 1&0&1&0&0&0&0&0\\ 1&0&0&1&1&0&0&0\\ 0&1&1&1&0&0&0&0\\ 1&1&1&1&0&1&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&1&0&0&0&1&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&1&0&0&0&1 \end{bmatrix}$$

We presently rephrase, in the context of our notation, a very useful theorem of A. Betten, R. Laue, and A. Wassermann in [1]. This will immediately yield a corollary concerning the question of isomorphism between the 10 large sets in $S = \{\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_9\}$.

Theorem 3.4. (Theorem 3.1 in [1]) *Let* $\mathcal{G}$ *be a finite group acting on a set* X*. Suppose that* $x_1, x_2 \in X$ *and* $g \in \mathcal{G}$ *such that* $x_1^g = x_2$*. Moreover, suppose that a Sylow subgroup* P *of* $\mathcal{G}$ *is contained in the stabilizers* $\mathcal{G}_{x_1}$ *and* $\mathcal{G}_{x_2}$*. Then,* $x_1^n = x_2$ *for some* $n \in N_{\mathcal{G}}(P)$*.*

1	222332233	65	311113121	129	122221313	193	233223323	257	111111111	321	123111112
2	111111111	66	211232221	130	333112123	194	233333323	258	132321213	322	311333321
3	211232221	67	233333333	131	223333232	195	122111113	259	111221311	323	211222321
4	133111112	68	111111111	132	211323231	196	223332232	260	132111112	324	223112133
5	132111112	69	232112123	133	132111112	197	133321213	261	233112122	325	123221213
6	311323331	70	322332323	134	323113123	198	311222331	262	232112133	326	233112133
7	333233333	71	323113133	135	323223233	199	311112131	263	111321311	327	233232223
8	332112122	72	322323333	136	123111113	200	322112133	264	211232231	328	211333321
9	133111112	73	111221311	137	311222231	201	322322323	265	311112131	329	232332322
10	211222331	74	122111113	138	133111112	202	211223321	266	211323221	330	223333232
11	311333221	75	311223321	139	211322221	203	332233323	267	223113132	331	111331311
12	133321313	76	233113132	140	111111111	204	123111112	268	223112123	332	211333221
13	311223331	77	322233223	141	332322232	205	322323333	269	211113131	333	233223232
14	122331213	78	223333332	142	232113132	206	133111112	270	322323232	334	122221212
15	211223221	79	232322322	143	332113123	207	333113123	271	322233233	335	111111111
16	333333223	80	111331211	144	111331211	208	211322331	272	111331211	336	323333323
17	222222332	81	111331311	145	323232223	209	223322332	273	123221213	337	111111111
18	232322323	82	211112131	146	323112123	210	332322233	274	323332222	338	232233233
19	111111111	83	222112122	147	111231311	211	211113131	275	122331312	339	211113121
20	223332232	84	222333232	148	123231313	212	123221213	276	222222232	340	223333322
21	233223332	85	211232321	149	133321212	213	111111111	277	232232323	341	133111113
22	222332323	86	332113133	150	132231212	214	233332233	278	223333332	342	133221212
23	322232223	87	111111111	151	323233223	215	311112121	279	332113132	343	233113132
24	223233322	88	133321313	152	322322323	216	332232222	280	233323322	344	311232221
25	322333322	89	111111111	153	211223331	217	132231313	281	332333223	345	233333232
26	211333331	90	332323232	154	223113132	218	332333223	282	111231311	346	223333332
27	132231212	91	332333332	155	223222222	219	132111112	283	332113122	347	332233222
28	323113132	92	233332222	156	333113123	220	223323232	284	223112123	348	111221311
29	111111111	93	123111113	157	111221311	221	211232331	285	211222221	349	222232333
30	311323321	94	122331312	158	322333233	222	323332322	286	333112123	350	323323323
31	222112133	95	222222232	159	223332322	223	222222222	287	111321211	351	323232232
32	132111113	96	132221213	160	133231212	224	332113123	288	223332233	352	111111111
33	311113121	97	332222323	161	222332223	225	311112121	289	132111112	353	132331212
34	233113133	98	322232233	162	311333331	226	111111111	290	111221211	354	311112131
35	322223223	99	133111112	163	211232331	227	211332321	291	132221213	355	222233223
36	133221313	100	322112133	164	332232233	228	111111111	292	223113133	356	322223333
37	133111112	101	233112122	165	133111112	229	322112132	293	311112121	357	122111113
38	232113132	102	211113131	166	122111112	230	332333333	294	211113121	358	222112132
39	222233222	103	133221312	167	333333332	231	122221212	295	132111112	359	311332231
40	333112132	104	211322221	168	332112132	232	111321211	296	111111111	360	311323321
41	111221211	105	323223233	169	223113122	233	133231212	297	322322222	361	322222223
42	111231211	106	111321211	170	232323223	234	311112121	298	322323333	362	223112122
43	322233232	107	223112132	171	123111112	235	123331212	299	333232233	363	332223333
44	111111111	108	111221211	172	323223233	236	132231312	300	332332333	364	111111111
45	233322233	109	311323331	173	232233333	237	111321211	301	122221313	365	311112131
46	133321312	110	133331312	174	111231211	238	333113122	302	211332221	366	311113131
47	111111111	111	133231212	175	123111113	239	323112122	303	222222232	367	122321213
48	111221311	112	333233233	176	111221311	240	222223332	304	311332321	368	122111112
49	211333231	113	223113133	177	233332322	241	333233333	305	211232321	369	311233321
50	311112121	114	323322223	178	222222333	242	132231313	306	333323322	370	311222221
51	123111113	115	111111111	179	211322221	243	123221213	307	311223321	371	111321311
52	323322222	116	232233332	180	133111112	244	322113123	308	133111113	372	333323232
53	323332322	117	332323332	181	211233331	245	323113122	309	111111111	373	222223322
54	333112133	118	211232221	182	211322221	246	233112122	310	223223332	374	311332331
55	222222322	119	311112121	183	322223222	247	232332232	311	322323232	375	332333332
56	232322233	120	211233321	184	211323231	248	123111113	312	223112123	376	233222232
57	323223223	121	132231313	185	311112121	249	122221312	313	232332332	377	322333333
58	311113121	122	222112122	186	322322223	250	211222231	314	111111111	378	211223331
59	122321312	123	232322332	187	132231213	251	111221311	315	311112121	379	232233332
60	333222323	124	133321213	188	111231211	252	332322323	316	222332223	380	322333323
61	111231311	125	111221311	189	322322223	253	211233331	317	232232323	381	333322322
62	333223322	126	222112132	190	111321211	254	223113123	318	332333333		
63	311332231	127	133321312	191	133231312	255	133231213	319	111111111		
64	133321313	128	123231213	192	232323333	256	311233231	320	111111111		

Corollary 3.5. *The ten Large Sets in $S = \{\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_9\}$ are pairwise non-isomorphic.*

Proof. Let $\mathcal{L}$ be the collection all large sets of type $LS[3][2,3,2^8]$, and $G = \langle\alpha\rangle$ be the Singer subgroup as defined earlier. Then, Γ acts on $\mathcal{L}$, and for any $\lambda \in \mathcal{L}$ the stabilizer Γ_λ is the full automorphism group of λ. In particular, for each $\lambda \in S$ we have that $P < G \leq \Gamma_\lambda$ where P is the Sylow-17 subgroup of G. Let $\beta, \gamma \in S$, $\beta \neq \gamma$, and suppose there is $g \in \Gamma$ such that $\beta^g = \gamma$. Then, by Theorem 3.4, there would exist an element $n \in N_\Gamma(P) = N_\Gamma(G)$ such that $\beta^n = \gamma$. But we know, by direct checking, that no element of the Frobenius group Φ_8, and therefore no element of $N_\Gamma(G) = G \cdot \Phi_8$ sends β to γ, a contradiction. □

3.4. Conclusions

Until 2014, the only large sets of geometric t-$[q^n, k, \lambda]$ designs known were for $t = 1$. In finite geometry, LS[N]-$[t, k, q^n]$ large sets with $t = 1$, are known as (k-1)-*parallelisms* of (k-1)-*spreads* in PG$(n-1, q)$. The first large set $\mathcal{L}_0$ of geometric 2-designs, a LS[3]-$[2,3,2^8]$, was constructed by the authors of [4]. In this paper we construct an additional nine pairwise different large sets $\mathcal{L}_1, \ldots, \mathcal{L}_9$ which are also different from $\mathcal{L}_0$. All these large sets are $[G]$-invariant, under the same Singer subgroup G of order 255. In fact, the large sets $\{\mathcal{L}_0, \ldots, \mathcal{L}_9\}$ are pairwise non-isomorphic.

3.5. Possible future work

The necessary conditions for the existence of a LS[3]-$[3,4,2^9]$ are satisfied and we are close to settling the question of existence of a LS[3]-$[3,4,2^9]$.

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