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The 3-GDDs of type g^3u^2

Research Article

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Abstract: A 3-GDD of type g^3u^2 exists if and only if g and u have the same parity, 3 divides u and $u \leq 3g$. Such a 3-GDD of type g^3u^2 is equivalent to an edge decomposition of $K_{g,g,g,u,u}$ into triangles.

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1. Introduction

A *group divisible design* (GDD) is a decomposition of the complete multipartite graph into complete subgraphs. The complete subgraphs used are the *blocks* of the GDD and are presented by giving the subset of the vertices they span. The partite sets are *groups*. Formally a $\mathscr{K}$-GDD is a triple $(V, \mathscr{B}, \mathscr{G})$ where

1. V is a finite set of *points*;
2. $\mathscr{B}$ is a collection of subsets of V, where $|B| \in \mathscr{K}$, for all $B \in \mathscr{B}$;
3. $\mathscr{G}$ is a partition of V into *groups*.
4. Every pair of points is in exactly one block or group.

The type of a GDD is the multiset of its group sizes. Thus a decomposition of $K_{g_0,g_1,g_2,\ldots,g_{t-1}}$ into complete subgraphs is a GDD of type $\{g_0, g_1, g_2, \ldots, g_{t-1}\}$. If the GDD has t_i groups of size g_i it is our custom to specify the type with the notation: $g_0^{t_0} g_1^{t_1} g_2^{t_2} \cdots g_\ell^{t_\ell}$. Also if $\mathscr{K} = \{k\}$ we write k-GDD instead of $\{k\}$-GDD. The blocks of a 3-GDD are usually called *triples* or *triangles*. For example a 3-GDD of type $4^3 2^1$ is a decomposition of $K_{4,4,4,2}$ into triangles.

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Theorem 1.1. *For a 3-GDD of type $g_1 g_2 \cdots g_s$ with $g_1 \geq \cdots \geq g_s \geq 1$, $s \geq 2$, and $v = \sum_{i=1}^{s} g_i$ to exist, necessary conditions include* (Colbourn [2])*:*

1. $\binom{v}{2} \equiv \sum_{i=1}^{s} \binom{g_i}{2} \pmod 3$;
2. $g_i \equiv v \pmod 2$ *for* $1 \leq i \leq s$;
3. $g_1 \leq \sum_{i=3}^{s} g_i$;
4. *whenever* $\alpha_i \in \{0,1\}$ *for* $1 \leq i \leq s$ *and* $v_0 = \sum_{i=1}^{s} \alpha_i g_i$,

$$v_0(v - v_0) \leq 2\left[\binom{v_0}{2} + \binom{v - v_0}{2} - \sum_{i=1}^{s} \binom{g_i}{2}\right]$$

5. $2g_2g_3 \geq g_1[g_2 + g_3 - \sum_{i=4}^{s} g_i]$; *and*
6. *if* $g_1 = \sum_{i=3}^{s} g_i$ *then* $2g_3g_4 \geq (g_1 - g_2)[g_3 + g_4 - \sum_{i=5}^{s} g_i]$.

These conditions are known to be sufficient when

1. (Wilson [8, 9]) $g_1 = \cdots = g_s$;
2. (Colbourn, Hoffman, and Rees [5]) $g_1 = \cdots = g_{s-1}$ *or* $g_2 = \cdots = g_s$;
3. (Colbourn, Cusack, and Kreher [3]) $1 \leq t \leq s$, $g_1 = \cdots = g_t$, *and* $g_{t+1} = \cdots = g_s = 1$;
4. (Bryant and Horsley [1]) $g_3 = \cdots = g_s = 1$; *and*
5. (Colbourn [2]) $\sum_{i=1}^{s} g_i \leq 60$.

Surprisingly, in no other cases are *necessary and sufficient conditions* known for any other class of 3-GDDs (of index 1). Partial results are known when $g_3 = \cdots = g_s = 2$ [6]. Theorem 1.1 establishes that no 3-GDD with two groups exists; every 3-GDD with three groups has $g_1 = g_2 = g_3$; and every 3-GDD with four groups has type g^4 or g^3u^1; moreover, the first and second sufficient conditions ensure that all such 3-GDDs exist. Turning to five groups, the situation is much less satisfactory. While Theorem 1.1 handles all types g^5, g^4u^1, and $g_1 \cdots g_5$ with $\sum_{i=1}^{5} g_i \leq 60$, many more cases are possible. Indeed it may happen that a 3-GDD with five groups has all groups of different sizes; for example, a 3-GDD of type $17^1 11^1 9^1 7^1 5^1$ exists [2]. Hence the general existence problem for five groups appears to be substantially more complicated than cases with fewer groups. We address one part of this problem, when there are only two group sizes.

The focus of this article is to prove

Theorem 1.2 (Main Theorem)**.** *A 3-GDD of type g^3u^2 exists if and only if $g \equiv u \pmod 2$, $u \equiv 0 \pmod 3$, and $u \leq 3g$.*

If a 3-GDD of type g^3u^2 exist, then $v = 3g + 2u \equiv 2u \pmod 3$ and $v \equiv g \pmod 3$. Thus it follows from Theorem 1.1 conditions (1) and (2) that $g \equiv u \pmod 2$ and $u \equiv 0 \pmod 3$. Condition 3 of Theorem 1.1 is exactly the necessary conditions for the existence of a 3-GDD of type g^3u^2 are established by Theorem 1.1. Sufficiency is proved in the sections that follow.

2. 3-GDDs of type g^3u^2

Let $\gamma_{ij\ell}$ be the number of triples that contain points of groups G_i, G_j, and G_ℓ. Elementary counting establishes that when $|G_1| = |G_2| = |G_3| = g$ and $|G_4| = |G_5| = u$, we have $\gamma_{123} = g^2 - \frac{1}{3}(u(3g - u))$, $\gamma_{124} = \gamma_{125} = \gamma_{134} = \gamma_{135} = \gamma_{234} = \gamma_{235} = \frac{1}{6}(u(3g - u))$, and $\gamma_{145} = \gamma_{245} = \gamma_{345} = \frac{1}{3}u^2$. An easy case arises when $u = 3g$:

Lemma 2.1. *There exists a* 3-GDD *of type* $g^3(3g)^2$*, for all* g.

Proof. A 3-GDD of type $(3g)^3$ exists. Partition one of the groups into three groups of size g on these groups place the triples of a 3-GDD of type g^3. □

A *one-factor* on a set S is a set of $|S|/2$ vertex-disjoint edges. A *holey one-factor* on a set S with *hole* H is a set of $(|S| - |H|)/2$ vertex-disjoint edges in which no edge is incident to a vertex in H. We use the following result.

Lemma 2.2. (Rees [7]) *Let* $h \geq 1$ *and* $0 \leq r \leq 2h$*,* $(h, r) \notin \{(1, 2), (3, 6)\}$*. There exists a* $\{2, 3\}$*-*GDD *of type* $(2h)^3$ *which is resolvable into* r *parallel classes of blocks of size* 3 *and* $4h - 2r$ *parallel classes of blocks of size* 2*. Consequently whenever* $0 \leq x \leq r$*, the edges of* $K_{2h,2h,2h}$ *can be partitioned into* $4h - 2r$ *one-factors,* $3x$ *holey one-factors (*x *for each group), and* $r - x$ *parallel classes of triples.*

Theorem 2.3. *If there exists a* 3-GDD *of type* x^3u^2 *with* $g \equiv x \pmod 2$ *and* $g \geq 2x+u$*, then there exists a* 3-GDD *of type* g^3u^2.

Proof. Write $h = \frac{g-x}{2}$. Without loss of generality, $u \neq 0$ so $u \geq 3$. Because $g \geq 2x + u$, then $h \geq \frac{x+u}{2} \geq 2$. When $h = 3$, we have $(g, x, u) \in \{(7, 1, 3), (9, 3, 3)\}$, and the required GDDs are from Theorem 1.1. Henceforth $h \notin \{1, 3\}$. Choose groups $\{G_i : 1 \leq i \leq 5\}$ with $|G_1| = |G_2| = |G_3| = g$ and $|G_4| = |G_5| = u$. For $i \in \{1, 2, 3\}$ partition G_i into parts $G_{i,1}$ and $G_{i,2}$ where $|G_{i,1}| = x$ and $|G_{i,2}| = g - x = 2h$. Place a 3-GDD of type x^3u^2 aligning the groups on $G_{1,1}$, $G_{2,1}$, $G_{3,1}$, G_4, and G_5. Now $r = 2h - u = g - x - u \geq 2x + u - x - u = x$. So use Lemma 2.2 with groups $G_{1,2}$, $G_{2,2}$, $G_{3,2}$ to construct a partition of $K_{2h,2h,2h}$ into $4h - 2r$ one-factors $\{F_y : y \in G_4 \cup G_5\}$; for $i \in \{1, 2, 3\}$, x holey one-factors $\{H_{i,x} : x \in G_{i,1}\}$ missing $G_{i,2}$; and $r - x$ parallel classes of triples. Include all $(2h)(r - x)$ triples in the $r - x$ parallel classes. Then for each $y \in G_4 \cup G_5$, adjoin y to each edge in F_y, forming $2u(3(2h))$ triples. Finally, for $i \in \{1, 2, 3\}$ and $x \in G_{i,1}$, adjoin point x to each edge in $H_{i,x}$ to form $6xh$ additional triples. □

Corollary 2.4. *There exists a* 3-GDD *of type* g^3u^2*, whenever* $g \geq \frac{5}{3}u$*,* $u \equiv 0 \pmod 3$*, and* $g \equiv u \pmod 2$.

Proof. Apply Theorem 2.3 with $x = u/3$. □

In the remainder, the expression *give weight* w *to the point* x means to replace x with a set of w new points $x_1, x_2, \ldots, x_w$; and if $S = \{s_1, s_2, \ldots, s_k\}$ is a set of points given weights $(w(s_i) : 1 \leq i \leq k)$, then to *place a* 3-GDD *of type* $\{w(s_1), w(s_2), \ldots, w(s_k)\}$ *on* S means to include all triples in a 3-GDD of type $\{w(s_1), w(s_2), \ldots, w(s_k)\}$ with groups $\{\{s_{i,1}, s_{i,2}, s_{i,3}, \ldots, s_{i,w(s_i)}\} : 1 \leq i \leq k\}$. A synonymous expression is to *fill the inflated block* with a 3-GDD of type $\{w(s_1), w(s_2), \ldots, w(s_k)\}$. This is illustrated in the following.

Lemma 2.5. *If a* 3-GDD *of type* $(g/w)^3(u/w)^2$ *exists, then a* 3-GDD *of type* g^3u^2 *also exists.*

Proof. Starting with a 3-GDD of type $(g/w)^3(u/w)^2$, give weight w to the points using a 3-GDD of type w^3, which always exists. □

Recall that a 5-GDD of type k^5 is equivalent to 3 mutually orthogonal Latin squares of order k, which are known to exist when $k \notin \{2, 3, 6, 10\}$ [4]. (When $k = 10$ existence remains uncertain, but they do not exist for $k \in \{2, 3, 6\}$.)

Lemma 2.6. *If there exists a* 5-GDD *of type* k^5*, and integers* g, u *with* $g \equiv u \equiv k \pmod 2$*,* $3k \leq g, u \leq 9k$*, and* $u \equiv 0 \pmod 3$*, then there exists a* 3-GDD *of type* g^3u^2.

Proof. Form a 5-GDD of type k^5 with groups $\overline{G_1}, \overline{G_2}, \overline{G_3}, \overline{G_4}, \overline{G_5}$. Let the points of $\overline{G_i}$ be $\{x_{i1}, \ldots, x_{ik}\}$, so that $\{x_{1k}, x_{2k}, x_{3k}, x_{4k}, x_{5k}\}$ is a block.

Write $g = 3a + 9(k-1-a) + b$ with $0 \le a \le k-1$ and $b \in \{3,5,7,9\}$. Write $u = 3c + 9(k-c)$ with $0 \le c \le k$. Set

$$w(x_{ij}) = \begin{cases} b & \text{if } 1 \le i \le 3 \text{ and } j = k, \\ 3 & \text{if } 1 \le i \le 3 \text{ and } 1 \le j \le a, \\ 3 & \text{if } 4 \le i \le 5 \text{ and } 1 \le j \le c, \\ 9 & \text{if } 1 \le i \le 3 \text{ and } a+1 \le j < k, \\ 9 & \text{if } 4 \le i \le 5 \text{ and } c+1 \le j \le k. \end{cases}$$

According to [2], there exist 3-GDDs of types 3^5, $5^1 3^4$, $7^1 3^4$, $9^1 3^4$, $9^1 5^1 3^3$, $5^3 3^2$, $9^1 7^1 3^3$, $7^3 3^2$, $9^2 3^3$, $9^2 5^1 3^2$, $9^2 7^1 3^2$, $9^3 3^2$, $9^2 5^3$, $9^3 5^1 3^1$, $9^3 7^1 3^1$, $9^2 7^3$, $9^4 3^1$, $9^4 5^1$, $9^4 7^1$, and 9^5. Each block of the 5-GDD of type k^5 has weights forming one of these types, so we place a 3-GDD on the points arising from each. □

Theorem 2.7. *Suppose $u \equiv 0 \pmod 3$, $u \equiv g \pmod 2$ and $3 \le u \le 3g$. Then a 3-GDD of type $g^3 u^2$ exists, except possibly when $3g + 2u > 60$ and*

$$g \in \{9, 10, 11, 13, 18, 20, 22, 30, 32, 34\} \text{ and } \frac{3}{5}g < u < 3g; \text{ or} \tag{1}$$

$$g \equiv 1 \pmod 3 \text{ and } u = 3g - 6; \text{ or} \tag{2}$$

$$u \in \{18, 30\} \text{ and } u < g < \frac{5}{3}u. \tag{3}$$

Proof. Using Lemma 2.1 and Corollary 2.4, assume that $\frac{3}{5}g < u < 3g$. Write $u = 3\ell$ and $g = 3m + r$, where $r \in \{0,1,2\}$; then $\ell \equiv m \pmod 2$ if and only if $r \in \{0,2\}$. To handle cases with $u \ge g$ and $g \notin S = \{2,4,6,8,9,10,11,13,18,20,22,30,32,34\}$, apply Lemma 2.6 with $k = m$ when $r \in \{0,2\}$ and $k = m-1$ when $r = 1$. When applied with $k = m-1$, $u \le 9m - 9$, leading to the possible exceptions in (2). When $g \in \{2,4,6,8\}$ and $u < 3g$, all required GDDs are from [2]. Thus when $u \ge g$, the possible exceptions are listed in (1) and (2).

To handle cases when $u \le g$ and $u \notin T = \{6, 9, 18, 30\}$, apply Lemma 2.6 with $k = \ell$. When $u \in \{6, 9\}$ and $u \le g \le \frac{5}{3}u$, all required GDDs are from [2]. Thus when $u \le g$, the possible exceptions are listed in (3). □

Lemma 2.8. *There exists a 3-GDD of type $13^3 15^2$.*

Proof. Begin with a 5-GDD of type 5^5. Fix a block B and give weights 1,1,1,3,3 to it. On the remaining points give weight 3. Fill the inflated blocks with 3-GDDs of type $1^3 3^1$, $1^1 3^4$, 3^5 from [2]. □

Theorem 2.9. *A 3-GDD of type $g^3 u^2$ exists if and only if $g \equiv u \pmod 2$ and $u \equiv 0 \pmod 3$ except possibly when $g \equiv 1 \pmod 3$, $g \ge 16$, and $u = 3g - 6$; or*

$$g^3 u^2 \in \left\{ \begin{array}{cccccccc} 9^3 21^2, & 10^3 24^2, & 11^3 15^2, & 11^3 21^2, & 11^3 27^2, & 13^3 21^2, & 13^3 27^2, & 13^3 33^2, \\ 18^3 42^2, & 18^3 48^2, & 20^3 42^2, & 20^3 48^2, & 20^3 54^2, & 22^3 42^2, & 22^3 48^2, & 22^3 54^2, \\ 22^3 60^2, & 30^3 84^2, & 32^3 78^2, & 32^3 90^2, & 34^3 84^2, & 34^3 96^2. & & \end{array} \right\}$$

Proof. Apply Lemma 2.1, Lemma 2.5, Theorem 2.3, and Theorem 2.7. Then apply Lemma 2.6 with $k = 4$ to handle types $18^3 u^1$ for $u \in \{12, 24\}$ and $22^3 u^1$ for $u \in \{24, 30, 36\}$; and with $k = 8$ to handle $30^3 u^1$ for $u \in \{24, 48, 72\}$, $32^3 u^1$ for $u \in \{30, 42, 54, 66\}$, and $34^3 u^1$ for $u \in \{24, 36, 48, 60, 72\}$. Apply Lemma 2.8 to handle $13^3 15^2$. □

3. Incomplete group divisible designs

Let K be a set of positive integers, each at least 2. An *incomplete group divisible design* (K-IGDD) of *type* $(g_1 : h_1)^{u_1} \cdots (g_s : h_s)^{u_s}$ is a quadruple $(V, \mathscr{B}, \mathscr{G}, H)$ where

1. V is a set of $\sum_{i=1}^{s} u_i g_i$ elements;
2. $H \subset V$, the *hole*, contains $\sum_{i=1}^{s} u_i h_i$ elements;
3. $\mathscr{G} = \{G_1, \ldots, G_m\}$ is a partition of V into $m = \sum_{i=1}^{s} u_i$ groups $G_1, \ldots G_m$ so that u_i of the groups have size g_i and contain h_i points of H, for $1 \le i \le s$;
4. $\mathscr{B}$ is a set of blocks with $|B| \in K$ whenever $B \in \mathscr{B}$, so that every pair of elements that are in the hole or in a group do not appear in a block, and every other pair occurs in exactly one block.

When $K = \{k\}$, we write k-IGDD.

Lemma 3.1. *Suppose that K is a set of odd positive integers. If a K-IGDD of type $(g_1 : h_1)^{u_1} \cdots (g_s : h_s)^{u_s}$ exists and $w \ge 2$, then a 3-IGDD of type $(wg_1 : wh_1)^{u_1} \cdots (wg_s : wh_s)^{u_s}$ exists.*

Proof. Give weight w to each point and fill with a 3-GDD of type w^k for $k \in K$. □

Corollary 3.2. *A 3-IGDD of type $(12 : 3)^1(6 : 3)^4$ exists.*

Proof. A $\{3, 5\}$-IGDD of type $(4 : 1)^1(2 : 1)^4$ exists with groups $\{\{d_i, x_i\} : 0 \le i \le 3\} \cup \{y, z_1, z_2, z_3\}$, hole $\{d_0, d_1, d_2, d_3, y\}$, and blocks $\{\{d_i, x_{(i+j) \bmod 4}, z_j\} : 0 \le i \le 3, 1 \le j \le 3\}$ and $\{x_0, x_1, x_2, x_3, y\}$. Apply Lemma 3.1 with $w = 3$. □

Lemma 3.3. *If a 3-IGDD of type $(3g : 3h)^3$ and a 3-IGDD of type $(g : h)^3$ exist, then a 3-IGDD of type $(3g : 3h)^2(g : h)^3$ exists.*

Proof. Fill one group of the 3-IGDD of type $(3g : 3h)^3$ with the 3-IGDD of type $(g : h)^3$. □

Corollary 3.4. *When $1 \le h \le \frac{1}{2}g$, a 3-IGDD of type $(3g : 3h)^2(g : h)^3$ exists. In particular, a 3-IGDD of type $(6 : 3)^2(2 : 1)^3$ and a 3-IGDD of type $(12 : 3)^2(4 : 1)^3$ exist.*

Proof. A 3-IGDD of type $(g : h)^3$ is equivalent to a latin square of side g with a subsquare of side h, which exist whenever $1 \le h \le \frac{1}{2}g$, [4]. □

Lemma 3.5. *A 3-IGDD of type $(4 : 1)^i(2 : 1)^{5-i}$ exists when $i \in \{0, 2\}$. Hence a 3-IGDD of type $(6 : 3)^5$ and a 3-IGDD of type $(12 : 3)^2(6 : 3)^3$ exist.*

Proof. When $i = 0$, form blocks $\{\{a_i, i+1, i+4\}, \{a_i, i+2, i+3\} : i \in \mathbb{Z}_5\}$ with groups $\{a_i, i\}$ and hole $\{a_i : i \in \mathbb{Z}_5\}$.

When $i = 2$, a solution follows:

Blocks: $\{5,7,10\}$, $\{5,6,11\}$, $\{4,9,10\}$, $\{4,8,12\}$, $\{4,7,13\}$, $\{3,8,10\}$, $\{3,6,12\}$, $\{3,4,11\}$, $\{2,9,12\}$, $\{2,7,11\}$, $\{2,6,13\}$, $\{2,5,8\}$, $\{1,8,11\}$, $\{1,7,12\}$, $\{1,4,6\}$, $\{1,2,10\}$, $\{0,9,11\}$, $\{0,8,13\}$, $\{0,6,10\}$, $\{0,5,12\}$, $\{0,3,7\}$, $\{0,2,4\}$.

Groups: $\{0,1\}$, $\{2,3\}$, $\{4,5\}$, $\{6,7,8,9\}$, $\{10,11,12,13\}$.

Hole: $\{1,3,5,9,13\}$.

Use Lemma 3.1 with weight 3 to obtain the specific IGDDs. □

Lemma 3.6. *There exist* 3-IGDDs *of type* $(4:1)^3(6:3)^i(12:3)^{2-i}$ *for* $i \in \{0,1,2\}$.

Proof. When $i = 0$, apply Corollary 3.4. When $i = 2$, start with points $\{x_j : x \in \mathbb{Z}_5, \quad j \in \mathbb{Z}_3\}$; elements with the same x-coordinate are in the same group of the IGDD. Place orbits of triples $\{0_0, 1_0, 2_0\}$, $\{0_0, 3_0, 4_0\}$, $\{1_0, 3_0, 4_1\}$, and $\{2_0, 3_0, 4_2\}$, developing the subscript modulo 3. Then the remaining pairs $\{x_i, y_j\}$ with $x \neq y$ can be partitioned into a holey 1-factor missing $\{x_0, x_1, x_2\}$ for $x \in \{0,1,2\}$ and three holey 1-factors missing $\{x_0, x_1, x_2\}$ for $x \in \{3,4\}$. Extending these 9 holey 1-factors gives the 9 points in the hole of the IGDD.

To construct a 3-IGDD of type $(4:1)^3(6:3)^1(12:3)^1$ first form seven sets of size 3: $\{A_i = \{a^i_j : j \in \mathbb{Z}_3\} : i \in \mathbb{Z}_3\}$, $B = \{b_j : j \in \mathbb{Z}_3\}$, and $\{C_i = \{c^i_j : j \in \mathbb{Z}_3\} : i \in \mathbb{Z}_3\}$. Let $H = \{\alpha_i, \beta_i, \gamma_i : i \in \mathbb{Z}_3\}$ be 9 additional points. We construct the 3-IGDD with groups:

$$\Big(A_0 \cup \{\alpha_0\}\Big), \Big(A_1 \cup \{\alpha_1\}\Big), \Big(A_2 \cup \{\alpha_2\}\Big), \Big(B \cup \{\beta_0, \beta_1, \beta_2\}\Big), \Big(C_0 \cup C_1 \cup C_2 \cup \{\gamma_0, \gamma_1, \gamma_2\}\Big)$$

and hole H. Now form

1. the triples of a 3-GDD of type 3^3 on groups $\{\beta_0, \beta_1, \beta_2\}$, A_i, and C_i for $i \in \mathbb{Z}_3$,
2. $\{\{\gamma_i, b_j, a^i_j\}, \{\gamma_i, a^{i+1}_j, a^{i+2}_j\} : i, j \in \mathbb{Z}_3\}$,
3. $\{\{\alpha_i, a^{i+1}_j, c^{i+2}_j\}, \{\alpha_i, a^{i+2}_j, c^{i+1}_j\}, \{\alpha_i, b_j, c^i_j\} : i, j \in \mathbb{Z}_3\}$,
4. $\{\{b_j, a^i_{j+1}, c^{i+1}_{j+2}\}, \{b_j, a^{i+1}_{j+2}, c^i_{j+1}\} : i, j \in \mathbb{Z}_3\}$,
5. $\{\{a^i_j, c^{i+1}_{j+2}, a^{i+2}_{j+1}\} : i, j \in \mathbb{Z}_3\}$,
6. $\{\{a^0_j, a^1_{j+1}, a^2_{j+2}\} : j \in \mathbb{Z}_3\}$.

It is an easy but tedious exercise to verify that these triples provide the desired IGDD. □

Lemma 3.7. *A* 3-IGDD *of type* $(5:1)^3(9:3)^2$ *exists.*

Proof. Form a set $X = \mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_2$ of points. Let $G_i = \{i\} \times \{0,1\} \times \mathbb{Z}_2$ for $i \in \mathbb{Z}_3$, and let $G_{j+1} = \mathbb{Z}_3 \times \{j\} \times \mathbb{Z}_2$ for $j \in \{2,3\}$. On X with groups $\{G_i : 0 \le i \le 4\}$ we construct a partition of pairs not in a group into one holey parallel class of ten pairs missing G_i for each $i \in \{0,1,2\}$; three parallel clases of nine pairs missing G_i for each $i \in \{3,4\}$; and 48 triples. Once constructed, extending holey parallel classes produces the desired IGDD.

First we make the triples. Form a 3-GDD of type 4^3 on $\mathbb{Z}_3 \times \mathbb{Z}_4$ having a parallel class on $\{\mathbb{Z}_3 \times \{j\} : j \in \mathbb{Z}_4\}$ and groups on $\{\{i\} \times \mathbb{Z}_4 : i \in \mathbb{Z}_3\}$. This has 12 triples; give weight 2 to form 48 triples.

For $i \in \mathbb{Z}_3$ let

$$F_i = \left\{ \begin{array}{lll} \{(i,2,0),(i,3,0)\}, & \{(i,2,1),(i,3,1)\}, & \\ \{(i+1,2,0),(i+1,3,1)\}, & \{(i+1,0,0),(i+1,2,1)\}, & \{(i+1,1,1),(i+1,3,0)\}, \\ \{(i+2,2,1),(i+2,3,0)\}, & \{(i+2,0,1),(i+2,2,0)\}, & \{(i+2,1,0),(i+2,3,1)\}, \\ \{(i+1,0,1),(i+2,0,0)\}, & \{(i+1,1,0),(i+2,1,1)\}. & \end{array} \right\}$$

Then F_i is a holey parallel class for G_i for $i \in \mathbb{Z}_3$.

For $i \in \mathbb{Z}_3$ and $\sigma \in \mathbb{Z}_2$ let $\overline{\sigma} = 1 - \sigma$ and let

$$H_{i\sigma} = \left\{ \begin{array}{lll} \{(i+1,\sigma,\sigma),(i+2,\sigma,\overline{\sigma})\}, & \{(i,\overline{\sigma},\sigma),(i+1,\overline{\sigma},\sigma)\}, & \{(i,\overline{\sigma},\overline{\sigma}),(i+2,\overline{\sigma},\overline{\sigma})\}, \\ \{(i,\sigma,\sigma),(i,2+\overline{\sigma},\sigma)\}, & \{(i+1,\sigma,\overline{\sigma}),(i+1,2+\overline{\sigma},\sigma)\}, & \{(i+2,\overline{\sigma},\sigma),(i+2,2+\overline{\sigma},\sigma)\}, \\ \{(i,\sigma,\overline{\sigma}),(i,2+\overline{\sigma},\overline{\sigma})\}, & \{(i+1,\overline{\sigma},\overline{\sigma}),(i+1,2+\overline{\sigma},\overline{\sigma})\}, & \{(i+2,\sigma,\sigma),(i+2,2+\overline{\sigma},\overline{\sigma})\}. \end{array} \right\}$$

Then $\{H_{i\sigma} : i \in \mathbb{Z}_3\}$ contains three holey parallel classes for $G_{3+\sigma}$ for $\sigma \in \mathbb{Z}_2$. □

4. Using incomplete group divisible designs

Theorem 4.1. *Let m, k be integers. If $5 \le m \le k \le 3m$, $m \equiv k \pmod 2$, and $m \notin \{6, 10\}$, then there exist 3-GDDs of type $(3m+1)^3(3k+3)^2$ and $(3m+3)^3(3k+3)^2$.*

Proof. There exists a 5-GDD of type m^5 that has a parallel class P of blocks (this is equivalent to three idempotent MOLS of side m, see [4]). Let G_1, G_2, G_3, G_4, G_5 be its groups. Give weight 3 to all points in $G_1 \cup G_2 \cup G_3$. In each of $\frac{3m-k}{2}$ of the blocks of P give weight 3 to the two points of the block in G_4 or G_5; for the remaining $\frac{k-m}{2}$ of the blocks of P, give weight 9.

Add a set H of 15 or 9 points distributed $3, 3, 3, 3, 3$ or $1, 1, 1, 3, 3$ to the groups to obtain group types $(3m+3)^3(3k+3)^2$ and $(3m+1)^3(3k+3)^2$ respectively. Fill blocks not in the parallel class P with a 3-GDD of type $9^i 3^{5-i}$, $i = 0, 1$ or 2 from [2]. Fill blocks that are in the parallel class P with a 3-IGDD of type $(6:3)^5$ and a 3-IGDD of type $(6:3)^3(12:3)^2$, or a 3-IGDD of type $(4:1)^3(6:3)^2$ and a 3-IGDD of type $(4:1)^3(12:3)^2$, all having hole H. Fill H with a 3-GDD of type 3^5 or a 3-GDD of type $1^3 3^2$ respectively. □

Corollary 4.2. *There exist 3-GDDs with $g \equiv 1 \pmod 3$, $g \ge 16$, $g \notin \{19, 31\}$, and $u = 3g - 6$; and when $g^3u^2 \in \left\{18^3 42^2, 18^3 48^2, 22^3 42^2, 22^3 48^2, 22^3 54^2, 22^3 60^2, 30^3 84^2, 34^3 84^2, 34^3 96^2\right\}$.*

Proof. Apply the first statement of Theorem 4.1 with $(m, k) = (\frac{g-1}{3}, g-3)$ when $g \equiv 1 \pmod 3$, $g \ge 16$, $g \notin \{19, 31\}$ and $u = 3g - 6$. Apply the first statement with $m = 7$ and $k \in \{13, 15, 17, 19\}$ to treat the cases with $g = 22$; and with $m = 11$ and $k \in \{27, 31\}$ to treat the cases with $g = 34$. Apply the second statement with $m = 5$ and $k \in \{13, 15\}$ to handle the cases with $g = 18$, and with $(m, k) = (9, 27)$ to handle $30^3 84^2$. □

It remains to treat

$$g^3u^2 \in \left\{ \begin{array}{llllllll} 9^3 21^2, & 10^3 24^2, & 11^3 15^2, & 11^3 21^2, & 11^3 27^2, & 13^3 21^2, & 13^3 27^2, & 13^3 33^2, \\ 19^3 51^2, & 20^3 42^2, & 20^3 48^2, & 20^3 54^2, & 31^3 87^2, & 32^3 78^2, & 32^3 90^2. & \end{array} \right\}$$

Next we extend Theorem 4.1:

Theorem 4.3. *Let $m \ge 5$ be an integer with $m \notin \{6, 10, 14, 18, 22\}$. Let $k \equiv m \pmod 2$ be an integer, where $m \le k \le 3m$. Let α be an integer with $1 \le \alpha \le m$, and let a be an integer for which $\alpha \le a \le 3\alpha$ and $a \equiv \alpha \pmod 2$. Then a 3-IGDD of type $((3m+a : a)^3(3k+3\alpha : 3\alpha)^2$ exists. If in addition a 3-GDD of type $a^3(3\alpha)^2$ exists, then a 3-GDD of type $(3m+a)^3(3k+3\alpha)^2$ exists.*

Proof. There are 4 MOLS of order m [4] and hence there exists a 5-GDD of type m^5 with α disjoint parallel classes $\{P_i : 1 \le i \le \alpha\}$. Let G_1, G_2, G_3, G_4, G_5 be its groups. Give weight 3 to all the points in $G_1 \cup G_2 \cup G_3$. In each of G_4 and G_5 give weight 3 to $\frac{3m-k}{2}$ of the points and weight 9 to the remaining $\frac{k-m}{2}$ points. Now for $1 \le i \le \alpha$, let H_i contain three new points in each of G_4 and G_5; and either three or one new points in each of G_1, G_2, G_3 according to whether $i \le \frac{a-\alpha}{2}$ or not.

Fill blocks not in $\bigcup_{i=1}^{\alpha} P_\alpha$ using a 3-GDD of type $9^i 3^{5-i}$ with $i \in \{0, 1, 2\}$. For each parallel class P_i, fill each block with an IGDD of type $(6:3)^3(12:3)^2$, $(6:3)^4(12:3)^1$, or $(6:3)^5$, that has hole H_i; these are from Corollary 3.2 and Lemma 3.5. This produces the 3-IGDD of type $((3m+a : a)^3(3k+3\alpha : 3\alpha)^2$. If a 3-GDD of type $a^3(3\alpha)^2$ exists, use it to fill the hole. □

Corollary 4.4. *There exist 3-GDDs of types $19^3 51^2$, 20^3u^2 for $u \in \{42, 48, 54\}$, $31^3 87^2$, and 32^3u^2 for $u \in \{78, 90\}$.*

Proof. Theorem 4.3 handles $19^3 51^2$ using $(m, \alpha, a) = (5, 2, 4)$ and $k = 15$; 20^3u^2 for $u \in \{42, 48, 54\}$ using $(m, \alpha, a) = (5, 3, 5)$and $k \in \{11, 13, 15\}$; $31^3 87^2$ using $(m, \alpha, a) = (9, 2, 4)$ and $k = 27$; and 32^3u^2 for $u \in \{78, 90\}$ using $(m, \alpha, a) = (9, 3, 5)$ and $k \in \{23, 27\}$. □

It remains only to treat a few cases with $g \leq 13$. A variant of Theorem 4.3 uses a different weighting:

Theorem 4.5. *Let $m \geq 5$ be an integer with $m \notin \{6, 10, 14, 18, 22\}$. Let α be an integer with $1 \leq \alpha \leq m$, and let a be an integer for which $\alpha \leq a \leq 3\alpha$ and $a \equiv \alpha \pmod 2$. Then a 3-IGDD of type $((m+\alpha : \alpha)^3(3m+a : a)^2$ exists. If in addition a 3-GDD of type $\alpha^3 a^2$ exists, then a 3-GDD of type $((m+\alpha)^3(3m+a)^2$ exists.*

Proof. Let G_1, G_2, G_3, G_4, G_5 be the groups of a 5-GDD of type m^5 with α disjoint parallel classes $\{P_i : 1 \leq i \leq \alpha\}$. Give weight 1 to all points in $G_1 \cup G_2 \cup G_3$, and weight 3 to all points in $G_4 \cup G_5$. Fill blocks not in $\bigcup_{i=1}^{\alpha} P_\alpha$ using a 3-GDD of type $1^3 3^2$.

For $1 \leq i \leq \alpha$, let H_i contain three new points in each of G_1, G_2, G_3; and either three or one new points in each of G_4 and G_5 according to whether $i \leq \frac{a-\alpha}{2}$ or not. For $1 \leq i \leq \frac{a-\alpha}{2}$, fill each block of P_i with an 3-IGDD of type $(2:1)^3(6:3)^2$ that has hole H_i. For $\frac{a-\alpha}{2} < i \leq \alpha$, fill each block of P_i with an 3-IGDD of type $(2:1)^3(4:1)^2$ that has hole H_i. This produces the 3-IGDD of type $((m+\alpha : \alpha)^3(3k+a : a)^2$. □

Corollary 4.6. *There exist 3-GDDs of types $9^3 21^2$, $10^3 24^2$, $11^3 27^2$, $13^3 27^2$, and $13^3 33^2$.*

Proof. Apply Theorem 4.5 with $(m, \alpha, a) = (5, 4, 6)$ to handle $9^3 21^2$; $(m, \alpha, a) = (7, 3, 3)$ to handle $10^3 24^2$; $(m, \alpha, a) = (7, 4, 6)$ to handle $11^3 27^2$; $(m, \alpha, a) = (7, 6, 6)$ to handle $13^3 27^2$; and $(m, \alpha, a) = (9, 4, 6)$ to handle $13^3 33^2$. □

Theorem 4.7. *If there exist 3-GDDs of types $g^3 u^2$ and $a^3 b^2$ and a 3-IGDD of type $(g+a : a)^3(u+b : b)^2$, then for all $w \geq 3$ there also exists a 3-GDD of type $(wg+a)^3(wu+b)^2$.*

Proof. Let $\{\overline{G_i} : 1 \leq i \leq 5\}$ be groups of size g, g, g, u, u respectively and set $\overline{\mathscr{G}} = \bigcup_{i=1}^{5} \overline{G_i}$. Let $\{H_i : 1 \leq i \leq 5\}$ be groups of size a, a, a, b, b respectively and set $\mathscr{H} = \bigcup_{i=1}^{5} H_i$. If $\overline{x} \in \overline{\mathscr{G}}$, then we denote by x, the w-element set $x = \overline{x} \times \mathbb{Z}_w$ and set $G_i = \bigcup_{\overline{x} \in \overline{G_i}} x$. We construct the 3-GDD of type $(wg+a)^3(wu+b)^2$ on groups $\{(G_i \cup H_i) : 1 \leq i \leq 5\}$.

If $\overline{x}, \overline{y}, \overline{z} \in \overline{\mathscr{G}}$, let $P(x, y, z) = \Big\{\{\overline{x} \times \{i\}, \overline{y} \times \{i\}, \overline{z} \times \{i\}\} : i \in \mathbb{Z}_w\Big\}$; this is a parallel class of triples. Because $w \geq 3$ there is an idempotent latin square of side w; consequently a 3-GDD of type w^3 can be constructed with groups x, y, z that contains the parallel class $P(x, y, z)$. Let $D(x, y, z)$ be the triples in this GDD that are *not* in $P(x, y, z)$.

Let $\mathbb{A}$ be a 3-IGDD of type $(g+a : a)^3(u+b : b)^2$ on the groups $(\overline{G_i} \cup H_i)$, $i = 1, 2, 3, 4, 5$ and hole $\mathscr{H}$. Let $\mathbb{B}$ be a 3-GDD of type $g^3 u^2$ on the groups $\overline{G_i}$, $i = 1, 2, 3, 4, 5$.

To construct the 3-GDD of type $(wg+a)^3(wu+b)^2$ we take the triples in: $\{\{h, \overline{x} \times \{i\}, \overline{y} \times \{i\}\} : i \in \mathbb{Z}_w\}$ whenever $\Big\{h, \overline{x}, \overline{y}\Big\}$ is a triple in $\mathbb{A}$ intersecting the hole $\mathscr{H}$ in the point h; $P(x, y, z)$ whenever $\Big\{\overline{x}, \overline{y}, \overline{z}\Big\}$ is a triple in $\mathbb{A}$ disjoint from the hole $\mathscr{H}$; $D(x, y, z)$ whenever $\Big\{\overline{x}, \overline{y}, \overline{z}\Big\}$ is a triple in $\mathbb{B}$; and a 3-GDD of type $a^3 b^2$ on the hole $\mathscr{H}$. □

Corollary 4.8. *There exists a 3-GDD of type $13^3 21^2$.*

Proof. There exist 3-GDDs of types $4^3 6^2$ and $1^3 3^2$ and a 3-IGDD of type $(5:1)^3(9:3)^2$ from Lemma 3.7. Apply Theorem 4.7 with $w = 3$. □

Theorem 4.9. *If there exist 3-GDDs of types $g^3 u^2$ and $a^3 b^2$ and a 3-IGDD of type $(g+a : a)^3(u+b : b)^2$, then for all $w \geq 3$ there also exists a 3-GDD of type $(wg+(w-1)a)^3(wu+(w-1)b)^2$.*

AP	BS	CR	EV	FW	HM	IO	KU	gG	hL	io	jq	kD	lN	mJ	nT	pQ	rX
BU	CO	DV	ER	FQ	HT	JS	LM	gm	ho	iN	jI	kW	lK	nG	pX	qP	rA
AM	BW	CN	DX	GV	HQ	IT	KR	gU	hm	iE	jF	kn	lO	oP	pL	qS	rJ
AU	CM	DQ	GN	HR	JW	KO	LS	gV	hT	iq	jE	kP	lm	nX	oF	pB	rI
AS	BV	CW	EQ	IX	JN	gM	hD	iL	jO	kF	lT	mK	nP	oR	pG	qH	rU
AT	EP	HS	IU	KX	LO	aG	bC	cD	dB	eW	fJ	mQ	nF	oV	pM	qN	rR
AW	BX	EM	FN	GQ	IR	KP	aS	bH	cO	dU	eD	fp	mT	nL	oC	qJ	rV
AO	BM	DW	FR	GP	HX	JV	LU	aE	br	cq	dQ	eC	fT	mI	nN	oS	pK
CV	DT	FM	HU	IW	JR	LQ	ap	bP	cX	dN	eA	fO	mS	nK	oE	qB	rG
DN	ET	FU	GM	JP	KW	LX	aQ	bV	cm	dI	eB	fS	nH	oA	pO	qR	rC
AX	BQ	DO	FV	IS	KM	LW	ag	bE	cG	dP	eR	fC	hN	iU	jT	kH	lJ
AN	BT	CP	DS	EW	FO	GU	IV	JX	KQ	aL	bM	cl	dg	ei	fk	hR	jH
AR	BO	CU	DM	FS	GT	HV	IN	LP	ai	bk	cW	dJ	eX	fl	gQ	hE	jK
BP	CT	DR	EU	GS	HN	JO	KV	aX	bW	ci	dh	eQ	fA	gI	jM	kL	lF
DP	EN	FX	GO	HW	JQ	KT	LV	aI	bh	cA	dC	ej	fU	gB	iM	kR	lS
aB	bJ	ck	dp	eL	fD	gK	hI	iA	jm	lE	nC	oH	qG	rF			
ao	bL	cp	di	eI	fH	gA	hG	jB	kK	lC	mE	nJ	qF	rD			
an	bF	cE	dr	eK	fg	hA	iB	jG	kC	lH	mL	oJ	pI	qD			
aF	bB	cC	dm	eG	fL	gJ	hr	iK	jn	kI	lA	oD	pH	qE			
aD	bl	cF	dL	eE	fj	gn	hB	iC	kA	mG	oK	pJ	qI	rH			
al	bo	cJ	dH	eg	fI	hF	iD	jA	kG	mC	nB	pE	qK	rL			
am	bG	cL	dK	en	fE	gF	hH	iI	jJ	ko	lD	pA	qC	rB			
aC	bA	cB	dF	eH	fK	gD	hJ	im	jo	kp	lG	nI	qL	rE			
aJ	bI	cK	dE	eo	fG	gC	hq	iH	jL	kB	lr	mA	nD	pF			
aV	bQ	cM	dn	el	fW	gR	hP	iX	jp	kN	mO	oU	qT	rS			
aP	bT	co	dO	eM	fr	gq	hW	iS	jR	kQ	lX	mN	nU	pV			
ar	bg	cS	dX	eV	fh	iR	jP	kT	lM	mU	nW	oQ	pN	qO			
aU	bX	cR	dk	eO	fP	gW	hS	ir	jV	ln	mM	oN	pT	qQ			
aT	bR	cU	dS	eN	fi	go	hO	jQ	km	lV	nM	pW	qX	rP			
aM	bO	cQ	dT	eq	fo	gP	hV	in	jN	kS	lU	mX	pR	rW			
aN	bj	cT	do	eS	fQ	gX	hU	iV	kO	lq	mW	nR	pP	rM			
ak	bN	cV	dM	em	fR	gp	hX	iQ	jU	lP	nS	oO	qW	rT			
aR	bn	cN	dq	eP	fX	gT	hM	iO	jS	kU	lp	mV	oW	rQ			
aHO	ahK	ajW	aqA	bKS	bip	bmD	bqU	cIP	cgH	chn	cjr	dAV	dGR	djD	dlW		
eFT	eJU	ehp	ekr	fBN	fmF	fnV	fqM	gES	gLN	grO	hCQ	iFP	iGW	iJT	jCX		
kEX	kJM	kqV	lIQ	lLR	loB	mBR	mHP	nAQ	nEO	oGX	oIM	oLT	pCS	pDU	rKN		

Figure 1. A partition of $K_{6,6,6,12,12}$

Proof. Let $\{\overline{G_i} : 1 \leq i \leq 5\}$ be groups of size g, g, g, u, u respectively and set $\overline{\mathscr{G}} = \bigcup_{i=1}^{5} \overline{G_i}$. Let $\{\overline{H}_i : 1 \leq i \leq 5\}$ be groups of size a, a, a, b, b respectively and set $\overline{\mathscr{H}} = \bigcup_{i=1}^{5} \overline{H}_i$. If $\overline{x} \in \overline{\mathscr{G}}$, then we denote by x, the w-element set $x = \overline{x} \times \mathbb{Z}_w$ and set $G_i = \bigcup_{\overline{x} \in \overline{G_i}} x$. If $\overline{h} \in \overline{\mathscr{H}}$, then we denote by h, the $(w-1)$-element set $x = \overline{x} \times (\mathbb{Z}_w \setminus \{0\})$ and set $H_i = \bigcup_{\overline{h} \in \overline{H_i}} h$. We construct the 3-GDD of type $(wg + (w-1)a)^3(wu + (w-1)b)^2$ on groups $\{(G_i \cup H_i) : 1 \leq i \leq 5\}$. $P(x, y, z)$ and $D(x, y, z)$ are as in the proof of Theorem 4.7.

Let $\mathbb{A}$ be a 3-IGDD of type $(g + a : a)^3(u + b : b)^2$ on the groups $(\overline{G_i} \cup \overline{H_i})$, $i = 1, 2, 3, 4, 5$ and hole $\overline{\mathscr{H}}$. Let $\mathbb{B}$ be a 3-GDD of type g^3u^2 on the groups $\overline{G_i}$, $i = 1, 2, 3, 4, 5$.

To construct the 3-GDD of type $(wg+(w-1)a)^3(wu+(w-1)b)^2$ we take the triples in: $\{\{\overline{h} \times \{j\}, \overline{x} \times \{i\}, \overline{y} \times \{i + j \bmod w\}\} : i, j \in \mathbb{Z}_w, j \neq 0\}$ whenever $\left\{\overline{h}, \overline{x}, \overline{y}\right\}$ is a triple in $\mathbb{A}$ intersecting the hole $\overline{\mathscr{H}}$ in the point $\overline{h}$; $D(x,y,z)$ whenever $\left\{\overline{x},\overline{y},\overline{z}\right\}$ is a triple in $\mathbb{A}$ disjoint from the hole $\overline{\mathscr{H}}$; $P(x,y,z)$ whenever $\left\{\overline{x},\overline{y},\overline{z}\right\}$ is a triple in $\mathbb{B}$; and a 3-GDD of type $((w-1)a)^3((w-1)b)^2$ on the hole $\bigcup_{i=1}^{5} H_i$. □

Corollary 4.10. *There exists a* 3-GDD *of type* $11^3 15^2$.

Proof. There exist 3-GDDs of types $3^3 3^2$ and $1^3 3^2$ and a 3-IGDD of type $(4:1)^3(6:3)^2$. Apply Theorem 4.9 with $w=3$. □

Lemma 4.11. *There exists a* 3-GDD *of type* $11^3 21^2$.

Proof. Figure 1 provides a partition of $K_{6,6,6,12,12}$ that was found using a hill-climbing algorithm on points `a-r` and `A-X` with groups $G_1 =$ `a-f`, $G_2 =$ `g-l`, $G_3 =$ `m-r`, $G_4 =$ `A-L`, and $G_5 =$ `M-X` into five holey parallel classes of pairs for each of G_1, G_2, and G_3; nine holey parallel classes of pairs for each of G_4 and G_5; and 48 triples. To form a 3-IGDD of type $(11:5)^3(21:9)^2$, extend the holey parallel classes. Then fill the hole with a 3-GDD of type $5^3 9^2$. □

This completes the proof of the Main Theorem.

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