

# On the parameters of a class of narrow sense primitive BCH codes

Research Article

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**Abstract:** The last few decades have seen an increase in the determination of the parameters of the primitive BCH codes. Indeed, BCH codes are powerful in terms of encoding and decoding. They are applied in several fields such as: satellite communications, cryptography, compact disk drives etc, and have good structural properties. Nevertheless, the dimension and the minimum distance of those codes are not known in general. In this paper, we present a class of narrow sense primitive BCH codes of designed distance  $\delta_4 = (q-1)q^{m-1} - 1 - q^{\lfloor \frac{m+3}{2} \rfloor}$ . Also, we investigate their Bose distance and dimension.

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## 1. Introduction

In coding theory, cyclic codes are considered as an important class of codes. They include a special subclass, discovered by Bose and Ray-Chaudhuri in 1960 [12], and independently by Hocquenghem in 1959 [2], known as BCH codes. Indeed, BCH codes have a good error-correcting capability. In many cases, BCH codes are the best linear codes, but the exact dimension and minimum weight are considered unsolved[11].

Recent results on the determination of the parameters of narrow sense primitive BCH codes can be found in [1, 3–8, 10, 14].

Let  $\mathbb{F}_q$  be the finite field of  $q$  elements, such that  $q$  is a prime power. We recall that an  $[n, k, d]$  linear code  $C$  over  $\mathbb{F}_q$  is a  $k$ -dimensional subspace of  $\mathbb{F}_q^n$ , where  $d$  is its Hamming minimum distance, and  $n$  is its length. An  $[n, k]$  code  $C$  is cyclic if it is linear and if any cyclic shift of a codeword is also a

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codeword, i.e., whenever  $(c_0, c_1, \dots, c_{n-1})$  is in  $C$  then so is  $(c_{n-1}, c_0, \dots, c_{n-2})$ . More information about cyclic codes can be found in [16] page 121. We identify a vector  $(c_0, \dots, c_{n-1}) \in \mathbb{F}_q^n$  with the polynomial  $c_0 + c_1x + \dots + c_{n-1}x^{n-1}$  in the ring  $R_n = \frac{\mathbb{F}_q[X]}{\langle x^n - 1 \rangle}$ . Thus, we can view an  $[n, k]$  cyclic code as a principal ideal of the ring  $R_n$  (see [16]). If  $C$  is not trivial there exists a unique monic polynomial  $g$  that generates the code  $C$ ,  $g$  is called the (standard) generator polynomial of the code  $C$ , and  $g$  divides the polynomial  $x^n - 1$ . The polynomial  $h$  defined by  $h(x) = \frac{x^n - 1}{g(x)}$  is called the (parity) check polynomial and gives information on the dual of the cyclic code  $C$ . Let  $\delta$  be an integer in  $\{0, \dots, n - 1\}$ , and  $\alpha$  be a primitive  $n^{\text{th}}$  root of unity. For any integer  $i$  such that  $0 \leq i \leq n - 1$ , let  $M^{(i)}(x)$  denote the minimal polynomial of  $\alpha^i$ . A cyclic code  $C$  is said to be a BCH code of designed distance  $\delta$ , if for some integer  $b \geq 0$  its generator polynomial noted  $g_{(q,m,\delta)}(x)$  is given by :

$$g_{(q,m,\delta)}(x) = lcm\left(M^{(b)}(x), M^{(b+1)}(x), \dots, M^{(b+\delta-2)}(x)\right),$$

where  $lcm$  is the least common multiple of these minimal polynomials. Therefore,  $g_{(q,m,\delta)}$  is the lowest degree monic polynomial over  $\mathbb{F}_q$  having  $\alpha^b, \alpha^{b+1}, \dots, \alpha^{b+\delta-2}$  as zeros, and a word  $c$  is in the code if and only if  $c(\alpha^b) = c(\alpha^{b+1}) = \dots = c(\alpha^{b+\delta-2}) = 0$ . In the case  $b = 1$ , we obtain the so called narrow sense BCH codes. The BCH codes of length  $n = q^m - 1$  are called the primitive BCH codes. The largest designed distance is called the Bose distance and denoted by  $d_B$ . For more results on the Bose distance of BCH codes see [6]. Let  $\tilde{g}_{(q,m,\delta)}(x) = (x - 1)g_{(q,m,\delta)}(x)$ .

Throughout this paper, we adopt the following notation :  $n = q^m - 1$ ,  $\delta_4 = (q - 1)q^{m-1} - 1 - q^{\lfloor \frac{m+3}{2} \rfloor}$ , and  $C_{(q,m,\delta)}$  denotes the narrow sense primitive BCH code of designed distance  $\delta$  with generator polynomial  $g_{(q,m,\delta)}$ , and  $\tilde{C}_{(q,m,\delta)}$  denotes the primitive BCH code with generator polynomial  $\tilde{g}_{(q,m,\delta)}(x)$ . According to authors in [4], the code  $\tilde{C}_{(q,m,\delta)}$  is a primitive code of designed distance  $\delta + 1$ , and since  $dim(\tilde{C}_{(q,m,\delta)}) = n - deg(\tilde{C}_{(q,m,\delta)})$ , we have  $dim(\tilde{C}_{(q,m,\delta)}) = dim(C_{(q,m,\delta)}) - 1$ . Let  $A_i$  be the number of codewords with Hamming weight  $i$ , the polynomial  $1 + A_1z + A_2z^2 + \dots + A_nz^n$  is called the weight enumerator of the code  $C_{(q,m,\delta)}$ .

The set  $A_1, A_2, \dots, A_n$  is called the weight distribution of the code  $C_{(q,m,\delta)}$ . Inspired by the results of [4], we prove that  $\delta_4$  is the fourth largest coset leader, and we study the parameters of the code  $C_{(q,m,\delta_4)}$ .

According to [5], author determined the first largest coset leader denoted by  $\delta_1 = (q - 1)q^{m-1} - 1$  and examined the parameters of the code  $C_{(q,m,\delta_1)}$ . Later, authors in [4] determined the second and the

third largest coset leaders modulo  $n$ , denoted respectively  $\delta_2 = (q - 1)q^{m-1} - 1 - q^{\lfloor \frac{m-1}{2} \rfloor}$  and  $\delta_3 = (q - 1)q^{m-1} - 1 - q^{\lfloor \frac{m+1}{2} \rfloor}$  and studied the parameters of  $C_{(q,m,\delta_2)}$  and  $C_{(q,m,\delta_3)}$ . With their weight distributions. The remainder of this paper is organized as follows. In section 2, preliminaries and notations are introduced. Section 3 is devoted to the exploration of the parameters of the codes  $C_{(q,m,\delta_4)}$  and  $\tilde{C}_{(q,m,\delta_4)}$ . Section 4 concludes the paper.

## 2. Notation and basic concepts

In order to present the most recent results and apply them to investigate the dimension and the Bose distance of our class of narrow sense primitive BCH codes, we present some auxiliary results on cyclotomic cosets. More details on cyclotomic cosets and code constructions can be found in [9].

**Definition 2.1** ([16], page 122). *The  $q$ -coset modulo  $n$  containing an element  $t$  is defined by  $C_t = \{t, tq, tq^2, \dots, tq^{l_t-1}\}$ , where  $l_t$  is the smallest integer such that  $tq^{l_t} \equiv t \pmod{n}$ .*

The smallest integer in  $C_t$  is called the coset leader of  $C_t$ . We denote by  $\Gamma_{(q,m)}$  the set of all coset leaders modulo  $n$ .

In [14], the authors give a general formula for computing the dimension of narrow sense primitive BCH code.

**Proposition 2.2** ([14], Proposition 2.1). *The dimension of the code  $C_{(q,m,\delta)}$  is equal to  $1 + \sum_{\substack{r \geq \delta \\ r \in \Gamma_{(q,m)}}} |C_r|$ .*

Cyclotomic cosets can be used to determine the Bose distance of narrow sense primitive BCH codes. The following proposition gives the connection between  $d_B$  and coset leaders. For more details about the proof we refer the readers to [13] page 3.

**Proposition 2.3** ([13], Proposition 4, page 3). *The Bose distance of the code  $C_{(q,m,\delta)}$  is a coset leader of a  $q$ -cyclotomic coset modulo  $n$ . Furthermore if  $\delta$  is a coset leader, then  $d_B = \delta$ .*

Next, we present some lemmas, that we will need later.

**Lemma 2.4** ([1], Lemma 2.1). *Let  $a$  and  $b$  be two positive distinct integers less than or equal to  $n$ . Let  $\sum_{j=0}^{m-1} a_j q^j$  and  $\sum_{j=0}^{m-1} b_j q^j$  be the  $q$ -adic expansions of  $a$  and  $b$  respectively. Set*

$$k = \min \left\{ i \in \mathbb{N} : 0 \leq i \leq m-1, a_{m-1-i} \neq b_{m-1-i} \right\}.$$

*Then we have  $a > b$  if and only if  $a_{m-1-k} > b_{m-1-k}$ .*

**Lemma 2.5.** *For non-negative integers  $a$  and  $j$  with  $0 \leq a \leq n-1$  such that  $a = \sum_{i=0}^{m-1} a_i q^i$  and  $1 \leq j \leq m-1$  we have*

$$\left[ q^j a \right]_n = (a_{m-(j+1)}, a_{m-(j+2)}, a_{m-(j+3)}, \dots, a_{m-j+2}, a_{m-j+1}, a_{m-j}),$$

*where  $[c]_n = (c_{m-1}, c_{m-2}, \dots, c_1, c_0)$  if  $c = c_0 + c_1 q + \dots + c_{m-2} q^{m-2} + c_{m-1} q^{m-1} \pmod{n}$ .*

**Proof.** Since

$$a = a_0 + a_1 q + a_2 q^2 + \dots + a_{m-(j+2)} q^{m-(j+2)} + a_{m-(j+1)} q^{m-(j+1)} + a_{m-j} q^{m-j} + a_{m-j+1} q^{m-j+1} + a_{m-j+2} q^{m-j+2} + \dots + a_{m-1} q^{m-1},$$

for all  $1 \leq j \leq m-1$ ,

$$q^j a = a_0 q^j + a_1 q^{j+1} + a_2 q^{j+2} + \dots + a_{m-(j+2)} q^{m-2} + a_{m-(j+1)} q^{m-1} + a_{m-j} + a_{m-j+1} q + a_{m-j+2} q^2 + \dots + a_{m-1} q^{m-1+j} \pmod{n}.$$

Hence,  $\left[ q^j a \right]_n = (a_{m-(j+1)}, a_{m-(j+2)}, a_{m-(j+3)}, \dots, a_{m-j+2}, a_{m-j+1}, a_{m-j})$ . □

**Lemma 2.6** ([4], Lemma 5, Lemma 7, Lemma 12). *Let*

$$\delta_1 = (q-1)q^{m-1} - 1, \delta_2 = (q-1)q^{m-1} - 1 - q^{\lfloor \frac{m-1}{2} \rfloor} \text{ and } \delta_3 = (q-1)q^{m-1} - 1 - q^{\lfloor \frac{m+1}{2} \rfloor}.$$

*Then we have :*

1.  $\delta_1$  is the first largest  $q$ -cyclotomic coset modulo  $n$  and  $|C_{\delta_1}| = m$ .
2.  $\delta_2$  is the second largest  $q$ -cyclotomic coset modulo  $n$ . Furthermore

$$\begin{cases} |C_{\delta_2}| = m & \text{if } m \text{ is odd,} \\ |C_{\delta_2}| = \frac{m}{2} & \text{if } m \text{ is even.} \end{cases}$$

3. For  $m \geq 4$ ,  $\delta_3$  is the third largest  $q$ -cyclotomic coset modulo  $n$  and  $|C_{\delta_3}| = m$ .

Below we give an overview about the recent results on the parameters of the codes  $C_{(q,m,\delta_1)}$ ,  $C_{(q,m,\delta_2)}$  and  $C_{(q,m,\delta_3)}$ .

**Theorem 2.7** ([5], theorem 13, page 5325). *The two cyclic codes  $C_{(q,m,\delta_1)}$  and  $R_q(1,m)^*$  are identical, and have parameters  $[q^m - 1, m+1, (q-1)q^{m-1} - 1]$  where  $R_q(1,m)^*$  is the first order punctured generalized Reed-Muller code of length  $n$ .*

According to authors in [16], a linear code  $C$  of length  $n$  over  $\mathbb{F}_q$  and minimum distance at least  $d$  is called optimal, if it has  $B_q(n, d)$  codewords, where  $B_q(n, d)$  is the largest number of codewords in the code  $C$ . There are other perspectives on optimizing a code, for more information readers can refer to [16] page 53.

According to authors in [5], the code  $C_{(q,m,\delta_1)}$  is optimal. The main results about the parameters of the codes  $C_{(q,m,\delta_2)}$ ,  $\tilde{C}_{(q,m,\delta_2)}$ ,  $C_{(q,m,\delta_3)}$ , and  $\tilde{C}_{(q,m,\delta_3)}$  are given in [4], and according to them we have the following theorems :

**Theorem 2.8** ([4], theorem 8, page 243). *The code  $\tilde{C}_{(q,m,\delta_2)}$  has parameters  $[n, \tilde{k}, \tilde{d}]$ , where  $n = q^m - 1$ ,  $\delta_2 = (q-1)q^{m-1} - 1 - q^{\lfloor \frac{m-1}{2} \rfloor}$ ,  $\tilde{d} \geq \delta_2 + 1$  and*

$$\tilde{k} = \begin{cases} 2m & \text{for odd } m, \\ \frac{3m}{2} & \text{for even } m. \end{cases}$$

In particular,

a- For  $q = 2$  and  $m$  any integer,  $\tilde{d} = \delta_2 + 1$ .

b- For  $q$  an odd prime,  $\tilde{d} = \delta_2 + 1$ .

The weight distribution in the case a, b are given [4].

**Theorem 2.9** ([4], theorem 11, page 250). *Let  $m \geq 2$  be an integer. The code  $C_{(q,m,\delta_2)}$  has parameters  $[n, k, d]$ , where  $n = q^m - 1$ ,  $\delta_2 = (q-1)q^{m-1} - 1 - q^{\lfloor \frac{m-1}{2} \rfloor}$ ,  $d \geq \delta_2$  and*

$$k = \begin{cases} 2m + 1 & \text{for odd } m, \\ \frac{3m}{2} + 1 & \text{for even } m. \end{cases}$$

Furthermore,  $d = \delta_2$  if  $q$  is prime.

According to authors in [4], the codes  $C_{(q,m,\delta_2)}$  and  $\tilde{C}_{(q,m,\delta_2)}$  are sometimes optimal, and sometimes have the same parameters as the best known linear codes in the tables of the best known linear codes maintained by Markus Grassl at <http://www.codetables.de>.

**Theorem 2.10** ([4], theorem 13, page 251). *Let  $m \geq 4$ . The code  $\tilde{C}_{(q,m,\delta_3)}$  has parameters  $[n, \tilde{k}, \tilde{d}]$ , where  $n = q^m - 1$ ,  $\delta_3 = (q-1)q^{m-1} - 1 - q^{\lfloor \frac{m-1}{2} \rfloor}$ ,  $\tilde{d} \geq \delta_3 + 1$  and*

$$\tilde{k} = \begin{cases} 3m & \text{for odd } m, \\ \frac{5m}{2} & \text{for even } m. \end{cases}$$

In particular,

a- When  $q = 2$  for any integer  $m$ ,  $\tilde{d} = \delta_3 + 1$ .

b- If  $q$  is an odd prime, and  $m \geq 4$  is even then  $\tilde{d} = \delta_3 + 1$ .

c- If  $q$  is an odd prime, and  $m \geq 5$  is odd then  $\tilde{d} = \delta_3 + 1$ .

The weight distributions in the cases a, b, c and d are given in [4].

**Theorem 2.11** ([4], theorem 15, page 254). Let  $m \geq 4$ . The code  $C_{(q,m,\delta_3)}$  has parameters  $[n, k, d]$ , where  $n = q^m - 1$ ,  $\delta_3 = (q - 1)q^{m-1} - 1 - q^{\lfloor \frac{m+1}{2} \rfloor}$ ,  $d \geq \delta_3$  and

$$k = \begin{cases} 3m + 1 & \text{for odd } m, \\ \frac{5m}{2} + 1 & \text{for even } m. \end{cases}$$

According to authors in [4], the codes  $C_{(q,m,\delta_3)}$  and  $\tilde{C}_{(q,m,\delta_3)}$  are sometimes optimal, and sometimes have the same parameters as the best known linear codes in the tables of the best linear known codes maintained by Markus Grassl at <http://www.codetables.de>.

### 3. Parameters of the code $C_{(q,m,\delta_4)}$

In order to investigate the parameters of  $C_{(q,m,\delta_4)}$ , we need the following two lemmas:

**Lemma 3.1.**  $\delta_4$  is a coset leader in the  $q$ -cyclotomic coset  $C_{\delta_4}$  and  $C_{\delta_4}$  has cardinality  $m$ .

**Proof.** To prove the lemma, we need to distinguish two cases :

- o **Case 1**,  $m$  is odd. Then,

$$\delta_4 = (q - 1)q^{m-1} - 1 - q^{\frac{m+3}{2}} = n - q^{m-1} - q^{\frac{m+3}{2}} = n - q^{\frac{m+3}{2}} \left( 1 + q^{\frac{m-5}{2}} \right)$$

Now, let us determine  $C_{\delta_4}$

$$\begin{aligned} \delta_4 &= n - q^{\frac{m+3}{2}} \left( 1 + q^{\frac{m-5}{2}} \right), \pmod{n}. \\ \delta_4 q &= n - q q^{\frac{m+3}{2}} \left( 1 + q^{\frac{m-5}{2}} \right), \pmod{n}. \\ &= n - \left( 1 + q^{\frac{m+5}{2}} \right), \pmod{n}. \\ \delta_4 q^2 &= n - q \left( 1 + q^{\frac{m+5}{2}} \right), \pmod{n}. \\ &\vdots \\ \delta_4 q^{\frac{m-5}{2}} &= n - q^{\frac{m-7}{2}} \left( 1 + q^{\frac{m+5}{2}} \right), \pmod{n}. \\ \delta_4 q^{\frac{m-3}{2}} &= n - q^{\frac{m-5}{2}} \left( 1 + q^{\frac{m+5}{2}} \right), \pmod{n}. \\ &= n - \left( 1 + q^{\frac{m-5}{2}} \right), \pmod{n}. \\ \delta_4 q^{\frac{m-1}{2}} &= n - q \left( 1 + q^{\frac{m-5}{2}} \right), \pmod{n}. \\ &\vdots \\ \delta_4 q^{m-3} &= n - q^{\frac{m-3}{2}} \left( 1 + q^{\frac{m-5}{2}} \right), \pmod{n}. \\ \delta_4 q^{m-2} &= n - q^{\frac{m-1}{2}} \left( 1 + q^{\frac{m-5}{2}} \right), \pmod{n}. \\ \delta_4 q^{m-1} &= n - q^{\frac{m+1}{2}} \left( 1 + q^{\frac{m-5}{2}} \right), \pmod{n}. \end{aligned}$$

If  $i$  and  $j$  are two distinct integers in the set  $\left\{0, 1, \dots, \frac{m-7}{2}\right\}$ , then we have

$$n - q^i \left(1 + q^{\frac{m+5}{2}}\right) = (q-1)q^{m-1} + \dots + (q-2)q^i + \dots + (q-2)q^{i+\frac{m+5}{2}} + \dots + (q-1)q + (q-1)$$

$$\text{and } n - q^j \left(1 + q^{\frac{m+5}{2}}\right) = (q-1)q^{m-1} + \dots + (q-2)q^j + \dots + (q-2)q^{j+\frac{m+5}{2}} + \dots + (q-1)q + (q-1).$$

It is clear, by lemma [2.4], we have  $n - q^i \left(1 + q^{\frac{m+5}{2}}\right) \neq n - q^j \left(1 + q^{\frac{m+5}{2}}\right)$ .

By the same reasoning when  $i$  and  $j$  are two distinct integers in the set  $\left\{0, \dots, \frac{m+3}{2}\right\}$ , we have that  $n - q^i \left(1 + q^{\frac{m-5}{2}}\right) \neq n - q^j \left(1 + q^{\frac{m-5}{2}}\right)$ . Thus

$$C_{\delta_4} = \left\{n - q^i \left(1 + q^{\frac{m+5}{2}}\right) : i = 0, 1, \dots, \frac{m-7}{2}\right\} \cup \left\{n - q^i \left(1 + q^{\frac{m-5}{2}}\right) : i = 0, \dots, \frac{m+3}{2}\right\}.$$

It is clear that  $|C_{\delta_4}| = m$ .

Observe that  $\delta_4$  is the smallest integer in  $C_{\delta_4}$ . Indeed, in the set  $\left\{n - q^i \left(1 + q^{\frac{m-5}{2}}\right) : i = 0, \dots, \frac{m+3}{2}\right\}$ ,  $\delta_4$  is the smallest integer. Let's compare  $\delta_4$  with  $n - q^{\frac{m-7}{2}} \left(1 + q^{\frac{m+5}{2}}\right)$ . We have

$$\delta_4 - \left(n - q^{\frac{m-7}{2}} \left(1 + q^{\frac{m+5}{2}}\right)\right) = q^{\frac{m-7}{2}} - q^{\frac{m+3}{2}} < 0.$$

Thus  $\delta_4$  is a coset leader modulo  $n$ .

- o **Case 2**,  $m$  is even. Then,

$$\delta_4 = (q-1)q^{m-1} - 1 - q^{\frac{m+2}{2}} = n - q^{m-1} - q^{\frac{m+2}{2}} = n - q^{\frac{m+2}{2}} \left(1 + q^{\frac{m-4}{2}}\right).$$

Let's determine  $C_{\delta_4}$ .

$$\begin{aligned} \delta_4 &= n - q^{\frac{m+2}{2}} \left(1 + q^{\frac{m-4}{2}}\right), \pmod{n}. \\ \delta_4 q &= n - qq^{\frac{m+2}{2}} \left(1 + q^{\frac{m-4}{2}}\right), \pmod{n}. \\ &= n - \left(1 + q^{\frac{m+4}{2}}\right), \pmod{n}. \\ \delta_4 q^2 &= n - q \left(1 + q^{\frac{m+4}{2}}\right), \pmod{n}. \\ &\vdots \\ \delta_4 q^{\frac{m-8}{2}} &= n - q^{\frac{m-10}{2}} \left(1 + q^{\frac{m+4}{2}}\right), \pmod{n}. \\ \delta_4 q^{\frac{m-6}{2}} &= n - q^{\frac{m-8}{2}} \left(1 + q^{\frac{m+4}{2}}\right), \pmod{n}. \\ \delta_4 q^{\frac{m-4}{2}} &= n - q^{\frac{m-6}{2}} \left(1 + q^{\frac{m+4}{2}}\right), \pmod{n}. \\ \delta_4 q^{\frac{m-2}{2}} &= n - q^{\frac{m-4}{2}} \left(1 + q^{\frac{m+4}{2}}\right), \pmod{n}. \\ &= n - \left(1 + q^{\frac{m-4}{2}}\right), \pmod{n}. \\ \delta_4 q^{\frac{m}{2}} &= n - q \left(1 + q^{\frac{m-4}{2}}\right), \pmod{n}. \\ &\vdots \\ \delta_4 q^{m-1} &= n - q^{\frac{m}{2}} \left(1 + q^{\frac{m-4}{2}}\right), \pmod{n}. \end{aligned}$$

Thus,

$$C_{\delta_4} = \left\{ n - q^i \left( 1 + q^{\frac{m-4}{2}} \right) : i = 0, 1, \dots, \frac{m+2}{2} \right\} \cup \left\{ n - q^i \left( 1 + q^{\frac{m+4}{2}} \right) : i = 0, \dots, \frac{m-6}{2} \right\}.$$

It is clear that  $|C_{\delta_4}| = m$ .

Observe that  $\delta_4$  is the smallest integer in the set  $\left\{ n - q^i \left( 1 + q^{\frac{m-4}{2}} \right) : i = 0, \dots, \frac{m+2}{2} \right\}$ .

Let's compare  $\delta_4$  with  $n - q^{\frac{m-6}{2}} \left( 1 + q^{\frac{m+4}{2}} \right)$ . We have

$$\delta_4 - \left( n - q^{\frac{m-6}{2}} \left( 1 + q^{\frac{m+4}{2}} \right) \right) = q^{\frac{m-6}{2}} - q^{\frac{m+2}{2}} < 0.$$

Thus  $\delta_4$  is a coset leader modulo  $n$ . □

Now we verify that  $\delta_4$  is the fourth largest coset leader modulo  $n$ .

**Lemma 3.2.** For  $m \geq 11$ ,  $\delta_4$  is the fourth largest coset leader modulo  $n$ .

*Proof.* To prove the lemma, we need to distinguish two cases :

o **Case 1,**  $m$  is odd. Then,

$\delta_4 = (q-1)q^{m-1} - 1 - q^{\frac{m+3}{2}}$ . We verify that there is no coset leader between  $\delta_3$  and  $\delta_4$ . Set  $s = \frac{m-1}{2}$ . Then,

$$\begin{aligned} \delta_4 &= (q-1)q^{m-1} - 1 - q^{\frac{m+3}{2}} \\ &= q^m - 1 - q^{m-1} - q^{s+2} \\ &= (q-1)(q^{m-1} + q^{m-2} + \dots + q + 1) - q^{m-1} - q^{s+2} \\ &= (q-2)q^{m-1} + (q-1)q^{m-2} + \dots + (q-2)q^{s+2} + (q-1)q^{s+1} + \\ &\quad (q-1)q^s + (q-1)q^{s-1} + \dots + (q-1)q + (q-1). \end{aligned}$$

With the same argument we have

$$\begin{aligned} \delta_3 &= (q-1)q^{m-1} - 1 - q^{\frac{m+1}{2}} \\ &= q^m - 1 - q^{m-1} - q^{s+1} \\ &= (q-2)q^{m-1} + (q-1)q^{m-2} + \dots + (q-1)q^{s+2} + (q-2)q^{s+1} + \\ &\quad (q-1)q^s + (q-1)q^{s-1} + \dots + (q-1)q + (q-1). \end{aligned}$$

Hence

$$\begin{aligned} \delta_3 - \delta_4 &= (q-1)q^{s+2} + (q-2)q^{s+1} - (q-2)q^{s+2} - (q-1)q^{s+1} \\ &= q^{s+2} - q^{s+1} \\ &= q^{s+1}(q-1). \end{aligned}$$

Since  $q^{s+1}(q-1) - 1 = (q-2)q^{s+1} + (q-1)q^s + \dots + (q-1)$ , any  $i \in \{1, \dots, q^{s+1}(q-1) - 1\}$  can be written as  $i = i_{s+1}q^{s+1} + i_s q^s + \dots + i_1 q + i_0$ , where  $0 \leq i_{s+1} \leq q-2$  and  $0 \leq i_l \leq q-1$ ;  $l \in \{0, \dots, s\}$ . Let  $K_i := \delta_3 - i$  for all  $i \in \{1, \dots, q^{s+1}(q-1) - 1\}$ . We are able to give the  $q$ -adic expansion of  $K_i$  and analyze it. Indeed, we have

$$\begin{aligned} K_i &= (q-2)q^{m-1} + (q-1)q^{m-2} + \dots + (q-1)q^{s+2} + (q-2)q^{s+1} + (q-1)q^s + (q-1)q^{s-1} + \\ &\quad \dots + (q-1)q - i_{s+1}q^{s+1} - i_s q^s - i_{s-1}q^{s-1} - \dots - i_1 q - i_0. \end{aligned}$$

Therefore,

$$K_i = (q-2)q^{m-1} + (q-1)q^{m-2} + \dots + (q-1)q^{s+2} + (q-2-i_{s+1})q^{s+1} + (q-1-i_s)q^s + (q-1-i_{s-1})q^{s-1} + \dots + (q-1-i_1)q + (q-1-i_0).$$

Now we need to verify that  $K_i$  cannot be a coset leader. To this end, we consider two subcases as follows.

– *Case 1* :  $q = 2$ . In this case, we have :

$$K_i = 2^{m-2} + \dots + 2^{s+2} + (1-i_s)2^s + \dots + (1-i_1)2 + (1-i_0).$$

Where  $i_0 \in \{0, 1\}$ .

–  $i_0 = 1$ . Then  $\frac{K_i}{2}$  and  $K_i$  are in the same  $q$ -cyclotomic coset modulo  $n$ , and since

$$K_i > \frac{K_i}{2}, K_i \text{ cannot be a coset leader.}$$

–  $i_0 = 0$ . Then,

$$K_i = 2^{m-2} + 2^{m-3} + \dots + 2^{s+2} + (1-i_s)2^s + (1-i_{s-1})2^{s-1} + \dots + (1-i_1)2 + 1.$$

Since  $i \neq 0$ , one of the  $i'_l$ 's must be nonzero. Let  $l$  denote the largest one such that  $i_l = 1$ . Thus, we have :

$$\begin{aligned} K_i &= 0 \times 2^{m-1} + 2^{m-2} + 2^{m-3} + \dots + 2^{s+2} + 0 \times 2^{s+1} + 2^s + \dots + 2^{l+1} + 0 \times 2^l + (1-i_{l-1})2^{l-1} + \dots + (1-i_1)2 + 1. \\ 2^{m-1-l} K_i &= 0 \times 2^{m-l-2} + 2^{m-l-3} + 2^{m-l-4} + \dots + 2^{s+1} + \dots + 2^{s-(l-1)} + 2^{s-(l+1)} + \dots + 2 + 1 + 0 \times 2^{m-1} + (1-i_{l-1})2^{m-2} + \dots + (1-i_1)2^{m-l} + 2^{m-l-1}. \\ &= 0 \times 2^{m-1} + (1-i_{l-1})2^{m-2} + \dots + (1-i_1)2^{m-l} + 2^{m-l-1} + 0 \times 2^{m-l-2} + 2^{m-l-3} + 2^{m-l-4} + \dots + 2^{s+2} + 2^{s+1} + \dots + 2^{s-(l-1)} + 0 \times 2^{s-l} + 2^{s-(l+1)} + \dots + 2 + 1. \end{aligned}$$

Using lemma [2.4], we confirm that  $2^{m-1-l} K_i < K_i$ . Hence,  $K_i$  cannot be a coset leader. In all cases there is no coset leader between  $\delta_4$  and  $\delta_3$ .

– *Case 2* :  $q > 2$ . Then,

$$K_i = (q-2)q^{m-1} + (q-1)q^{m-2} + \dots + (q-1)q^{s+2} + (q-2-i_{s+1})q^{s+1} + (q-1-i_s)q^s + (q-1-i_{s-1})q^{s-1} + \dots + (q-1-i_1)q + (q-1-i_0).$$

– If  $i_{s+1} \geq 1$ , then we verify that  $q^{m-1-(s+1)} K_i < K_i$ .

By lemma [2.5] we have  $[q^{m-2-s} K_i]_n = ((q-2-i_{s+1}); (q-1-i_s); (q-1-i_{s-1}); \dots; (q-1-i_1); (q-1-i_0); 0; (q-2); (q-1); \dots; (q-1))$ .

Since  $1 \leq i_{s+1} \leq q-2$ ,  $q-2-i_{s+1} < q-2$ , and by lemma [2.4],  $K_i$  cannot be a coset leader.

– If  $i_l \geq 2$  for some  $l$  with  $l \in \{0, \dots, s\}$ , let  $K_i = (q-2)q^{m-1} + (q-1)q^{m-2} + \dots + (q-1)q^{s+2} + (q-2-i_{s+1})q^{s+1} + (q-1-i_s)q^s + (q-1-i_{s-1})q^{s-1} + \dots + (q-1-i_l)q^l + \dots + (q-1-i_1)q + (q-1-i_0)$ .

$[q^{m-1-l} K_i]_n = ((q-1-i_l); (q-1-i_{l-1}); (q-1-i_{l-2}); \dots; (q-1-i_1); (q-1-i_0); 0; (q-2); (q-1); \dots; (q-1); (q-2-i_{s+1}); (q-1-i_s); \dots; (q-1-i_{l-1}))$ .

Since  $i_l \geq 2$ ,  $q-1-i_l \leq q-3 < q-2$ , and by lemma [2.4],  $K_i$  cannot be a coset leader.



- We now assume that  $i_l \in \{0, 1\}$  for all  $0 \leq l \leq s-1$  and  $i_{s+1} = 0$ . Since  $i \geq 1$ , at least one of the  $i'_l$ 's must be 1. Let  $l$  denote the largest one such that  $i_l = 1$ . Then we have :

$$\begin{aligned} K_i &= (q-2)q^{m-1} + (q-1)q^{m-2} + \dots + (q-1)q^{s+2} + (q-2)q^{s+1} + \\ &\quad (q-1)q^s + (q-1)q^{s-1} + \dots + (q-1)q^{l+1} + (q-2)q^l + \\ &\quad (q-1-i_{l-1})q^{l-1} + \dots + (q-1-i_1)q + (q-1-i_0) \\ q^{m-1-l} K_i &= (q-2)q^{m-2-l} + (q-1)q^{m-3-l} + \dots + (q-1)q^{s+1-l} + (q-1)q^{s-l} + \\ &\quad (q-1)q^{s-(l+1)} + \dots + (q-1) + (q-2)q^{m-1} + (q-1-i_{l-1})q^{m-2} \\ &\quad + \dots + (q-1-i_1)q^{m-l} + (q-1-i_0)q^{m-1-l} \\ q^{m-1-l} K_i &= (q-2)q^{m-1} + (q-1-i_{l-1})q^{m-2} + \dots + (q-1-i_1)q^{m-l} + \\ &\quad (q-1-i_0)q^{m-1-l} + (q-2)q^{m-2-l} + (q-1)q^{m-3-l} + \dots + (q-1)q^s + \\ &\quad \dots + (q-1)q^{s-(l-1)} + (q-1)q^{s-l} + (q-1)q^{s-(l+1)} + \dots + (q-1). \end{aligned}$$

By lemma [2.4],  $K_i$  cannot be a coset leader.

- **Case 2**,  $m$  is even. Then,

$\delta_4 = (q-1)q^{m-1} - 1 - q^{\frac{m+2}{2}}$ . We verify that there is no coset leader between  $\delta_3$  and  $\delta_4$ .

Set  $s = \frac{m-2}{2}$ . Then,

$$\begin{aligned} \delta_4 &= (q-1)q^{m-1} - 1 - q^{s+2} \\ &= q^m - 1 - q^{m-1} - q^{s+2} \\ &= (q-1)(q^{m-1} + q^{m-2} + \dots + q + 1) - q^{m-1} - q^{s+2} \\ &= (q-2)q^{m-1} + (q-1)q^{m-2} + \dots + (q-2)q^{s+2} + (q-1)q^{s+1} + \\ &\quad (q-1)q^s + (q-1)q^{s-1} + \dots + (q-1)q + (q-1) \end{aligned}$$

With the same argument we have :

$$\begin{aligned} \delta_3 &= (q-1)q^{m-1} - 1 - q^{\frac{m}{2}} \\ &= q^m - 1 - q^{m-1} - q^{\frac{m}{2}} \\ &= (q-2)q^{m-1} + (q-1)q^{m-2} + \dots + (q-1)q^{\frac{m}{2}} + (q-2)q^{\frac{m-1}{2}} + \\ &\quad (q-1)q^{\frac{m-1}{2}} + (q-1)q^{\frac{m-3}{2}} + \dots + (q-1)q + (q-1) \end{aligned}$$

It is similar to the odd case that we follow the same steps for the remainder of the proof. □

Now we are able to investigate the parameters of the code  $C_{(q,m,\delta_4)}$  and the code  $\tilde{C}_{(q,m,\delta_4)}$  :

**Theorem 3.3.** Let  $m \geq 11$ . The code  $\tilde{C}_{(q,m,\delta_4)}$  has parameters  $[n, k, d]$ , where  $n = q^m - 1$ ,  $d \geq \delta_4 + 1$  and

$$k = \begin{cases} 4m & \text{if } m \text{ is odd,} \\ \frac{7m}{2} & \text{if } m \text{ is even.} \end{cases}$$

**Proof.** For the value of the dimension we apply the proposition [2.2]. Finally the BCH bound ensures the bound on the minimum distance. □

**Corollary 3.4.** For an odd integer  $m$ ,  $m \geq 11$ , the code  $\tilde{C}_{(2,m,\delta_4)}$  has parameters  $[n, k, d]$ , where  $n = 2^m - 1$ ,  $k = 4m$  and  $d = 2^{m-1} - 2^{\frac{m+3}{2}}$ .

Weight $w$	Number of words of weight $w$
0	1
$2^{m-1} + 2^{\frac{m-1}{2}}$	$(2^{m-1} - 2^{\frac{m-1}{2}}) (151 \times 2^{2m-3} + 25 \times 2^m + 2^5) \frac{2^{m-1}}{45}$
$2^{m-1} - 2^{\frac{m-1}{2}}$	$(2^{m-1} + 2^{\frac{m-1}{2}}) (151 \times 2^{2m-3} + 25 \times 2^m + 2^5) (\frac{2^{m-1}}{45})$
$2^{m-1} + 2^{\frac{m+1}{2}}$	$(2^{m-2} - 2^{\frac{m-1}{2}}) (23 \times 2^{m-5} + 1) (2^{m-1} - 1) (\frac{2^m - 1}{9})$
$2^{m-1} - 2^{\frac{m+1}{2}}$	$(2^{m-2} + 2^{\frac{m-1}{2}}) (23 \times 2^{m-5} + 1) (2^{m-1} - 1) (\frac{2^m - 1}{9})$
$2^{m-1} + 2^{\frac{m+3}{2}}$	$(2^{m-6} - 2^{\frac{m-7}{2}}) (2^{m-3} - 1) (\frac{2^m - 1}{45})$
$2^{m-1} - 2^{\frac{m+3}{2}}$	$(2^{m-6} + 2^{\frac{m-7}{2}}) (2^{m-3} - 1) (\frac{2^m - 1}{45})$
$2^{m-1}$	$2^{4m} - 1 - \sum_{j \neq 0, 2}^{m-1} A_j$

**Table 1.** Weight distribution of the code  $\tilde{C}_{(2,m,\delta_4)}$

**Proof.** Since  $m$  is odd, and  $m \geq 11$ . Then the dimension of the code  $\tilde{C}_{(2,m,\delta_4)}$  is  $4m$ . According to Kassami in [15] theorem 16 page 24, the code  $\tilde{C}_{(2,m,\delta_4)}$  is the same as the code defined under [15] lemma 9 page 15, which has the weight distribution shown in table 1. And since there is a codeword with weight  $\delta_4$ . Then, the minimum distance of the code  $\tilde{C}_{(2,m,\delta_4)}$  is exactly  $\delta_4$ .  $\square$

According to author in [15] page 24. The dual code of the code  $\tilde{C}_{(2,m,\delta_4)}$  is a subcode of the dual code of the code  $\tilde{C}_{(2,m,\delta_4)}$ , which has minimum distance 7.

**Example 3.5.** Let  $(q, m) = (2, 11)$ . Then  $\delta_4 = 895$ , and  $\tilde{C}_{(q,m,\delta_4)}$  has parameters  $[2047, 44, 896]$ , with the weight enumerator  $1 + 3574742979584z^{1056} + 3805371558912z^{992} + 164511003360z^{1088} + 186445803808z^{960} + 332260852z^{1152} + 427192524z^{896} + 9860355245375z^{1024}$ .

**Theorem 3.6.** Let  $m \geq 11$ . The code  $C_{(q,m,\delta_4)}$  has parameters  $[n, k, d]$ , where  $n = q^m - 1$ ,  $d \geq d_B$  such that  $d_B = \delta_4$ , and

$$k = \begin{cases} 4m + 1 & \text{if } m \text{ is odd,} \\ \frac{7m}{2} + 1 & \text{if } m \text{ is even.} \end{cases}$$

**Proof.** For the value of the dimension we apply the proposition [2.2], and by proposition [2.3] we find that  $d_B = \delta_4$  since  $\delta_4 \in \Gamma_{(q,m)}$ . Finally BCH bound ensures the bound on the minimum distance.  $\square$

## Conclusion and further works

By this work we initiate our first reaserch in studying the parameters of narrow sense primitive BCH codes. Our idea in this work was inspired from the ideas proposed by authors in [4]. Thus we give a similar demonstration as the one proposed in [4] to find the fourth largest coset leader modulo  $q^m - 1$ , and then investigate the parameters of the code  $C_{(q,m;\delta_4)}$  and the code  $\tilde{C}_{(q,m,\delta_4)}$ . The investigation of the weight distribution of the code  $\tilde{C}_{(2,m,\delta_4)}$  for odd  $m \geq 11$  was presented by Tadao Kassami in [15]. In

a future work we plan to study the weight distributions of the codes  $C_{(q,m,\delta_4)}$  and  $\tilde{C}_{(q,m,\delta_4)}$  by adopting the theory of quadratic forms over the finite field and the theory of association schemes. We also plan to attack some open problems proposed by Cunsheng Ding in [4] about the weight distribution of the extended codes  $\tilde{C}_{(q,m,\delta_2)}$  and  $\tilde{C}_{(q,m,\delta_3)}$ .

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