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On the isomorphism of unitary subgroups of noncommutative group algebras*

Research Article

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Abstract: Let FG be the group algebra of a finite p-group G over a field F of characteristic p. Let $\circledast$ be an involution of the group algebra FG which arises form the group basis G. The upper bound for the number of non-isomorphic $\circledast$-unitary subgroups is the number of conjugacy classes of the automorphism group G with all the elements of order two. The upper bound is not always reached in the case when G is an abelian group, but for non-abelian case the question is open. In this paper we present a non-abelian p-group G whose group algebra FG has sharply less number of non-isomorphic $\circledast$-unitary subgroups than the given upper bound.

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1. Introduction

Let FG be the group algebra of the group G over a field F. Let $\circledast$ be an involution of the group algebra FG. We say that the algebra involution $\circledast$ arises from the group G when $\circledast$ is an antiautomorphism on G. This antiautomorphism of G may also be called involution (for more details see in [19]). In this case the algebra involution $\circledast$ is the linear extension of the group involution $\circledast$ defined on G. A group algebra is always an algebra with involution, because the canonical $*$-involution of FG (the linear extension of the involution on G which sends each element of G to its inverse) exists for every F and G. The canonical involution $*$ on FG is a simple example of an algebra involution that arises from the group basis G.

Let $V(FG)$ denote the normalized unit group of FG, that is, the subgroup of the unit group of FG containing all units with augmentation 1. An element $u \in V(FG)$ is called $\circledast$-unitary if $u^{-1} = u^{\circledast}$. The set of all $\circledast$-unitary units of FG forms a subgroup of $V(FG)$, which is called $\circledast$-unitary subgroup and is denoted by $V_{\circledast}(FG)$. Interest in the unitary subgroups arose in algebraic topology and unitary K-theory

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introduced by Novikov [20]. The $*$-unitary subgroup is an actively investigated subgroup and it plays an important role of studying the structure of $V(FG)$ for more details we refere the reader to Bovdi's paper [9]).

Let L be a finite Galois extension of F with Galois group G, where F is a finite field of characteristic two. A relation between the self-dual normal basis of L over F and the $*$-unitary subgroup of FG was discovered by Serre [21]. It was shown in [2] that the $*$-unitary subgroup of a group algebra determines the group basis G when it is a finite abelian p-group and F is a finite field of characteristic p. The structure of the unitary subgroups was studied in several papers (see [3], [4], [5], [12], [14], [15], [16], [17] and [22]).

Let F be a field of characteristic p and G a nonabelian locally finite p-group. The groups G when $V_*(FG)$ is normal in $V(FG)$ are listed in [12]. Bovdi and Szakács [10] described the structure of the group $V_*(FG)$ when G is a finite abelian p-group and F is a finite field of characteristic p. They also constructed a basis for $V_*(FG)$ in [11].

The order of the unitary subgroup $V_*(FG)$ is determined for finite p-groups and finite fields of characteristic p, if p is an odd prime (see in [13]). The order of $V_*(FG)$ when $p = 2$ is an open question. It was determined only for some group classes (see in [1], [8] and [13]). The structure of $V_*(F_2G)$, where G is a 2-group of maximal class of order 8 or 16 and F_2 is the field of two elements has been established in [6]. Additionally, the structures of $V_*(FQ_8)$ and $V_*(FD_8)$ are established in [16] and [18] respectively, where F is a finite field of characteristic 2, Q_8 is the quaternion group of order 8 and D_8 is the dihedral group of order 8.

In the case when f is a homomorphism of G into the multiplicative group of the commutative ring K all the groups G whose f-unitary subgroup coincides with the unit group of KG are established in [9]. In [8] the invariants of the $\circledast$-unitary subgroup of FG are presented, when G is a finite abelian p-group, F is a field of p elements (p is an odd prime) and $\circledast$ is an involutory automorphism of G. In [3] an upper bound for the non-isomorphic $\circledast$-unitary subgroups is given, when $\circledast$ arises from G. The upper bound coincides the number of conjugacy classes of the automorphism group G with all the elements of order two including the identity map. In the case, when G is an abelian p-group the upper bound is not always sharp. A counterexample can be found in [4]. For non-abelian groups this question is open. In this paper we gave an example for a non-abelian p-group whose group algebra FG has less non-isomorphic $\circledast$-unitary subgroups than the given upper bound.

2. Involutions and unitary subgroups

Let F be a finite field and G is either the dihedral group of order 8 or the quaternion group of order 8. In this section we show that the number of non-isomorphic $\circledast$-unitary subgroups of FG with respect to the involutions which arise from G is equals to the upper bound mentioned in the introduction.

Let $Aut\ G\{2\}$ be the set of all automorphism of G with the identity map. The composition of two antiautomorphisms $\circledast$ and $*$ of the group G is an automorphism of order two. Therefore, $\circledast$ can be considered as a composition of an automorphism of order two and the canonical involution, that is, $\circledast = \phi \circ *$, where $\phi \in Aut\ G\{2\}$. We say that the involutions $\circledast_1 = \phi_1 \circ *$ and $\circledast_2 = \phi_2 \circ *$ are similar if ϕ_1 is conjugate to ϕ_2 in $Aut\ G$. We need the following lemma.

Lemma 2.1. *[3, Proposition 7] Let G be a group and F a field and let $\circledast_1$ and $\circledast_2$ be involutions of FG which arise from G. If $\circledast_1$ is similar to $\circledast_2$, then*

$$V_{\circledast_1}(FG) \cong V_{\circledast_2}(FG).$$

Let G be a finite group and let Λ_2 denote the number of all distinct conjugacy classes of $Aut\ G\{2\}$. As a consequence of the previous lemma we have the following corollary.

Corollary 2.2. *[3, Corollary 8] Let G be a finite group and F a field. The number of non-isomorphic unitary subgroups of $V(FG)$ with respect to the involutions which arise from G is at most Λ_2.*

In this section we show that the upper bound Λ_2 is sharp for all the non-abelian groups of order 8. Moreover, we establish the structure of all non-isomorphic $\circledast$-unitary subgroups for these groups.

First, let us consider the dihedral group D_8 of order 8. It is well known that $D_8 \cong Aut\ D_8$ and $Aut\ G\{2\}$ is the union of four distinct conjugacy classes, that is, $\Lambda_2 = 4$. Throughout this section we will use Lemma 2.4 in [1] free.

Lemma 2.3. *The number of non-isomorphic unitary subgroups of FD_8 with respect to the involutions which arise from D_8 is equals to Λ_2, where $|F| = 2^n \geq 2$.*

Proof. It was shown in [18] that $V_*(FD_8) \cong {C_2}^{5n} \rtimes {C_2}^n$.

According to Lemma 2.1 it is enough to establish the structure of $V_\circledast(FD_8)$ when the involution $\circledast$ links to different conjugacy classes in $Aut\ G\{2\}$. Then $C_{\sigma_1} = \{\sigma_1\}$, C_{σ_2}, C_{σ_3} and C_{σ_4} are the distinct conjugacy classes of $Aut\ G\{2\}$, where σ_1 is the identity map and

$$\sigma_2 : \begin{cases} a \mapsto a^3 \\ b \mapsto ab \end{cases} \qquad \sigma_3 : \begin{cases} a \mapsto a^3 \\ b \mapsto a^2b \end{cases} \qquad \sigma_4 : \begin{cases} a \mapsto a \\ b \mapsto a^2b \end{cases}.$$

Case σ_2. Let $\alpha = \sum_{i=0}^{3} a^i(\alpha_i + \beta_i b) \in FD_8$ where $\alpha_i, \beta_j \in F$. Then α is $\circledast$-unitary if and only if $\alpha\alpha^\circledast = 1$. A straightforward computation shows that $\alpha\alpha^\circledast$ equals to

$$\begin{aligned}(\alpha_0 + \alpha_2)^2 + &(\beta_1 + \beta_3)^2 a + (\alpha_1 + \alpha_3)^2 a^2 + \\ &(\beta_0 + \beta_2)^2 a^3 + \delta_1(1 + a)b + \delta_2(a^2 + a^3)b,\end{aligned}$$

where $\delta_1 = \alpha_0(\beta_0 + \beta_1) + \alpha_1(\beta_1 + \beta_2) + \alpha_2(\beta_2 + \beta_3) + \alpha_3(\beta_0 + \beta_3)$ and $\delta_2 = \alpha_0(\beta_2 + \beta_3) + \alpha_1(\beta_3 + \beta_0) + \alpha_2(\beta_0 + \beta_1) + \alpha_3(\beta_1 + \beta_2)$.

Clearly $\alpha\alpha^\circledast = 1$ if and only if $\alpha_0 + \alpha_2 = 1$, $\beta_0 = \beta_2$, $\alpha_1 = \alpha_3$ and $\beta_1 = \beta_3$. Therefore $\delta_1 = \delta_2 = \beta_0 + \beta_1 = 0$, that is, $\beta_0 = \beta_1$ and every $\circledast$-unitary element can be written as

$$\alpha_0 + \alpha_1 a + (1 + \alpha_0)a^2 + \alpha_1 a^3 + \beta_0 b + \beta_0 ab + \beta_0 a^2 b + \beta_0 a^3 b.$$

Therefore $V_\circledast(FD_8) \cong {C_2}^{3n}$.

Case σ_3. Let $\alpha = \sum_{i=0}^{3} a^i(\alpha_i + \beta_i b) \in FD_8$, where $\alpha_i, \beta_j \in F$. Then

$$\alpha^\circledast\alpha = (\alpha_0 + \alpha_2 + \beta_1 + \beta_3)^2 + (\alpha_1 + \alpha_3 + \beta_0 + \beta_2)^2 a^2 + \delta(1 + a^2)b,$$

where $\delta = (\alpha_0 + \alpha_2)(\beta_0 + \beta_2) + (\alpha_1 + \alpha_3)(\beta_1 + \beta_3)$. Clearly $\alpha^\circledast\alpha = 1$ if and only if $\alpha_0 + \alpha_2 + \beta_1 + \beta_3 = 1$, $\beta_0 = \beta_2$ and $\alpha_1 = \alpha_3$. Therefore every element of $V_\circledast(FD_8)$ is central or it can be written in the form either $ab + x_1$ or $a^3b + x_2$, where $x_1, x_2 \in \zeta(V(FD_8))$. Since the exponent of $\zeta(V(FD_8))$ is two we have proved that $V_\circledast(FD_8) \cong {C_2}^{5n}$.

Case σ_4. Let $\alpha = \sum_{i=0}^{3} a^i(\alpha_i + \beta_i b) \in FD_8$, where $\alpha_i, \beta_j \in F$. Then

$$\begin{aligned}\alpha\alpha^\circledast &= (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3)^2 + (\delta_0 + \delta_1)(a + a^3) + (\beta_0 + \beta_1 + \beta_2 + \beta_3)^2 a^2 \\ &\quad + (\delta_2 + \delta_3)(1 + a^2)b + (\delta_4 + \delta_5)(1 + a^2)ab,\end{aligned}$$

where

$$\begin{aligned}\delta_0 &= (\alpha_0 + \alpha_2)(\alpha_1 + \alpha_3), \quad \delta_1 = (\beta_0 + \beta_2)(\beta_1 + \beta_3), \\ \delta_2 &= (\alpha_0 + \alpha_2)(\beta_0 + \beta_2), \quad \delta_3 = (\alpha_1 + \alpha_3)(\beta_1 + \beta_3), \\ \delta_4 &= (\alpha_0 + \alpha_2)(\beta_1 + \beta_3), \quad \delta_5 = (\alpha_1 + \alpha_3)(\beta_0 + \beta_2).\end{aligned}$$

Therefore, $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 1$, $\beta_0 + \beta_1 + \beta_2 + \beta_3 = 0$, $\delta_0 + \delta_1 = 0$, $\delta_2 + \delta_3 = 0$ and $\delta_4 + \delta_5 = 0$. Since $\beta_1 + \beta_3 = \beta_0 + \beta_2$ we conclude that $\delta_4 = \delta_2$ and $\delta_5 = \delta_3$. Moreover, $0 = \delta_2 + \delta_3 = (\alpha_0 + \alpha_1 + \alpha_2 +$

$\alpha_3)(\beta_0 + \beta_2) = \beta_0 + \beta_2$ and we have that $\beta_0 = \beta_2$, $\beta_1 = \beta_3$, $\delta_1 = 0$ and $\delta_0 = 0$. Thus, every $\circledast$-unitary element can be written as either

$$a^3 + \alpha_0\widehat{C} + \alpha_1\widehat{C}a + \beta_0\widehat{C}b + \beta_1\widehat{C}ab, \quad \text{if } \alpha_2 = \alpha_0,$$

or

$$a^2 + \alpha_0\widehat{C} + \alpha_1\widehat{C}a + \beta_0\widehat{C}b + \beta_1\widehat{C}ab, \quad \text{if } \alpha_2 = 1 + \alpha_0.$$

Let us denote by N the central elementary abelian subgroup $\langle\, 1+\alpha_0\widehat{C}+\alpha_1\widehat{C}a+\beta_0\widehat{C}b+\beta_1\widehat{C}ab \mid \alpha_i, \beta_i \in F \,\rangle$. Evidently, $a^2 \in N$. Since a, a^3 belong to the $\circledast$-unitary subgroup we have proved that $V_\circledast(FD_8) \cong C_4 \times {C_2}^{4n-1}$. ∎

It is well-known that $Aut\ Q_8 \cong S_4$, where S_4 is the symmetric group of order 24. It follows that $\Lambda_2 = 3$.

Lemma 2.4. *The number of non-isomorphic unitary subgroups of FQ_8 with respect to the involutions which arise from D_8 equals Λ_2, where $|F| = 2^n \geq 2$.*

Proof. Let σ_1 be the identity automorphism of Q_8. A straightforward computation shows that $Aut\ G\{2\} = C_{\sigma_1} \cup C_{\sigma_2} \cup C_{\sigma_3}$, where

$$\sigma_2 : \begin{cases} a \mapsto b \\ b \mapsto a \end{cases} \qquad \sigma_3 : \begin{cases} a \mapsto a^3 \\ b \mapsto b \end{cases}.$$

It was shown in [16] that $V_\circledast(FQ_8) \cong Q_8 \times C_2^{4n-1}$. Let us consider the following two cases.

Case $i = 2$. Let $\alpha = \sum_{i=0}^{3} a^i(\alpha_i + \beta_i b) \in FQ_8$, where $\alpha_i, \beta_j \in F$. Then

$$\begin{aligned}\alpha^\circledast\alpha = {} & (\alpha_0+\alpha_2)^2 + \delta_1 a + (\beta_0+\beta_2)^2a^2 + \delta_2 a^3 + (\beta_1+\beta_3)^2 b + \delta_1 ab + \\ & (\alpha_1+\alpha_3)^2a^2b + \delta_2 a^3 b,\end{aligned}$$

where

$$\begin{aligned}\delta_1 &= \alpha_0(\alpha_1+\beta_1) + \alpha_2(\alpha_3+\beta_3) + \beta_0(\alpha_1+\beta_3) + \beta_2(\alpha_3+\beta_1),\\ \delta_2 &= \alpha_0(\alpha_3+\beta_3) + \alpha_2(\alpha_1+\beta_1) + \beta_0(\alpha_3+\beta_1) + \beta_2(\alpha_1+\beta_3).\end{aligned}$$

Evidently, $\alpha^\circledast\alpha = 1$ if and only if $\alpha_0 + \alpha_2 = 1$, $\beta_0 = \beta_2$, $\alpha_1 = \alpha_3$ and $\beta_1 = \beta_3$. They imply that $\delta_1 = \delta_2 = \alpha_1 + \beta_1 = 0$, and so $\alpha_1 = \beta_1$.

Therefore every $\circledast$-unitary element can be written as

$$a^2 + \alpha_0\widehat{C}a^2 + \alpha_1\widehat{C}a + \beta_0\widehat{C}b + \alpha_1\widehat{C}ab.$$

Thus $V_\circledast(FQ_8) \cong {C_2}^{3n}$.

Case $i = 3$. Let $\alpha = \sum_{i=0}^{3} a^i(\alpha_i + \beta_i b) \in FQ_8$, where $\alpha_i, \beta_j \in F$. Then

$$\alpha\alpha^\circledast = (\alpha_0+\alpha_2+\beta_0+\beta_2)^2 + (\alpha_1+\alpha_3+\beta_1+\beta_3)^2a^2 + \delta(1+a^2)b,$$

where $\delta = (\alpha_0+\alpha_2)(\beta_0+\beta_2) + (\alpha_1+\alpha_3)(\beta_1+\beta_3)$. Let $S_\circledast = \{\alpha\alpha^\circledast \mid \alpha \in V(FQ_8)\}$. Clearly, $S_\circledast$ is a subgroup of $\zeta(V(FQ_8))$, therefore $\psi : V(FQ_8) \to S_\circledast$ (given by $x \mapsto xx^\circledast$) is a homomorphism with kernel $V_\circledast(FQ_8)$. Thus

$$|V_\circledast(FQ_8)| = \frac{|V(FQ_8)|}{|S_\circledast|} = \frac{2^{7n}}{2^{2n}} = 2^{5n}.$$

Let $n = 1$ and $G_{\circledast} = \{g \in G \,|\, g^{\circledast} = g^{-1}\}$. It is easy to see that $G_{\circledast} = \langle\ b\ \rangle$ and $V_{\circledast}(FQ_8)$ is a subgroup of $G_{\circledast} \cdot N$, where N is an elementary abelian group. Since $G_{\circledast} \cong C_4$, we get that $V_{\circledast}(FQ_8) \cong C_4 \times C_2^3$.

Suppose that $n > 1$ and let ω_1 and ω_2 be elements of the unit group of F satisfying that $\omega_1 \neq 1$ and $\omega_1 + \omega_2 = 1$. It is easy to see that b and $\omega_1 + a + \omega_2 b + ab$ are elements of $V_{\circledast}(FQ_8)$, but they are not commute. Therefore $V_{\circledast}(FQ_8)$ is not an abelian group.

According to Theorem 2 in [7], the exponent of $V(FQ_8)$ is 4. Since b is a $\circledast$-unitary element with exponent 4 it follows that the exponent of $V_{\circledast}(FQ_8)$ is 4. Since $|\zeta(V(FQ_8))| = 2^{4n}$ and $x^2 \in \zeta(V(FQ_8))$ for all $x \in V(FQ_8)$ we have proved that

$$V_{\circledast}(FQ_8)/\zeta(V(FQ_8)) \cong C_2^n.$$

Therefore $V_{\circledast}(FQ_8)$ is a central extension of C_2^n by ${C_2}^{4n}$. □

3. Isomorphic unitary subgroups of noncommutative group algebra with non similar involutions

In this section we present a non-abelian group whose group algebra has sharply less number of non-isomorphic $\circledast$-unitary subgroups than the given upper bound given in Corollary 2.2.

Let $H_{16} = \langle\ a, c \mid a^4 = b^2 = c^2 = 1,\ (a, b) = 1,\ (a, c) = b,\ (b, c) = 1\ \rangle$ be and let F be a finite field with $|F| = 2^n$. The automorphism group of H_{16} is isomorphic to the following group

$$\begin{aligned}\langle\ \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_4^2 = \sigma_5^2 = 1,\ (\sigma_1, \sigma_2) = \sigma_4,\ (\sigma_1, \sigma_3) = 1,\\ (\sigma_2, \sigma_3) = \sigma_5\ \rangle,\end{aligned}$$

where

$$\sigma_1 =: \begin{cases} a \mapsto a \\ b \mapsto a^2 b \\ c \mapsto c \end{cases} \quad \sigma_2 : \begin{cases} a \mapsto a \\ b \mapsto bc \\ c \mapsto c \end{cases} \quad \sigma_3 : \begin{cases} a \mapsto ab \\ b \mapsto b \\ c \mapsto c. \end{cases}$$

Let us consider the following two automorphisms of order two in $Aut\ H_{16}$

$$\tau_1 = \sigma_1\sigma_2\sigma_5 : \begin{cases} a \mapsto ac \\ b \mapsto a^2 bc \\ c \mapsto c \end{cases} \quad \text{and} \quad \tau_2 = (\sigma_1, \sigma_2) : \begin{cases} a \mapsto a^3 c \\ b \mapsto bc \\ c \mapsto c. \end{cases}$$

The conjugacy class of τ_1 is $C_{\tau_1} = \{\sigma_1\sigma_2\sigma_5,\ \sigma_1\sigma_2\sigma_4\}$ and τ_2 is a central element of the automorphism group.

Theorem 3.1. *Let $\circledast_1 = \tau_1 \circ *$ and $\circledast_2 = \tau_2 \circ *$ be involutions of H_{16} and let F be a finite field with $|F| = 2^n$ $(n \geq 1)$. Then $\circledast_1$ is not similar to $\circledast_2$ and $V_{\circledast_1}(FH_{16}) \cong V_{\circledast_2}(FH_{16})$.*

Proof. First, we establish the structure of $V_{\circledast_1}(FH_{16})$. Since every element of FH_{16} can be written as

$$\begin{aligned} x =&\alpha_0 + \alpha_1 a + \alpha_2 a^2 + \alpha_3 a^3 + \alpha_4 b + \alpha_5 ab + \alpha_6 a^2 b + \alpha_7 a^3 b + \\ &(\alpha_8 + \alpha_9 a + \alpha_{10} a^2 + \alpha_{11} a^3 + \alpha_{12} b + \alpha_{13} ab + \alpha_{14} a^2 b + \alpha_{15} a^3 b)c \end{aligned} \tag{1}$$

we have

$$\begin{aligned} xx^{\circledast} =&(\alpha_0 + \alpha_2 + \alpha_8 + \alpha_{10})^2 + (\alpha_5 + \alpha_7 + \alpha_{13} + \alpha_{15})^2 a^2 + \delta_1(a + a^3 c) + \\ &\delta_2(a^3 + ac) + \delta_3(b + a^2 bc) + \delta_4(ab + abc) + \delta_5(a^2 b + bc) + \delta_6(a^3 b + a^3 bc) + \\ &(\alpha_1 + \alpha_3 + \alpha_9 + \alpha_{11})^2 c + (\alpha_4 + \alpha_6 + \alpha_{12} + \alpha_{14})^2 a^2 c, \end{aligned}$$

where

$$\begin{aligned}
\delta_1 &= (\alpha_0+\alpha_{10})(\alpha_1+\alpha_{11})+(\alpha_2+\alpha_8)(\alpha_3+\alpha_9)+(\alpha_4+\alpha_{14})(\alpha_5+\alpha_{15})+(\alpha_6+\alpha_{12})(\alpha_7+\alpha_{13})\\
\delta_2 &= (\alpha_0+\alpha_{10})(\alpha_3+\alpha_9)+(\alpha_2+\alpha_8)(\alpha_1+\alpha_{11})+(\alpha_4+\alpha_{14})(\alpha_7+\alpha_{13})+(\alpha_6+\alpha_{12})(\alpha_5+\alpha_{15})\\
\delta_3 &= (\alpha_0+\alpha_{10})(\alpha_4+\alpha_{14})+(\alpha_1+\alpha_{11})(\alpha_5+\alpha_{15})+(\alpha_2+\alpha_8)(\alpha_6+\alpha_{12})+(\alpha_3+\alpha_9)(\alpha_7+\alpha_{13})\\
\delta_4 &= (\alpha_0+\alpha_8)(\alpha_5+\alpha_{13})+(\alpha_2+\alpha_{10})(\alpha_7+\alpha_{15})+(\alpha_4+\alpha_{12})(\alpha_3+\alpha_{11})+(\alpha_6+\alpha_{14})(\alpha_1+\alpha_9)\\
\delta_5 &= (\alpha_0+\alpha_{10})(\alpha_6+\alpha_{12})+(\alpha_1+\alpha_{11})(\alpha_7+\alpha_{13})+(\alpha_3+\alpha_9)(\alpha_5+\alpha_{15})+(\alpha_4+\alpha_{14})(\alpha_2+\alpha_8)\\
\delta_6 &= (\alpha_0+\alpha_8)(\alpha_7+\alpha_{15})+(\alpha_2+\alpha_{10})(\alpha_5+\alpha_{13})+(\alpha_1+\alpha_9)(\alpha_4+\alpha_{12})+(\alpha_3+\alpha_{11})(\alpha_6+\alpha_{14}).
\end{aligned}$$

Evidently, x belongs to $V_{\circledast_1}(FH_{16})$ if and only if $xx^{\circledast_1}=1$. Therefore $\alpha_0+\alpha_2+\alpha_8+\alpha_{10}=1$, $\alpha_5+\alpha_7+\alpha_{13}+\alpha_{15}=0$, $\alpha_1+\alpha_3+\alpha_9+\alpha_{11}=0$, $\alpha_4+\alpha_6+\alpha_{12}+\alpha_{14}=0$ and $\delta_1=\delta_2=\delta_3=\delta_4=\delta_5=\delta_6=0$.

Since $\alpha_2+\alpha_8=1+\alpha_0+\alpha_{10}$ and $\alpha_4+\alpha_{14}=\alpha_6+\alpha_{12}$ we have that

$$\begin{aligned}
\delta_1 &= (\alpha_3+\alpha_9)+(\alpha_0+\alpha_{10})(\alpha_1+\alpha_{11}+\alpha_3+\alpha_9)+(\alpha_4+\alpha_{14})(\alpha_5+\alpha_{15}+\alpha_7+\alpha_{13})=\alpha_3+\alpha_9,\\
\delta_2 &= (\alpha_1+\alpha_{11})+(\alpha_0+\alpha_{10})(\alpha_3+\alpha_9+\alpha_1+\alpha_{11})+(\alpha_4+\alpha_{14})(\alpha_7+\alpha_{13}+\alpha_5+\alpha_{15})=\alpha_1+\alpha_{11},\\
\delta_3 &= (\alpha_6+\alpha_{12})+(\alpha_0+\alpha_{10})(\alpha_4+\alpha_{14}+\alpha_6+\alpha_{12})+(\alpha_1+\alpha_{11})(\alpha_5+\alpha_{15}+\alpha_7+\alpha_{13})=\alpha_6+\alpha_{12},\\
\delta_4 &= (\alpha_7+\alpha_{15})+(\alpha_0+\alpha_8)(\alpha_5+\alpha_{13}+\alpha_7+\alpha_{15})+(\alpha_4+\alpha_{12})(\alpha_3+\alpha_{11}+\alpha_1+\alpha_9)=\alpha_7+\alpha_{15},\\
\delta_5 &= (\alpha_4+\alpha_{14})+\alpha_0+\alpha_{10})(\alpha_6+\alpha_{12}+\alpha_4+\alpha_{14})+(\alpha_1+\alpha_{11})(\alpha_7+\alpha_{13}+\alpha_5+\alpha_{15})=\alpha_4+\alpha_{14},\\
\delta_6 &= (\alpha_5+\alpha_{13})+(\alpha_0+\alpha_8)(\alpha_7+\alpha_{15}+\alpha_5+\alpha_{13})+(\alpha_1+\alpha_9)(\alpha_4+\alpha_{12}+\alpha_6+\alpha_{14})=\alpha_5+\alpha_{13}.
\end{aligned}$$

Therefore

$$x=\alpha_0+\alpha_2a^2+\alpha_8c+\alpha_{10}a^2c+\alpha_1\widehat{C}a+\alpha_4\widehat{C}b+\alpha_5\widehat{C}ab,$$

where $\widehat{C}=1+a^2+c+a^2c$. As a consequence $V_{\circledast_1}(FH_{16})$ is a central subgroup of $V(FH_{16})$.

Let $N=\langle\ 1+\beta_1\widehat{C}a,\ 1+\beta_2\widehat{C}b,\ 1+\beta_3\widehat{C}ab\ |\ \beta_i\in F\ \rangle$ be. Evidently, $N\cong C_2^{3n}$. Since $a^2\widehat{C}=c\widehat{C}=a^2c\widehat{C}=\widehat{C}$ we conclude that $N\cong a^2N\cong cN\cong a^2cN$. Since $a^2N\cdot cN=a^2cN$ and the pairwise intersections of N,a^2N,a^2cN are $\{1\}$ we have proved that $V_{\circledast}(FG)\cong N\times a^2N\times cN$. Thus $V_{\circledast_1}(FG)\cong C_2^{9n}$.

Now, we establish the structure of $V_{\circledast_2}(FH_{16})$. Let $x\in FH_{16}$ be. Using formula (1) we can compute the product

$$\begin{aligned}
xx^{\circledast} = {} & (\alpha_0+\alpha_2+\alpha_8+\alpha_{10})^2+(\alpha_5+\alpha_7+\alpha_{13}+\alpha_{15})^2a^2+\delta_1(a+ac)+\\
& \delta_2(a^3+a^3c)+\delta_3(b+bc)+\delta_4(ab+abc)+\delta_5(a^2b+a^2bc)+\delta_6(a^3b+a^3bc)+\\
& (\alpha_4+\alpha_6+\alpha_{12}+\alpha_{14})^2c+(\alpha_1+\alpha_3+\alpha_9+\alpha_{11})^2a^2c,
\end{aligned}$$

where

$$\begin{aligned}
\delta_1 &= (\alpha_0+\alpha_8)(\alpha_9+\alpha_1)+(\alpha_2+\alpha_{10})(\alpha_{11}+\alpha_3)+(\alpha_4+\alpha_{12})(\alpha_5+\alpha_{13})+(\alpha_6+\alpha_{14})(\alpha_7+\alpha_{15}),\\
\delta_2 &= (\alpha_0+\alpha_8)(\alpha_{11}+\alpha_3)+(\alpha_2+\alpha_{10})(\alpha_9+\alpha_1)+(\alpha_4+\alpha_{12})(\alpha_7+\alpha_{15})+(\alpha_6+\alpha_{12})(\alpha_5+\alpha_{15}),\\
\delta_3 &= (\alpha_0+\alpha_8)(\alpha_{12}+\alpha_4)+(\alpha_2+\alpha_{10})(\alpha_{14}+\alpha_6)+(\alpha_1+\alpha_9)(\alpha_{15}+\alpha_7)+(\alpha_3+\alpha_{11})(\alpha_{13}+\alpha_5),\\
\delta_4 &= (\alpha_0+\alpha_8)(\alpha_{13}+\alpha_5)+(\alpha_2+\alpha_{10})(\alpha_{15}+\alpha_7)+(\alpha_4+\alpha_{12})(\alpha_1+\alpha_9)+(\alpha_6+\alpha_{14})(\alpha_3+\alpha_{11}),\\
\delta_5 &= (\alpha_0+\alpha_8)(\alpha_{14}+\alpha_6)+(\alpha_1+\alpha_9)(\alpha_{13}+\alpha_5)+(\alpha_2+\alpha_{10})(\alpha_{12}+\alpha_4)+(\alpha_3+\alpha_{11})(\alpha_{15}+\alpha_7),\\
\delta_6 &= (\alpha_0+\alpha_8)(\alpha_{15}+\alpha_7)+(\alpha_2+\alpha_{10})(\alpha_{13}+\alpha_5)+(\alpha_1+\alpha_9)(\alpha_{14}+\alpha_6)+(\alpha_3+\alpha_{11})(\alpha_{12}+\alpha_4).
\end{aligned}$$

Keeping in mind that x belongs to $V_{\circledast_2}(FH_{16})$, it follows that $xx^{\circledast_2}=1$. Therefore $\alpha_0+\alpha_2+\alpha_8+\alpha_{10}=1$, $\alpha_5+\alpha_7+\alpha_{13}+\alpha_{15}=0$, $\alpha_1+\alpha_3+\alpha_9+\alpha_{11}=0$, $\alpha_4+\alpha_6+\alpha_{12}+\alpha_{14}=0$ and $\delta_1=\delta_2=\delta_3=\delta_4=\delta_5=\delta_6=0$.

Since $\alpha_0+\alpha_8=1+\alpha_2+\alpha_{10}$ and $\alpha_4+\alpha_{12}=\alpha_6+\alpha_{14}$ we have that $\delta_1=\alpha_3+\alpha_{11}$, $\delta_2=\alpha_1+\alpha_9$, $\delta_3=\alpha_6+\alpha_{14}$, $\delta_4=\alpha_7+\alpha_{15}$, $\delta_5=\alpha_4+\alpha_{12}$ and $\delta_6=\alpha_5+\alpha_{13}$. Therefore $\alpha_1=\alpha_3=\alpha_9=\alpha_{11}$, $\alpha_6=\alpha_4=\alpha_{12}=\alpha_{14}$ and $\alpha_5=\alpha_7=\alpha_{13}=\alpha_{15}$.

According to the above calculations we get that every $x \in V_{\circledast_2}(FH_{16})$ can be written as

$$x = \alpha_0 + \alpha_2 a^2 + \alpha_8 c + \alpha_{10} a^2 c + \alpha_1 \widehat{C} a + \alpha_4 \widehat{C} b + \alpha_5 \widehat{C} ab,$$

where $\widehat{C} = 1 + a^2 + c + a^2 c$, so $V_{\circledast_2}(FH_{16})$ is a central subgroup of $V(FH_{16})$.

Let $N = \langle\, 1+\beta_1\widehat{C}a,\ 1+\beta_2\widehat{C}b,\ 1+\beta_3\widehat{C}ab \mid \beta_i \in F \,\rangle$ be. Clearly, $N \cong C_2^{3n}$ and $N \cong a^2N \cong cN \cong a^2cN$ because $a^2\widehat{C} = c\widehat{C} = a^2c\widehat{C} = \widehat{C}$. Since $a^2N \cdot cN = a^2cN$ and the pairwise intersections of N, a^2N, a^2cN are $\{1\}$ we have proved that $V_{\circledast}(FG) \cong N \times a^2N \times cN$. Therefore we have $V_{\circledast_1}(FG) \cong V_{\circledast_2}(FG) \cong C_2^{9n}$ and the proof is completed. □

Corollary 3.2. *The number of non-isomorphic unitary subgroups of FH_{16} with respect to the involutions which arise from H_{16} is less than $\Lambda_2 = 11$, where $|F| = 2^n \geq 2$.*

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