J. Algebra Comb. Discrete Appl.
9(3) • 161–174

Received: 11 January 2021
Accepted: 31 March 2022

Journal of Algebra Combinatorics Discrete Structures and Applications

A note on two-dimensional cyclic and constacyclic codes

Research Article

Om Prakash, Shikha Patel

Abstract: During the study of the two-dimensional cyclic (TDC) codes of length $n = ls$ over a finite field $\mathbb{F}_q$ where $s = 2^k$, Sepasdar and Khashyarmanesh (2016, [11]) arose a problem that the technique used by them to characterize TDC codes of length $n = ls$ does not work for TDC codes of length $3l$. It naturally motivates us to study the TDC codes of other lengths together with $3l$. Further, (λ_1, λ_2)-constacyclic codes are the generalization of constacyclic codes. Thus, we study two-dimensional cyclic codes of length $3l$ and (λ_1, λ_2)-constacyclic codes of length $2l$, respectively over finite fields. Here, the generating set of polynomials for these two-dimensional codes and their duals are obtained. Finally, with the help of our derived results, we have constructed many MDS codes corresponding to the two-dimensional codes.

2010 MSC: 12E20, 94B05, 94B15, 94B60

Keywords: Cyclic codes, Two-dimensional cyclic codes, Constacyclic codes, Dual codes, Generator matrix

1. Introduction

Cyclic codes were introduced by Prange [9] in 1957 and have been studied extensively till now. These codes have been studied over several finite rings and produce a huge amount of new and optimal codes, refer [2, 3, 8, 14, 15]. One of the powerful generalizations of cyclic codes over a finite field is constacyclic codes. This class of linear codes has a rich algebraic structure and is easy to recognize and implement. Note that one can specify cyclic codes and λ-constacyclic codes of length n over the finite field $\mathbb{F}_q$ by ideals of the polynomial ring $R := \mathbb{F}_q[u]/\langle u^n - 1\rangle$ and $R' := \mathbb{F}_q[u]/\langle u^n - \lambda\rangle$, respectively. Since the above rings are principal ideal rings, the cyclic and constacyclic codes are generated by a unique polynomial. These generator polynomials help us to find the important parameters of the codes.

In 1975, Ikai et al. [5] observed the two-dimensional cyclic (TDC) codes as a generalized class of cyclic codes. Moreover, just after 2 years, Imai [4] introduced the concept of binary two-dimensional

Om Prakash (Corresponding Author), Shikha Patel; Department of Mathematics, Indian Institute of Technology Patna, Patna–801106, Bihar, India (email: om@iitp.ac.in, shikha_1821ma05@iitp.ac.in).

cyclic codes. The two-dimensional theory has many applications in the analysis and generation of two-dimensional periodic arrays that help us to construct the two-dimensional feedback shift register with a minimum number of storage devices. Therefore, the study of these codes over finite rings has got the attention of many researchers and hence many new techniques have been discovered to produce cyclic codes over the finite commutative rings with better parameters, we refer [7, 10, 11, 14, 15].

In 2014, Xiuli and Hongyan [13] generalized the concept as two-dimensional skew cyclic codes over a finite field. In 2016, Sepasdar and Khashyarmanesh [11] further studied two-dimensional cyclic codes corresponding to the ideals of the ring $\mathscr{R}' := \mathbb{F}[x, y]/\langle x^l - 1, y^{2^k} - 1\rangle$. In 2019, Sharma and Bhaintwal [12] studied the structural behaviour of two-dimensional skew cyclic codes over the ring $\mathbb{F}_q + u\mathbb{F}_q$ with $u^2 = 1$. In [11], Sepasdar and Khashyarmanesh studied the algebraic structure of TDC codes of length $n = l2^k$ over the finite field. Moreover, they claimed that the method used by them is not applicable for TDC codes of length $3l$. These works motivate us to attempt over the problem raised by them in [11] and extend the study for (λ_1, λ_2)-constacyclic codes as well. Further, we also study the algebraic properties of two-dimensional cyclic and constacyclic codes and their duals.

The article is structured as follows: Section 2 contains some basic definitions and notations. Section 3 presents the study of two-dimensional cyclic codes and their duals of length $3l$. Here, we obtain the generating polynomials and the generator matrices of these codes. Similarly, in Section 4, we characterize the structure of two-dimensional (λ_1, λ_2)-constacyclic codes and their duals and calculate the generating polynomials. As an application of our results, some examples are presented in Section 5 while Section 6 concludes the paper.

2. Notation and background

For a prime p and an integer $m \geq 1$, let $\mathbb{F}_q$ be a finite field where $q = p^m$ and $\lambda_1, \lambda_2 \in \mathbb{F}_q^*$. Throughout the paper, $\mathcal{R}$ and $\mathcal{R}'$ represent the quotient ring $\mathbb{F}_q[u, v]/\langle u^l - 1, v^3 - 1\rangle$ and $\mathbb{F}_q[u, v]/\langle u^l - \lambda_1, v^2 - \lambda_2\rangle$, respectively. Now, we recall some definitions for two-dimensional codes of $\mathbb{F}_q^{ls}$.

Definition 2.1. *A two-dimensional code $\mathcal{C} \subseteq \mathbb{F}_q^{ls}$ is a set of $l \times s$ arrays over $\mathbb{F}_q$. These arrays are known as codewords (or code arrays). A two-dimensional code $\mathcal{C}$ is said to be linear if it is a subspace of the ls-dimensional linear space $\mathbb{F}_q^{ls}$. Later, in 1977, Imai [4] introduced the notion of the binary two-dimensional cyclic codes as follows:*

Definition 2.2. *Let $\mathcal{C} \subseteq \mathbb{F}_q^{ls}$ be a linear code of length $n = ls$ over $\mathbb{F}_q$ whose codewords are viewed as $l \times s$ arrays, i.e., $c \in \mathcal{C}$ is written as*

$$c = \begin{pmatrix} c_{0,0} & c_{0,1} & \cdots & c_{0,s-1} \\ c_{1,0} & c_{1,1} & \cdots & c_{1,s-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{l-1,0} & c_{l-1,1} & \cdots & c_{l-1,s-1} \end{pmatrix}.$$

Each codeword $c \in \mathcal{C}$ written in above matrix form is also defined by its polynomial representation $c(u, v) = \sum_{i=0}^{l-1} \sum_{j=0}^{s-1} c_{i,j} u^i v^j = f_0(u) + f_1(u)v + \cdots + f_{s-1}(u)v^{s-1} \in \mathbb{F}_q[u, v]/\langle u^l - 1, v^s - 1\rangle$, where $f_j(u) = c_{0,j} + c_{1,j}u + \cdots + c_{l-1,j}u^{l-1} \in \mathbb{F}_q[u]/\langle u^l - 1\rangle$. These representations provide an explicit algebraic description for two-dimensional linear codes.

- *If $\mathcal{C}$ is closed under row-shift and column-shift of codewords, then $\mathcal{C}$ is called a two-dimensional cyclic code of length ls over $\mathbb{F}_q$, i.e., if for every $l \times s$ array $c = (c_{ij}) \in \mathcal{C}$, we have*

$$\begin{pmatrix} c_{0,s-1} & c_{0,0} & \cdots & c_{0,s-2} \\ c_{1,s-1} & c_{1,0} & \cdots & c_{1,s-2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{l-1,s-1} & c_{l-1,0} & \cdots & c_{l-1,s-2} \end{pmatrix} \in \mathcal{C}$$

and

$$\begin{pmatrix} c_{l-1,0} & c_{l-1,1} & \cdots & c_{l-1,s-1} \\ c_{0,0} & c_{0,1} & \cdots & c_{0,s-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{l-2,0} & c_{l-2,1} & \cdots & c_{l-2,s-1} \end{pmatrix} \in \mathcal{C}.$$

- *Let $\lambda_1 \in \mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. Then $\mathcal{C}$ is said to be a column λ_1-constacyclic code of length ls if for every $l \times s$ array $c = (c_{ij}) \in \mathcal{C}$, we have*

$$\begin{pmatrix} \lambda_1 c_{l-1,0} & \lambda_1 c_{l-1,1} & \cdots & \lambda_1 c_{l-1,s-1} \\ c_{0,0} & c_{0,1} & \cdots & c_{0,s-1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{l-2,0} & c_{l-2,1} & \cdots & c_{l-2,s-1} \end{pmatrix} \in \mathcal{C}.$$

- *Let $\lambda_2 \in \mathbb{F}_q^*$. Then $\mathcal{C}$ is said to be a row λ_2-constacyclic code of length ls if for every $l \times s$ array $c = (c_{ij}) \in \mathcal{C}$, we have*

$$\begin{pmatrix} \lambda_2 c_{0,s-1} & c_{0,0} & \cdots & c_{0,s-2} \\ \lambda_2 c_{1,s-1} & c_{1,0} & \cdots & c_{1,s-2} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_2 c_{l-1,s-1} & c_{l-1,0} & \cdots & c_{l-1,s-2} \end{pmatrix} \in \mathcal{C}.$$

- *Let $\lambda_1, \lambda_2 \in \mathbb{F}_q^*$. If $\mathcal{C}$ is both column λ_1-constacyclic and row λ_2-constacyclic, then $\mathcal{C}$ is said to be a two-dimensional (λ_1, λ_2)-constacyclic code of length ls. Clearly, when $\lambda_1 = \lambda_2 = 1$, then two-dimensional (λ_1, λ_2)-constacyclic code coincides with two-dimensional cyclic (TDC) code.*

It is noted that there is a one-one correspondence between cyclic codes of length l over $\mathbb{F}_q$ and ideals of the polynomial ring $\mathbb{F}_q[u]/\langle u^l - 1\rangle$. Similarly, in case of TDC codes, there is also a one-one correspondence between TDC codes of length ls over $\mathbb{F}_q$ and ideals of the polynomial ring $\mathbb{F}_q[u,v]/\langle u^l - 1, v^s - 1\rangle$. Hence, a TDC code $\mathcal{C} \subseteq \mathbb{F}_q^{ls}$ of length $n = ls$ over the finite field $\mathbb{F}_q$ can be viewed as an ideal of the quotient ring $\mathbb{F}_q[u,v]/\langle u^l - 1, v^s - 1\rangle$. Further, for $\lambda_1, \lambda_2 \in \mathbb{F}_q^*$, a two-dimensional (λ_1, λ_2)-constacyclic code $\mathcal{C} \subseteq \mathbb{F}_q^{ls}$ of length $n = ls$ over the finite field $\mathbb{F}_q$ is an ideal of the quotient ring $\mathbb{F}_q[u,v]/\langle u^l - \lambda_1, v^s - \lambda_2\rangle$. Recall that for $c = (c_0, c_1, ..., c_{n-1}) \in \mathcal{C} \subseteq \mathbb{F}_q^n$, the Hamming weight $w_H(c)$ is equal to the number of non-zero components of c and for any two codewords c and c' of $\mathcal{C}$, the Hamming distance is defined as $d_H(c, c') = w_H(c - c')$. Also, the Hamming distance for the code $\mathcal{C}$ is

$$d_{\mathcal{C}} = min\ \{d(c, c') \mid c,\ c' \in \mathcal{C}, c \neq c'\}.$$

Let $c = (c_0, c_1, ..., c_{n-1})$ and $c' = (c'_0, c'_1, \dots, c'_{n-1})$ be two elements of $\mathcal{C}$. Then the inner product of c and c' in $\mathbb{F}_q^n$ is defined as $c \cdot c' = \sum_{j=0}^{n-1} c_j c'_j$. The dual code of $\mathcal{C}$ is $\mathcal{C}^\perp = \{c \in \mathbb{F}_q^n \mid c \cdot c' = 0,\ for\ all\ c' \in \mathcal{C}\}$. It is well known that a code $\mathcal{C}$ is self-orthogonal if $\mathcal{C} \subseteq \mathcal{C}^\perp$ and self dual if $\mathcal{C} = \mathcal{C}^\perp$.

Now, we recall the construction of TDC codes of length $2l$ over $\mathbb{F}_q$ from [11], which is as follows: Let M be a non-zero ideal of $\mathscr{R} = \mathbb{F}_q[u,v]/\langle u^l - 1, v^2 - 1\rangle$ and $char(\mathbb{F}_q) \neq 2$. It is known that

$$\mathbb{F}_q[u,v]/\langle u^l - 1, v^2 - 1\rangle \cong (\mathbb{F}_q[u]/\langle u^l - 1\rangle)[v]/\langle v^2 - 1\rangle.$$

Any arbitrary element of M can be uniquely written as $g(u,v) = g_0(u) + g_1(u)v$, where $g_i(u) \in \mathbb{F}_q[u]/\langle u^l - 1\rangle$ for $i = 0, 1$. Here, we find a set of generator polynomials for M. To get the generator polynomials, we use the following identity in $\mathscr{R}$.

$$g(u,v) = 2^{-1}\left(g(u,v)(1+v) + g(u,v)(1-v)\right) \tag{1}$$

where

(X) $g(u,v)(1+v) = (g_0(u) + g_1(u))(1+v)$,

(Y) $g(u,v)(1-v) = (g_0(u) - g_1(u))(1-v)$.

Next, we consider the following ideals M_1 and M_2 of $\mathbb{F}_q[u]/\langle u^l - 1\rangle$,

$$M_1 = \{f(u) \in \mathbb{F}_q[u]/\langle u^l - 1\rangle \mid f(u)(1+v) \in M\},$$
$$M_2 = \{f(u) \in \mathbb{F}_q[u]/\langle u^l - 1\rangle \mid f(u)(1-v) \in M\}.$$

Since $\mathbb{F}_q[u]/\langle u^l - 1\rangle$ is a principal ideal ring, M_1 and M_2 are principal ideals. Thus, there exist unique monic polynomials $p_1(u)$ and $p_2(u)$ in $\mathbb{F}_q[u]/\langle u^l - 1\rangle$ such that $M_i = \langle p_i(u)\rangle$ for $i = 1, 2$. Also, $p_1(u)$ and $p_2(u)$ are divisors of $u^l - 1$. Therefore, there exists $p_i'(u)$ such that $p_i(u)p_i'(u) = u^l - 1$ for $i = 1, 2$.
Now, from (X), we have $g_0(u) + g_1(u) \in M_1$, and hence

$$g_0(u) + g_1(u) = p_1''(u)p_1(u),$$

for some polynomial $p_1''(u) \in \mathbb{F}_q[u]/\langle u^l - 1\rangle$.

Also, from (Y), we have $g_0(u) - g_1(u) \in M_2$, i.e., there exists a polynomial $p_2''(u)$ in $\mathbb{F}_q[u]/\langle u^l - 1\rangle$ such that

$$g_0(u) - g_1(u) = p_2''(u)p_2(u).$$

Therefore, from Equation (1), we have

$$\begin{aligned} g(u,v) &= 2^{-1}\Big(g(u,v)(1+v) + g(u,v)(1-v)\Big) \\ &= 2^{-1}\Big((g_0(u) + g_1(u))(1+v) + (g_0(u) - g_1(u))(1-v)\Big) \\ &= 2^{-1}\Big(p_1''(u)p_1(u)(1+v) + p_2''(u)p_2(u)(1-v)\Big). \end{aligned}$$

Since $g(u,v)$ is an arbitrary element of M and hence M is an ideal of $\mathscr{R}$ generated by the polynomials $p_1(u)(1+v)$ and $p_2(u)(1-v)$, i.e.,

$$M = \langle p_1(u)(1+v),\ p_2(u)(1-v)\rangle. \tag{2}$$

These polynomials $p_1(u)(1+v)$ and $p_2(u)(1-v)$ are generators of M or of the corresponding two-dimensional cyclic code of length $2l$. The above discussed generating set of polynomials for TDC codes of length $2l$ will be used in the next Section 3 for the construction of TDC codes of length $3l$.

3. Two-dimensional cyclic codes of length $3l$

In this section, we study two-dimensional cyclic (TDC) codes of length $3l$ over $\mathbb{F}_q$ with $char(\mathbb{F}_q) \neq 2, 3$. Here, our main target is to find the generator polynomials for these codes to explore the structural properties and their duals.

3.1. Generator matrix

Let I be a non-zero ideal of $\mathcal{R} = \mathbb{F}_q[u,v]/\langle u^l - 1, v^3 - 1\rangle$. It is known that

$$\mathbb{F}_q[u,v]/\langle u^l - 1, v^3 - 1\rangle \cong (\mathbb{F}_q[u]/\langle u^l - 1\rangle)[v]/\langle v^3 - 1\rangle.$$

Therefore, each element $g(u,v) \in I$ can be uniquely written as $g(u,v) = g_0(u) + g_1(u)v + g_2(u)v^2$, where $g_i(u) \in \mathbb{F}_q[u]/\langle u^l - 1\rangle$ for $i = 0, 1, 2$. In order to obtain the set of generator polynomials of I, we have the following identity in $\mathcal{R}$:

$$g(u,v) = 3^{-1}\left(g(u,v)(1+v+v^2) + g(u,v)(1-v) + g(u,v)(1-v^2)\right) \tag{3}$$

where

(a) $g(u,v)(1+v+v^2) = (g_0(u) + g_1(u) + g_2(u))(1+v+v^2)$;

(b) $g(u,v)(1-v) = (g_0(u) - g_2(u) + (g_1(u) - g_2(u))v)(1-v)$;

(c) $g(u,v)(1-v^2) = (g_0(u) - g_2(u) + (g_1(u) - g_2(u))v)(1-v^2)$.

Next, we consider the ideals I_1 in $\mathbb{F}_q[u]/\langle u^l-1\rangle$ and I_2, I_3 in $\mathbb{F}_q[u,v]/\langle u^l-1, v^2-1\rangle$, as follows:

$$I_1 = \{f(u) \in \mathbb{F}_q[u]/\langle u^l-1\rangle \mid f(u)(1+v+v^2) \in I\};$$
$$I_2 = \{f(u,v) \in \mathbb{F}_q[u,v]/\langle u^l-1, v^2-1\rangle \mid f(u,v)(1-v) \in I\};$$
$$I_3 = \{f(u,v) \in \mathbb{F}_q[u,v]/\langle u^l-1, v^2-1\rangle \mid f(u,v)(1-v^2) \in I\}.$$

Since $\mathbb{F}_q[u]/\langle u^l-1\rangle$ is a principal ideal ring, there exists a unique monic polynomial $p_0(u) \in \mathbb{F}_q[u]/\langle u^l-1\rangle$ such that $I_1 = \langle p_0(u)\rangle$. Also, $p_0(u)$ is a divisor of u^l-1, there exists $p_0'(u)$ such that $p_0(u)p_0'(u) = u^l-1$. Further, in I_2 and I_3, $f(u,v) \in \mathbb{F}_q[u,v]/\langle u^l-1, v^2-1\rangle$. Hence, from Equation (2), we can find the generators of I_2 and I_3, i.e., I_2 is generated by two polynomials $p_1(u)(1+v)$, $p_2(u)(1-v)$ and I_3 is generated by the polynomials $p_3(u)(1+v)$ and $p_4(u)(1-v)$, respectively for some monic polynomials $p_1(u)$, $p_2(u)$, $p_3(u)$ and $p_4(u)$ in $\mathbb{F}_q[u]/\langle u^l-1\rangle$.

Therefore, (a) gives us $g_0(u) + g_1(u) + g_2(u) \in I_1$. Hence,

$$g_0(u) + g_1(u) + g_2(u) = p_0(u)p_0''(u)$$

for some polynomial $p_0''(u) \in \mathbb{F}_q[u]/\langle u^l-1\rangle$. Thus,

$$g(u,v)(1+v+v^2) = p_0(u)p_0''(u)(1+v+v^2). \tag{4}$$

Further, from (b), $g_0(u) - g_2(u) + (g_1(u) - g_2(u))v \in I_2$. Also,

$$g_0(u) - g_2(u) + (g_1(u) - g_2(u))v = p_1(u)p_1''(u)(1+v) + p_2(u)p_2''(u)(1-v)$$

for some polynomials $p_1''(u)$ and $p_2''(u)$ in $\mathbb{F}_q[u]/\langle u^l-1\rangle$. Hence,

$$g(u,v)(1-v) = (p_1(u)p_1''(u)(1+v) + p_2(u)p_2''(u)(1-v))(1-v). \tag{5}$$

Again, by (c), $g_0(u) - g_2(u) + (g_1(u) - g_2(u))v \in I_3$, i.e.,

$$g_0(u) - g_2(u) + (g_1(u) - g_2(u))v = p_3(u)p_3''(u)(1+v) + p_4(u)p_4''(u)(1-v)$$

for some polynomials $p_3''(u)$ and $p_4''(u)$ in $\mathbb{F}_q[u]/\langle u^l-1\rangle$. Hence,

$$g(u,v)(1-v^2) = (p_3(u)p_3''(u)(1+v) + p_4(u)p_4''(u)(1-v))(1-v^2). \tag{6}$$

From Equations (4), (5) and (6), Equation (3) can be viewed as

$$\begin{aligned}
g(u,v) = 3^{-1}\Big(&p_0(u)p_0''(u)(1+v+v^2) + (p_1(u)p_1''(u)(1+v) + p_2(u)p_2''(u)(1-v))\\
&(1-v) + (p_3(u)p_3''(u)(1+v) + p_4(u)p_4''(u)(1-v))(1-v^2)\Big)\\
= 3^{-1}\Big(&p_0(u)p_0''(u)(1+v+v^2) + p_1(u)p_1''(u)(1-v^2) + p_2(u)p_2''(u)\\
&(1+v^2-2v) + p_3(u)p_3''(u)(1+v)(1-v^2) + p_4(u)p_4''(u)(1-v)\\
&(1-v^2)\Big).
\end{aligned}$$

As $g(u,v)$ was arbitrary in I and $g(u,v)$ is written as a linear combination of the elements $p_0(u)(1+v+v^2)$, $p_1(u)(1-v^2)$, $p_2(u)(1+v^2-2v)$, $p_3(u)(1+v)(1-v^2)$, $p_4(u)(1-v)(1-v^2)$ over $\mathbb{F}_q[u]/\langle u^l-1\rangle$. Therefore, we have

$$I = \langle p_0(u)(1+v+v^2), p_1(u)(1-v^2), p_2(u)(1+v^2-2v), p_3(u)\\ (1+v)(1-v^2), p_4(u)(1-v)(1-v^2)\rangle.$$

Since $p_4(u)(1-v)(1-v^2)\in I$, then $p_4(u)(1-v)(1-v^2)(1-v^2)\in I$ and this implies $p_4(u)(1-v)(1-v^2)^2 = 3p_4(u)(1-v^2)\in I$ and hence $p_4(u)(1-v^2)\in I$. Also, from the definition of I_3, $p_4(u)\in I_3$. Hence, there exist polynomials $m(u)$ and $m'(u)$ in $\mathbb{F}_q[u]/\langle u^l-1\rangle$ such that

$$p_4(u) = m(u)p_3(u)(1+v) + m'(u)p_4(u)(1-v). \tag{7}$$

Comparing the degree of v in Equation (7), we get

$$p_4(u) = m(u)p_3(u) + m'(u)p_4(u) \quad \text{and} \quad m(u)p_3(u) = m'(u)p_4(u),$$

i.e., $p_4(u) = 2m(u)p_3(u)$, implies $p_3(u)|p_4(u)$. As $p_3(u)(1+v)(1-v^2)\in I$, so $p_3(u)(1+v)(1-v^2)(1+v^2)\in I$ and this gives $p_3(u)(1-v^2)\in I$. In the same manner, we have $p_4(u)|p_3(u)$. Therefore, $p_3(u) = p_4(u)$. Hence,

$$I_3 = \langle p_3(u)(1+v), p_4(u)(1-v)\rangle = \langle p_3(u)\rangle.$$

Thus,

$$I = \langle p_0(u)(1+v+v^2), p_1(u)(1-v^2), p_2(u)(1+v^2-2v), p_3(u)(1-v^2)\rangle$$

Further, we note that $p_1(u)(1-v^2)\in I$ and hence $p_1(u)\in I_3$. Therefore, there exist polynomials $m_1(u)$ and $m_1'(u)$ in $\mathbb{F}_q[u]/\langle u^l-1\rangle$ such that

$$p_1(u) = m_1(u)p_3(u)(1+v) + m_1'(u)p_3(u)(1-v). \tag{8}$$

Comparing the degree of v in Equation (8), we get

$$p_1(u) = m_1(u)p_3(u) + m_1'(u)p_3(u) \quad \text{and} \quad m_1(u)p_3(u) = m_1'(u)p_3(u),$$

i.e., $p_1(u) = 2m_1(u)p_3(u)$, and hence $p_3(u)|p_1(u)$. Next, we have $p_3(u)(1-v^2)\in I$. This implies that $p_3(u)(1-v^2)(1+v^2) = p_3(u)(1-v)\in I$, i.e., $p_3(u)\in I_2$. Thus, there exist polynomials $m_2(u)$ and $m_2'(u)$ in $\mathbb{F}_q[u]/\langle u^l-1\rangle$ such that

$$p_3(u) = m_2(u)p_1(u)(1+v) + m_2'(u)p_2(u)(1-v). \tag{9}$$

Comparing the degree of v in Equation (9), we have

$$p_3(u) = m_2(u)p_1(u) + m_2'(u)p_2(u) \quad \text{and} \quad m_2(u)p_1(u) = m_2'(u)p_2(u),$$

i.e., $p_3(u) = 2m_2(u)p_1(u)$, implies $p_1(u)|p_3(u)$ and hence $p_1(u) = p_3(u)$. Therefore,

$$I = \langle p_0(u)(1+v+v^2), p_1(u)(1-v^2), p_2(u)(1+v^2-2v)\rangle.$$

Moreover, $p_2(u)v^2(v-1)(1+v^2-2v)\in I$ as $p_2(u)(1+v^2-2v)\in I$. This implies $p_2(u)3(1-v)\in I$ and hence $p_2(u)(1-v)\in I$. Finally, the set of generator polynomials for I (the associated TDC code) is

$$\left\{p_0(u)(1+v+v^2), p_1(u)(1-v^2), p_2(u)(1-v)\right\}.$$

Also, any element of I can be written as an $\mathbb{F}_q[u]/\langle u^l-1\rangle$-combination of

$$\{p_0(u)(1+v+v^2), p_1(u)(1-v^2), p_2(u)(1+v^2-2v)\}.$$

In view of the above demonstration, the following theorem determines the generator matrix for the TDC code of length $3l$.

Theorem 3.1. *Suppose $\mathcal{C}$ is a TDC code of length $n = 3l$ and its generating set of polynomials is*

$$\{p_0(u)(1+v+v^2), p_1(u)(1-v^2), p_2(u)(1-v)\}$$

with deg $p_i(u) = a_i$ for $i = 0, 1, 2$. Then

$$\{p_0(u)(1+v+v^2), up_0(u)(1+v+v^2), \dots, u^{l-a_0-1}p_0(u)(1+v+v^2), p_1(u)(1-v^2),$$
$$up_1(u)(1-v^2), \dots, u^{l-a_1-1}p_1(u)(1-v^2), p_2(u)(1+v^2-2v), up_2(u)(1+v^2-2v), \dots, u^{l-a_2-1}$$
$$p_2(u)(1+v^2-2v)\},$$

is an independent set, and hence elements of this set form the rows of the generator matrix of the code $\mathcal{C}$.

Proof. Here, it is enough to show that the above set is independent. For this, we consider

$$m_0(u)p_0(u)(1+v+v^2) + m_1(u)p_1(u)(1-v^2) + m_2(u)p_2(u)(1+v^2-2v) = 0, \tag{10}$$

for some polynomials $m_i(u) \in \mathbb{F}_q[u]/\langle u^l - 1\rangle$ with *deg* $m_i(u) \leq l - a_i - 1$ for $i = 0, 1, 2$.

Claim: $m_i(u) = 0$ for $i = 0, 1, 2$.
From Equation (10), there exist polynomials $a(u, v)$ and $b(u, v)$ in $\mathbb{F}_q[u, v]$ such that

$$\begin{aligned} &m_0(u)p_0(u)(1+v+v^2) + m_1(u)p_1(u)(1-v^2) + m_2(u)p_2(u)(1+v^2-2v) \\ &= a(u,v)(u^l-1) + b(u,v)(v^3-1) \end{aligned} \tag{11}$$

in $\mathbb{F}_q[u, v]$. Let $a(u, v) = \sum_{i=0}^{t} a_i(u)v^i$ and $b(u, v) = \sum_{i=0}^{t'} b_i(u)v^i$ in $\mathbb{F}_q[u, v]$ and m' be the maximum degree of v in the right hand side of Equation (11). Then we have the following table for coefficients of each v^i in the right hand side of Equation (11).

Index	Coefficient
i=0	$a_0(u)(u^l-1) - b_0(u)$
i=1	$a_1(u)(u^l-1) - b_1(u)$
i=2	$a_2(u)(u^l-1) - b_2(u)$
i=3	$a_3(u)(u^l-1) - b_3(u) + b_0(u)$
i=4	$a_4(u)(u^l-1) - b_4(u) + b_1(u)$
i=5	$a_5(u)(u^l-1) - b_5(u) + b_2(u)$
$\vdots$	$\vdots$
i=$m'-1$	$a_{m'-1}(u)(u^l-1) + b_{m'-4}(u)$
i=m'	$a_{m'}(u)(u^l-1) + b_{m'-3}(u)$

Since the maximum degree of v in the left hand side of Equation (11) is 2. Therefore, for $i \geq 3$, the coefficients of v^i in right hand side of Equation (11) must be zero. Finally, we have the following equalities:

$$\begin{aligned} b_{m'-3}(u) &= -a_{m'}(u)(u^l-1), \\ b_{m'-4}(u) &= -a_{m'-1}(u)(u^l-1), \\ b_{m'-5}(u) &= -a_{m'-2}(u)(u^l-1), \\ b_{m'-6}(u) &= -a_{m'-3}(u)(u^l-1), \\ &\vdots \\ b_0(u) &= -a_3(u)(u^l-1) + b_3(u). \end{aligned}$$

It can be easily seen that the coefficients of each v^i in right hand side of Equation (11) is zero or has the factor $u^l - 1$. Hence, comparing the coefficients of v^i's in both side of Equation (11), we get the following equations:

$$\begin{aligned}
&m_0(u)p_0(u) + m_1(u)p_1(u) + m_2(u)p_2(u) = 0,\\
&m_0(u)p_0(u) - 2m_2(u)p_2(u) = 0,\\
&m_0(u)p_0(u) - m_1(u)p_1(u) + m_2(u)p_2(u) = 0.
\end{aligned}$$

Since $char(\mathbb{F}_q) \neq 2, 3$, the above equations give $m_i(u) = 0$ for $i = 0, 1, 2$. Thus, we get the required result. □

3.2. Generator matrix of $\mathcal{C}^\perp$

Our next aim is to find a generator matrix for the dual of a TDC code. Towards this, we recall the following proposition.

Proposition 3.2. *[4] The dual of a TDC code is also a TDC code.*

Since $dim(\mathcal{C}) + dim(\mathcal{C}^\perp) = 3l$, by Theorem 3.1, $dim(\mathcal{C}) = 3l - a_0 - a_1 - a_2$. Therefore, $dim(\mathcal{C}^\perp) = a_0 + a_1 + a_2$. Recall that the reciprocal polynomial $f^*(u)$ of $f(u)$ with $deg(f(u)) = k$ is defined as $f^*(u) := u^k f(1/u)$. Further, if $g(u)$ is a generator polynomial of $\mathcal{C}$, then $g(u)$ is a factor of $u^l - 1$ and so

$$u^l - 1 = g(u)h(u)$$

for some polynomial $h(u)$. This polynomial $h(u)$ is called the check polynomial of $\mathcal{C}$ and also $h^*(u)$ is a codeword in $\mathcal{C}^\perp$.

The following theorem gives the generator matrix for the dual $\mathcal{C}^\perp$.

Theorem 3.3. *Let $\mathcal{C}$ be a TDC code of length $n = 3l$ with generator polynomials $p_0(u)(1 + v + v^2)$, $p_1(u)(1 - v^2)$ and $p_2(u)(1 + v^2 - 2v)$ such that deg $p_i(u) = a_i$ and also, $p_i(u)p_i'(u) = u^l - 1$ for $i = 0, 1, 2$. Then*

$$\mathbb{H} = \begin{pmatrix}
(p_0')^*(u)(1 + v + v^2)\\
u(p_0')^*(u)(1 + v + v^2)\\
\vdots\\
u^{a_0-1}(p_0')^*(u)(1 + v + v^2)\\
(p_1')^*(u)(1 - v^2)\\
u(p_1')^*(u)(1 - v^2)\\
\vdots\\
u^{a_1-1}(p_1')^*(u)(1 - v^2)\\
(p_2')^*(u)(1 + v^2 - 2v)\\
u(p_2')^*(u)(1 + v^2 - 2v)\\
\vdots\\
u^{a_2-1}(p_2')^*(u)(1 + v^2 - 2v)
\end{pmatrix}$$

is the generator matrix of $\mathcal{C}^\perp$.

Proof. With the help of a similar argument used to prove Theorem 3.1, one can easily show that the rows of $\mathbb{H}$ are independent. Also, $(p_i')^*(u) \in \mathcal{C}^\perp$ and $(1 + v + v^2)^* = 1 + v + v^2$, $(1 - v^2)^* = v^2 - 1$ and $(1 + v^2 - 2v)^* = 1 + v^2 - 2v$. Hence, $(p_0')^*(u)(1 + v + v^2)$, $(p_1')^*(u)(1 - v^2)$ and $(p_2')^*(u)(1 + v^2 - 2v)$ are codewords in $\mathcal{C}^\perp$. Therefore, the result follows from [6]. □

4. Two-dimensional (λ_1, λ_2)-constacyclic codes of length $2l$

In this section, we study (λ_1, λ_2)-constacyclic code $\mathcal{C}$ of length $2l$ over $\mathbb{F}_q$ with $char(\mathbb{F}_q) \neq 2$.

4.1. Generator matrix of $\mathcal{C}$

Suppose J is a non-zero ideal of the ring $\mathcal{R}' = \mathbb{F}_q[u,v]/\langle u^l - \lambda_1, v^2 - \lambda_2\rangle$ and $\alpha \in \mathbb{F}_q^*$ such that $\lambda_2 = \alpha^2$ in $\mathbb{F}_q^*$. As

$$\mathbb{F}_q[u,v]/\langle u^l - \lambda_1, v^2 - \lambda_2\rangle \cong (\mathbb{F}_q[u]/\langle u^l - \lambda_1\rangle)[v]/\langle v^2 - \alpha^2\rangle,$$

an arbitrary element of J can be uniquely written as $g(u,v) = g_0(u) + g_1(u)v$ where $g_i(u) \in \mathbb{F}_q[u]/\langle u^l - \lambda_1\rangle$ for $i = 0, 1$. First we find a set of generator polynomials for J. In order to get the generator polynomials, we use the following identity in $\mathcal{R}'$.

$$g(u,v) = 2^{-1}\alpha^{-1}\left(g(u,v)(\alpha+v) + g(u,v)(\alpha - v)\right) \tag{12}$$

where

(A) $g(u,v)(\alpha + v) = (g_0(u) + \alpha g_1(u))(\alpha + v)$;

(B) $g(u,v)(\alpha - v) = (g_0(u) - \alpha g_1(u))(\alpha - v)$.

Next, we consider the two ideals J_1 and J_2 of $\mathbb{F}_q[u]/\langle u^l - \lambda_1\rangle$ where

$$J_1 = \{f(u) \in \mathbb{F}_q[u]/\langle u^l - \lambda_1\rangle \mid f(u)(\alpha + v) \in J\};$$
$$J_2 = \{f(u) \in \mathbb{F}_q[u]/\langle u^l - \lambda_1\rangle \mid f(u)(\alpha - v) \in J\}.$$

Since $\mathbb{F}_q[u]/\langle u^l - \lambda_1\rangle$ is a principal ideal ring, J_1 and J_2 are principal ideals. Hence, there exist unique monic polynomials $p_1(u)$ and $p_2(u)$ in $\mathbb{F}_q[u]/\langle u^l - \lambda_1\rangle$ such that $J_i = \langle p_i(u)\rangle$ for $i = 1, 2$. Also, $p_1(u)$ and $p_2(u)$ are divisors of $u^l - \lambda_1$. Therefore, there exists $p_i'(u)$ such that $p_i(u)p_i'(u) = u^l - \lambda_1$ for $i = 1, 2$.

Now, from (A), we have $g_0(u) + \alpha g_1(u) \in J_1$, and hence

$$g_0(u) + \alpha g_1(u) = p_1''(u)p_1(u)$$

for some polynomial $p_1''(u) \in \mathbb{F}_q[u]/\langle u^l - \lambda_1\rangle$.

Also from (B), we have $g_0(u) - \alpha g_1(u) \in J_2$, i.e., there exists a polynomial $p_2''(u)$ in $\mathbb{F}_q[u]/\langle u^l - \lambda_1\rangle$ such that

$$g_0(u) - \alpha g_1(u) = p_2''(u)p_2(u).$$

Therefore, we have the following equality in $\mathcal{R}'$:

$$\begin{aligned} g(u,v) &= 2^{-1}\alpha^{-1}\Big(g(u,v)(\alpha+v) + g(u,v)(\alpha - v)\Big) \\ &= 2^{-1}\alpha^{-1}\Big((g_0(u) + \alpha g_1(u))(\alpha + v) + (g_0(u) + \alpha g_1(u))(\alpha - v)\Big) \\ &= 2^{-1}\alpha^{-1}\Big(p_1''(u)p_1(u)(\alpha + v) + p_2''(u)p_2(u)(\alpha - v)\Big). \end{aligned}$$

Since $g(u,v)$ is an arbitrary element of J, hence

$$J = \langle p_1(u)(\alpha + v),\ p_2(u)(\alpha - v)\rangle.$$

The polynomials $p_1(u)(\alpha+v)$ and $p_2(u)(\alpha-v)$ are generators of J or of the corresponding two-dimensional (λ_1, λ_2)-constacyclic code $\mathcal{C}$. Further, based on the above construction, we have the following theorem.

Theorem 4.1. *Let J be an ideal of $\mathbb{F}_q[u]/\langle u^l-\lambda_1, v^2-\lambda_2\rangle$ with generating set $\{p_1(u)(\alpha+v), p_2(u)(\alpha-v)\}$ where $p_1(u)$ and $p_2(u)$ are monic polynomials in $\mathbb{F}_q[u]/\langle u^l - \lambda_1\rangle$ with $deg(p_1(u)) = c$ and $deg(p_2(u)) = d$. Then every element of J has the form*

$$c_1(u)p_1(u)(\alpha + v) + c_2(u)p_2(u)(\alpha - v),$$

where $c_1(u)$ and $c_2(u)$ are polynomials in the ring $\mathbb{F}_q[u]/\langle u^l-\lambda_1\rangle$ with $deg(c_1(u)) < l-c$ and $deg(c_2(u)) < l - d$.

Lemma 4.2. *[6, Theorem 7.2.12] Let $l = tp^a$ and $\lambda_1^{p^a} = \lambda_1$ where p is the characteristic of $\mathbb{F}_q$ and does not divide t. Assume $u^t - \lambda_1 = \prod_{i=1}^{r} p_i(u)$ where $p_i(u)$ are distinct monic irreducible polynomials for $i = 1, 2, \ldots, r$. Then $u^l - \lambda_1 = (u^t - \lambda_1)^{p^a} = \prod_{i=1}^{r} p_i(u)^{p^a}$. Thus, the number of monic divisors of $u^l - \lambda_1 \in \mathbb{F}_q[u]$ is $(p^a + 1)^r$.*

Theorem 4.3. *The number of two-dimensional (λ_1, λ_2)-constacyclic codes of length $n = 2l$ with exactly two generators is*

$$((p^a + 1)^r)\left((p^a + 1)^r - 1\right),$$

where $u^l - \lambda_1 = \prod_{i=1}^{r} p_i(u)^{p^a}$ and $\lambda_1^{p^a} = \lambda_1$ for some distinct monic irreducible polynomials $p_i(u)$ for $i = 1, 2, \ldots, r$.

Proof. Initially, we show that for distinct divisors $m_1(u)$ and $m_2(u)$ of $u^l - \lambda_1$, the polynomials $m_1(u)(\alpha + v)$ and $m_2(u)(\alpha - v)$ are the generator polynomials of some two-dimensional (λ_1, λ_2)-constacyclic code. For this, we suppose J is an ideal of $\mathbb{F}_q[u, v]/\langle u^l - \lambda_1, v^2 - \lambda_2\rangle$ generated by $\{m_1(u)(\alpha+v), m_2(u)(\alpha-v)\}$ and consider the corresponding two-dimensional (λ_1, λ_2)-constacyclic code $\mathcal{C}$ of $\mathbb{F}_q[u, v]/\langle u^l - \lambda_1, v^2 - \lambda_2\rangle$.
We can assume $p_1(u)(\alpha + v)$ and $p_2(u)(\alpha - v)$ are the generator polynomials of $\mathcal{C}$ as we discussed in the starting of this section. Also, $p_1(u)(\alpha + v) \in J$, there exist polynomials $g_1(u, v)$ and $g_2(u, v)$ in $\mathbb{F}_q[u, v]/\langle u^l - \lambda_1, v^2 - \lambda_2\rangle$ such that

$$\begin{aligned} p_1(u)(\alpha + v) = g_1(u, v)m_1(u)(\alpha + v) + g_2(u, v)m_2(u)(\alpha - v) + a(u, v)(u^l - \lambda_1) \\ + b(u, v)(v^2 - \lambda_2). \end{aligned}$$

Next, consider the evaluation map $\psi_1 : \mathbb{F}_q[u, v] \to \mathbb{F}_q[u]$ defined by

$$\psi_1(g(u, v)) = g(u, \alpha).$$

We have

$$2\alpha p_1(u) = 2\alpha g_1(u, \alpha)m_1(u) + a(u, \alpha)(u^l - \lambda_1).$$

Since $\mathrm{char}(\mathbb{F}_q) \neq 2$ and $m_1(u)$ is a divisor of $u^l - \lambda_1$, i.e., $m_1(u)|p_1(u)$. As

$$J_1 = \{f(u) \in \mathbb{F}_q[u]/\langle u^l - \lambda_1\rangle \mid f(u)(\alpha + v) \in J\} =< p_1(u) >,$$

and $m_1(u) \in J_1$, so $p_1(u)|m_1(u)$, and hence $p_1(u) = m_1(u)$. Similarly, we can prove that $p_2(u) = m_2(u)$. Moreover, if $m_1(u) = m_2(u)$, then $p_1(u) = p_2(u)$ and $J =< p_1(u) >$. Thus, the number of two-dimensional (λ_1, λ_2)-constacyclic codes of length $n = 2l$ with exactly two generators is

$$((p^a + 1)^r)\left((p^a + 1)^r - 1\right).$$

□

Now, we provide the generator matrix for a two-dimensional (λ_1, λ_2)-constacyclic code of length $n = 2l$.

Theorem 4.4. *Let $\mathcal{C}$ be a two-dimensional (λ_1, λ_2)-constacyclic code of length $n = 2l$ with generator polynomials $p_1(u)(\alpha + v)$ and $p_2(u)(\alpha - v)$ where $deg(p_1(u)) = c$ and $deg(p_2(u)) = d$. Then*

$$\mathbb{G} = \begin{pmatrix} p_1(u)(\alpha+v) \\ up_1(u)(\alpha+v) \\ \vdots \\ u^{l-c-1}p_1(u)(\alpha+v) \\ p_2(u)(\alpha-v) \\ up_2(u)(\alpha-v) \\ \vdots \\ u^{l-d-1}p_2(u)(\alpha-v) \end{pmatrix}$$

is the generator matrix of $\mathcal{C}$.

Proof. Under the same notations given in Theorem 4.1, suppose $c_1(u)p_1(u)(\alpha+v)+c_2(u)p_2(u)(\alpha-v)$ and $d_1(u)p_1(u)(\alpha+v) + d_2(u)p_2(u)(\alpha - v)$ are two elements of J such that

$$c_1(u)p_1(u)(\alpha + v) + c_2(u)p_2(u)(\alpha - v) = d_1(u)p_1(u)(\alpha + v) + d_2(u)p_2(u)(\alpha - v).$$

Then we find the following in $\mathbb{F}_q[u, v]$:

$$\begin{aligned}(c_1(u) - d_1(u))p_1(u)(\alpha + v) + (c_2(u) - d_2(u))p_2(u)(\alpha - v) = e(u, v)(u^l - \lambda_1)\\ + e'(u, v)(v^2 - \lambda_2)\end{aligned}$$

for some $e(u, v)$, $e'(u, v)$ in $\mathbb{F}_q[u, v]$. Now, we define the evaluation map $\psi_m : \mathbb{F}_q[u, v] \to \mathbb{F}_q[u]$ by

$$\psi_m(g(u, v)) = g(u, m)$$

for $m = -\alpha, \alpha$. Then

$$2\alpha(c_1(u) - d_1(u))p_1(u) = e(u, \alpha)(u^l - \lambda_1),$$

and

$$2\alpha(c_2(u) - d_2(u))p_2(u) = e(u, -\alpha)(u^l - \lambda_1).$$

Since degree of u in the right hand side of the above equalities is at least l whereas the corresponding degree in the left hand side is at most $l - 1$. Therefore, $c_i(u) = d_i(u)$ for $i = 1, 2$. Hence, J has $q^{l-c}q^{l-d}$ elements. Thus, the dimension of the corresponding two-dimensional (λ_1, λ_2)-constacyclic code is $2l - c - d$. □

4.2. Generator matrix of $\mathcal{C}^\perp$

It is known that $dim(\mathcal{C}) + dim(\mathcal{C}^\perp) = 2l$, then from Theorem 4.4, we have $dim(\mathcal{C}) = 2l - c - d$. Therefore,

$$dim(\mathcal{C}^\perp) = c + d.$$

Further, if $g(u)$ is a generator polynomial of a λ_1-constacyclic code $\mathcal{C}$, then $g(u)$ is a factor of $u^l - \lambda_1$ and so

$$u^l - \lambda_1 = g(u)h(u)$$

for some polynomial $h(u)$ and $h^*(u)$ is a codeword in $\mathcal{C}^\perp$.

The following theorem gives the generator matrix for the dual $\mathcal{C}^\perp$.

Theorem 4.5. *Let $\mathcal{C}$ be a two-dimensional (λ_1, λ_2)-constacyclic code of length $n = 2l$ with generator polynomials $p_1(u)(\alpha + v)$ and $p_2(u)(\alpha - v)$ such that deg $p_1(u) = c$, $p_1(u)p_1'(u) = u^l - \lambda_1$, deg $p_2(u) = d$ and $p_2(u)p_2'(u) = u^l - \lambda_1$. Then*

$$\mathbb{H} = \begin{pmatrix} (p_1')^*(u)(1+v\alpha) \\ u(p_1')^*(u)(1+v\alpha) \\ \vdots \\ u^{c-1}(p_1')^*(u)(1+v\alpha) \\ (p_2')^*(u)(1-v\alpha) \\ u(p_2')^*(u)(1-v\alpha) \\ \vdots \\ u^{d-1}(p_1')^*(u)(1-v\alpha) \end{pmatrix}$$

is the generator matrix of $\mathcal{C}^\perp$.

Proof. One can easily check that the rows of $\mathbb{H}$ are independent by using the method given in Theorem 4.4. Now, $(p_i')^*(u) \in \mathcal{C}^\perp$ for $i = 1, 2$; $(\alpha + v)^* = 1 + v\alpha$ and $(\alpha - v)^* = v\alpha - 1$. Therefore, $(p_1')^*(u)(1+v\alpha)$ and $(p_2')^*(u)(1 - v\alpha)$ are codewords in $\mathcal{C}^\perp$. Hence, the result follows from [6]. $\square$

5. Examples

A linear code $\mathcal{C}$ of length n over $\mathbb{F}_q$ with minimum distance d and dimension k is represented by $[n, k, d]$. If $\mathcal{C}$ is an $[n, k, d]$ code, then from the Singleton bound, its minimum distance is bounded above by

$$d \leq n - k + 1.$$

A code meeting the above bound is known as maximum-distance-separable (MDS). We call a code almost MDS if its minimum distance is one unit less than the MDS. A code is called optimal if it has the highest possible minimum distance for its length and dimension. We provide here a few examples in which all computations are carried out by using Magma software [1].

Example 5.1. *We consider a two-dimensional cyclic code of length* 6 *over $\mathbb{F}_7$. In $\mathbb{F}_7$, we have*

$$u^2 - 1 = (u + 1)(u + 6).$$

Suppose $p_0(u) = p_1(u) = u + 1$ and $p_2(u) = u + 6 = u - 1$, then from Theorem 3.1, the generator matrix of the TDC code is given by

$$\mathbb{G} = \begin{pmatrix} (u+1)(1+v+v^2) \\ (u+1)(1-v^2) \\ (u-1)(1+v^2-2v) \end{pmatrix}$$

and codewords corresponding to rows of that matrix are $c_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, $c_2 = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix}$, $c_3 = \begin{pmatrix} -1 & 2 & -1 \\ 1 & -2 & 1 \end{pmatrix}$. *Hence,*

$$\mathbb{G} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 & 0 & -1 \\ -1 & 2 & -1 & 1 & -2 & 1 \end{pmatrix}.$$

Here, the obtained TDC code $[6, 3, 4]$ is an MDS code.

Table 1. Two-dimensional cyclic and (λ_1, λ_2)-constacyclic codes

q	(λ_1, λ_2)	(l, n)	$p_0(u)$	$p_1(u)$	$p_2(u)$	Obtained TDC/ constacyclic code	Remark
7	$(1,1)$	$(2,6)$	$u+1$	$u+1$	$u-1$	$[6,3,4]$	MDS
5	$(1,1)$	$(3,9)$	$u-1$	$u-1$	$1+u+u^2$	$[9,5,4]$	Optimal
5	$(1,1)$	$(14,42)$	$u+1$	$u+1$	$u+4$	$[42,39,2]$	Optimal
11	$(1,1)$	$(4,12)$	$u+1$	$u-1$	u^2+1	$[12,8,4]$	Almost MDS
9	$(1,1)$	$(5,10)$		$u+2$	u^2+w^3u+1	$[10,7,4]$	MDS
5	$(3,4)$	$(3,6)$		$u+3$	u^2+2u+4	$[6,3,4]$	MDS

Example 5.2. *We consider a two-dimensional* $(3,4)$*-constacyclic code of length* 6 *over* $\mathbb{F}_5$*. In* $\mathbb{F}_5$*, we have*

$$u^3 - 3 = (u+3)(u^2+2u+4)$$

and also $\alpha = 2$*. Suppose* $p_1(u) = u+3$ *and* $p_2(u) = u^2+2u+4$*, then from Theorem 4.4, the generator matrix of the two-dimensional* $(3,4)$*-constacyclic code is given by*

$$\mathbb{G} = \begin{pmatrix} (u+3)(2+v) \\ u(u+3)(2+v) \\ (u^2+2u+4)(2-v) \end{pmatrix}$$

and codewords corresponding to rows of that matrix are
$c_1 = \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 0 & 0 \end{pmatrix}$, $c_2 = \begin{pmatrix} 0 & 0 \\ 1 & 3 \\ 2 & 1 \end{pmatrix}$, $c_3 = \begin{pmatrix} 3 & -4 \\ 4 & -2 \\ 2 & -1 \end{pmatrix}$*. Hence,*

$$\mathbb{G} = \begin{pmatrix} 1 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 2 & 1 \\ 3 & -4 & 4 & -2 & 2 & -1 \end{pmatrix}.$$

Here, the obtained two-dimensional $(3,4)$*-constacyclic code* $[6,3,4]$ *is an MDS code.*

6. Conclusion

In this paper, we have obtained the generating set of polynomials for the ideals corresponding to the two-dimensional cyclic codes of length $3l$ and (λ_1, λ_2)-constacyclic codes of length $n = 2l$, respectively. In future, we would like to develop a technique by which we can find the generating set of polynomials for two-dimensional codes of arbitrary length.

Acknowledgment: The authors are thankful to the Department of Science and Technology, Govt. of India for financial support under Ref No. DST/INSPIRE/03/2016/001445 and Indian Institute of Technology Patna for providing the research facilities.

References

[1] W. Bosma, J. Cannon, Handbook of magma functions, Univ. of Sydney, Sydney (1995).
[2] I. F. Blake, Codes over certain rings, Inf. Control. 20(4) (1972) 396–404.
[3] I. F. Blake, Codes over integer residue rings, Inf. Control. 29(4) (1975) 295–300.
[4] H. Imai, A theory of two-dimensional cyclic codes, Inf. Control. 34(1) (1977) 1–21.
[5] T. Ikai, H. Kosako, Y. Kojima, Two-dimensional cyclic codes, Electron. Comm. Japan 57(4) (1974/75) 27–35.
[6] S. Ling, C. Xing, Coding theory: a first course, Cambridge University Press, Cambridge (2004).
[7] S. Patel, O. Prakash, Repeated-root bidimensional (μ, ν)-constacyclic codes of length $4p^t.2^r$, Int. J. Inf. Coding Theory 5(3/4) (2020) 266–289.
[8] S. Patel, O. Prakash, H. Islam, Cyclic codes over $M_4(\mathbb{F}_2 + u\mathbb{F}_2)$, Cryptogr. Commun. (2022).
[9] E. Prange, Cyclic error-correcting codes in two symbols, Air Force Cambridge Research Center, Cambridge, MA, Tech. Rep. AFCRC-TN, (1957) 57–103.
[10] Z. Rajabi, K. Khashyarmanesh, Repeated-root two-dimensional constacyclic codes of length $2p^s.2^k$, Finite Fields their Appl. 50 (2018) 122–137.
[11] Z. Sepasdar, K. Khashyarmanesh, Characterizations of some two-dimensional cyclic codes correspond to the ideals of $\mathbb{F}[x, y]/\langle x^s - 1, y^{2^k} - 1\rangle$, Finite Fields their Appl. 41 (2016) 97–112.
[12] A. Sharma, M. Bhaintwal, A class of $2D$ skew-cyclic codes over $\mathbb{F}_q + u\mathbb{F}_q$, Appl. Algebra Eng. Commun. Comput. 30(6) (2019) 71–90.
[13] L. Xiuli, L. Hongyan, 2-D skew-cyclic codes over $\mathbb{F}_q[x, y; \rho, \theta]$, Finite Fields their Appl. 25 (2014) 49–63.
[14] B. Yildiz, N. Aydin, On cyclic codes over $\mathbb{Z}_4 + u\mathbb{Z}_4$ and their $\mathbb{Z}_4$-images, Int. J. Inf. Coding Theory 2(4) (2014) 226–237.
[15] X. Zheng, B. Kong, Cyclic codes and $\lambda_1 + \lambda_2 u + \lambda_3 v + \lambda_4 uv$-constacyclic codes over $\mathbb{F}_p + u\mathbb{F}_p + v\mathbb{F}_p + uv\mathbb{F}_p$, Appl. Math. Comput. 306 (2017) 86–91.