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Some upper and lower bounds for D_α-energy of graphs

Research Article

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Abstract: The generalized distance matrix of a connected graph G, denoted by $D_\alpha(G)$, is defined as $D_\alpha(G) = \alpha Tr(G) + (1-\alpha)D(G), \quad 0 \le \alpha \le 1$. Here, $D(G)$ is the distance matrix and $Tr(G)$ represents the vertex transmissions. Let $\partial_1 \ge \partial_2 \ge \cdots \ge \partial_n$ be the eigenvalues of $D_\alpha(G)$ and let $W(G)$ be the Wiener index. The generalized distance energy of G can be defined as $E^{D_\alpha}(G) = \sum_{i=1}^{n} \left| \partial_i - \frac{2\alpha W(G)}{n} \right|$. In this paper, we develop some new theory regarding the generalized distance energy $E^{D_\alpha}(G)$ for a connected graph G. We obtain some sharp upper and lower bounds for $E^{D_\alpha}(G)$ connecting a wide range of parameters in graph theory including the maximum degree Δ, the Wiener index $W(G)$, the diameter d, the transmission degrees, and the generalized distance spectral spread $D_\alpha S(G)$. We characterized the special graph classes that attain the bounds.

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1. Introduction

We consider simple connected graphs in this paper. Let $G = (V(G), E(G))$ be a graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$ being its vertex set and $E(G)$ its edge set. $n = |V(G)|$ is called the *order* and $m = |E(G)|$ the *size*. The *neighborhood* of a vertex v is the collection of vertices adjacent to it and is denoted by $N(v)$. $d_G(v)$ or simply d_v is the *degree* of v, meaning the cardinality of its neighborhood. A *regular* graph has all degrees the same. The adjacency matrix $A(G) = A = (a_{ij})$ of G has (i,j)-element one if v_i is

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adjacent to v_j and zero if not. Hence, A is an n by n symmetric matrix. The degree diagonal matrix is $\mathrm{Deg}(G) = \mathrm{diag}(d_1, d_2, \ldots, d_n)$. Two well known matrices associated with A are the Laplacian $L(G) = \mathrm{Deg}(G) - A(G)$ and the signless Laplacian $Q(G) = \mathrm{Deg}(G) + A(G)$. They are symmetric and positive semi-definite. Their spectra are organized as $0 = \mu_n \leq \mu_{n-1} \leq \cdots \leq \mu_1$ and $0 \leq q_n \leq q_{n-1} \leq \cdots \leq q_1$, respectively.

The length of a shortest path between two vertices u and v is commonly known as the *distance* and is denoted by d_{uv}. If we take maximum among all such distances in a graph, we have the *diameter*. The matrix $D(G) = (d_{uv})_{u,v \in V(G)}$ is called the *distance matrix* of G. For a comprehensive survey of recent results on distance matrix and its spectrum, we refer the reader to [6]. The sum of the distances from v to all other vertices in G is called the *transmission* and it is denoted by $Tr_G(v) = \sum\limits_{u \in V(G)} d_{uv}$. If $Tr_G(v) = k$ for all v, then G is called *k-transmission regular*. The well known *Wiener index* is defined by $W(G) = \frac{1}{2} \sum\limits_{v \in V(G)} Tr_G(v)$ and is also called *transmission* of a graph. The transmission $Tr_G(v_i)$ or shortly Tr_i for a vertex v_i is also called the *transmission degree*. The *transmission degree sequence* of G is $\{Tr_1, Tr_2, \ldots, Tr_n\}$.

The distance Laplacian and the distance signless Laplacian matrices of connected graphs have been introduced in M. Aouchiche and P. Hansen [7]. Two matrices are of utmost relevance: the *distance Laplacian matrix* $D^L(G) = Tr(G) - D(G)$ and the *distance signless Laplacian matrix* $D^Q(G) = Tr(G) + D(G)$. Here, $Tr(G) = \mathrm{diag}(Tr_1, Tr_2, \ldots, Tr_n)$ characterizes the vertex transmission of G. Many important spectral properties of these matrices have been intensively explored in the recent years; see e.g. [4, 5, 7, 8].

An effort has been made in [9] to merge the different spectra theory of distance matrix, distance Laplacian etc. by introducing the so-called *generalized distance matrix* $D_\alpha(G)$, where $D_\alpha(G) = \alpha Tr(G) + (1-\alpha)D(G)$, for $0 \leq \alpha \leq 1$. Note that $D_0(G) = D(G)$, $\quad 2D_{\frac{1}{2}}(G) = D^Q(G)$, $\quad D_1(G) = Tr(G)$ and $D_\alpha(G) - D_\beta(G) = (\alpha - \beta)D^L(G)$. Therefore, the spectral properties of individual graph matrices can be reproduced from the spectral theory of generalized distance matrix. We will re-arrange the eigenvalues of $D_\alpha(G)$ as $\partial_1 \geq \partial_2 \geq \cdots \geq \partial_n$. The largest eigenvalue ∂_1 is referred to as the *generalized distance spectral radius*. When no confusion will be caused, we simply write $\partial(G)$. The spectral properties of $D_\alpha(G)$ have attracted much more attention of the researchers. For some recent works we refer to [1–3, 9, 11, 22, 23] and the references therein.

The topic of graph energy [13] was put forward by Ivan Gutman. It is rooted in the theory of mathematical chemistry. Assume the adjacency spectrum of a graph G is represented by $\lambda_1, \lambda_2, \ldots, \lambda_n$. The graph *energy* is defined as $E(G) = \sum\limits_{i=1}^{n} |\lambda_i|$ [14]. Graph energy has been intensively studied in mathematical chemistry and some early bounds for $E(G)$ have been reported in e.g. [17]. Energy-like graph invariants with respect to other matrices (in addition to the adjacency matrix) have been discussed in [5, 10, 12, 15, 16, 21, 24].

This paper aims to study a new energy-like quantity on the basis of the eigenvalues of generalized distance matrix $D_\alpha(G)$. We define auxiliary eigenvalues Θ_i corresponding to the generalized distance eigenvalues as $\Theta_i = \partial_i - \frac{2\alpha W(G)}{n}$. The following definition of generalized distance energy is given in [3], which is inspired by the distance Laplacian energy $E^L(G)$ and the distance signless Laplacian $E^Q(G)$. Namely,

$$E^{D_\alpha}(G) = \sum_{i=1}^{n} \left| \partial_i - \frac{2\alpha W(G)}{n} \right| = \sum_{i=1}^{n} |\Theta_i|,$$

which is the average deviation of the generalized distance eigenvalues.

Let t be the largest positive integer satisfying $\partial_t \geq \frac{2\alpha W(G)}{n}$. Let $M_k(G) = \sum_{i=1}^{k} \partial_i$ be the sum of k

largest generalized distance eigenvalues. It is shown in [3] that

$$E^{D_\alpha}(G) = 2\left(M_t - \frac{2\alpha t W(G)}{n}\right) = 2\max_{1\le j\le n}\left(\sum_{i=1}^{j}\partial_i - \frac{2\alpha j W(G)}{n}\right).$$

It can be seen that $\sum_{i=1}^n \Theta_i = 0$. Noting $\sum_{i=1}^n \partial_i = 2\alpha W(G)$ and $\sum_{i=1}^{n}\partial_i^2 = \operatorname{trace}[D_\alpha(G)]^2 = 2(1-\alpha)^2\sum_{1\le i<j\le n}(d_{ij})^2 + \alpha^2\sum_{i=1}^{n}Tr_i^2$, in view of Θ_i we obtain $\sum_{i=1}^n \Theta_i^2 = 2\zeta$, where $\zeta = (1-\alpha)^2\sum_{1\le i<j\le n}^{n}(d_{ij})^2 + \frac{\alpha^2}{2}\sum_{i=1}^{n}Tr_i^2 - \frac{2\alpha^2W^2(G)}{n}$. Moreover, $E^{D_0}(G) = E^D(G)$ and $2E^{D_{\frac{1}{2}}}(G) = E^Q(G)$. The generalized distance graph energy can be viewed as merging the theories of distance graph energy and distance signless Laplacian graph energy. It is therefore appealing to investigate $E^{D_\alpha}(G)$ and explore the influence of the graph structure and parameter α on $E^{D_\alpha}(G)$. In [22], the authors defined the *generalized distance spectral spread* as $D_\alpha S(G) = \partial_1 - \partial_n$. Some lower and upper bounds for the parameter $D_\alpha S(G)$ can be found in [22].

In this paper, we continue on this line and study the generalized distance energy $E^{D_\alpha}(G)$ of a connected graph G. We establish some sharp upper and lower bounds for $E^{D_\alpha}(G)$ using graph-theoretic parameters including Wiener index $W(G)$, maximum degree Δ, minimum degree δ, diameter d, as well as generalized distance spectral spread $D_\alpha S(G)$ and transmission degrees. We characterize the graphs that attain our bounds.

2. New bounds on generalized distance energy of graphs

We start this section with some essential lemmas which will be used along the paper. The following lemma characterizes the graphs with two generalized distance eigenvalues.

Lemma 2.1. *A connected graph G has two different $D_\alpha(G)$ eigenvalues if and only if G is complete.*

Proof. The proof is similar to the proof given in [15, Lemma 2], and is excluded. □

Lemma 2.2. [9] *Suppose that G is an n-vertex connected graph. Then*

$$\partial(G) \ge \frac{2W(G)}{n},$$

with equality holding if and only if G is transmission regular.

Lemma 2.3. [18] *Suppose that A is an $n\times n$ non-negative matrix with the spectral radius $\lambda(A)$ and row sums $r_1, r_2, \dots, r_n$. We have $\min_{1\le i\le n} r_i \le \lambda(A) \le \max_{1\le i\le n} r_i$. If A is irreducible, then one of the equalities holds if and only if all the row sums of A are equal.*

Since the i-th row sum of $D_\alpha(G)$ is $r_i = \alpha Tr_i + (1-\alpha)\sum_{j=1}^n d_{ij} = Tr_i$, by Lemma 2.3 we have the following statement.

Corollary 2.4. *Suppose G is an n-vertex simple connected graph. Let $Tr_{\max}$ and $Tr_{\min}$ be the largest and the smallest transmissions of G, respectively. We have $Tr_{\min} \le \partial(G) \le Tr_{\max}$. Furthermore, any equality above holds if and only if G is transmission regular.*

Our first aim is to giving some upper bounds for $E^{D_\alpha}(G)$. The following theorem presents an upper bound for $E^{D_\alpha}(G)$. Some parameters like Wiener index $W(G)$ and transmission degrees are used.

Theorem 2.5. *Suppose G is a graph of order n. We have*

$$E^{D_\alpha}(G) \le \sqrt{n\left(2(1-\alpha)^2\sum_{1\le i<j\le n}(d_{ij})^2+\alpha^2\sum_{i=1}^{n}Tr_i^2-\frac{4\alpha^2W^2(G)}{n}\right)}. \tag{1}$$

The above equality holds if and only if G is a complete graph.

Proof. Let $\partial_1 \ge \partial_2 \ge \cdots \ge \partial_n$ be the eigenvalues of $D_\alpha(G)$. By applying the Cauchy-Schwarz inequality to these $n-1$ vectors $(1,1,\dots,1)$ and $\left(\left|\partial_2-\frac{2\alpha W(G)}{n}\right|,\left|\partial_3-\frac{2\alpha W(G)}{n}\right|,\dots,\left|\partial_n-\frac{2\alpha W(G)}{n}\right|\right)$ we obtain,

$$\begin{aligned}\left(E^{D_\alpha}(G)-\left(\partial_1-\frac{2\alpha W(G)}{n}\right)\right)^2 \le\ & (n-1)\Bigg(2(1-\alpha)^2\sum_{1\le i<j\le n}(d_{ij})^2+\alpha^2\sum_{i=1}^{n}Tr_i^2\\ &-\frac{4\alpha^2W^2(G)}{n}-\left(\partial_1-\frac{2\alpha W(G)}{n}\right)^2\Bigg).\end{aligned} \tag{2}$$

Then

$$\begin{aligned}&E^{D_\alpha}(G)\le \partial_1-\frac{2\alpha W(G)}{n}\\ &+\sqrt{(n-1)\left(2(1-\alpha)^2\sum_{1\le i<j\le n}(d_{ij})^2+\alpha^2\sum_{i=1}^{n}Tr_i^2-\frac{4\alpha^2W^2(G)}{n}-\left(\partial_1-\frac{2\alpha W(G)}{n}\right)^2\right)}.\end{aligned}$$

The function

$$f(x)=x+\sqrt{(n-1)\left(2(1-\alpha)^2\sum_{1\le i<j\le n}(d_{ij})^2+\alpha^2\sum_{i=1}^{n}Tr_i^2-\frac{4\alpha^2W^2(G)}{n}-x^2\right)}$$

decreases if and only if $x \ge \sqrt{\frac{2(1-\alpha)^2\sum_{1\le i<j\le n}(d_{ij})^2+\alpha^2\sum_{i=1}^{n}Tr_i^2-\frac{4\alpha^2W^2(G)}{n}}{n}}$. Therefore

$$E^{D_\alpha}(G)\le f\left(\sqrt{\frac{2(1-\alpha)^2\sum_{1\le i<j\le n}(d_{ij})^2+\alpha^2\sum_{i=1}^{n}Tr_i^2-\frac{4\alpha^2W^2(G)}{n}}{n}}\right).$$

Hence the result follows.

Now, suppose equality holds in (1). Then

$$\partial_1=\sqrt{\frac{2(1-\alpha)^2\sum_{1\le i<j\le n}(d_{ij})^2+\alpha^2\sum_{i=1}^{n}Tr_i^2-\frac{4\alpha^2W^2(G)}{n}}{n}}+\frac{2\alpha W(G)}{n}.$$

From equality in (2), we get $\left|\partial_2-\frac{2\alpha W(G)}{n}\right|=\left|\partial_3-\frac{2\alpha W(G)}{n}\right|=\cdots=\left|\partial_n-\frac{2\alpha W(G)}{n}\right|$ and hence

$$\begin{aligned}\sum_{i=2}^{n}\left(\partial_i-\frac{2\alpha W(G)}{n}\right)^2 =\ & \Bigg(2(1-\alpha)^2\sum_{1\le i<j\le n}(d_{ij})^2+\alpha^2\sum_{i=1}^{n}Tr_i^2-\frac{4\alpha^2W^2(G)}{n}\\ &-\left(\partial_1-\frac{2\alpha W(G)}{n}\right)^2\Bigg),\end{aligned}$$

that is,

$$(n-1)\left(\partial_i - \frac{2\alpha W(G)}{n}\right)^2 = \left(2(1-\alpha)^2 \sum_{1\le i<j\le n} (d_{ij})^2 + \alpha^2 \sum_{i=1}^{n} Tr_i^2 - \frac{4\alpha^2 W^2(G)}{n} - \left(\partial_1 - \frac{2\alpha W(G)}{n}\right)^2\right).$$

Therefore for $i = 2, \dots, n$, we get

$$\left|\partial_i - \frac{2\alpha W(G)}{n}\right| = \sqrt{\frac{2(1-\alpha)^2 \sum_{1\le i<j\le n}(d_{ij})^2 + \alpha^2 \sum_{i=1}^{n} Tr_i^2 - \frac{4\alpha^2 W^2(G)}{n} - \left(\partial_1 - \frac{2\alpha W(G)}{n}\right)^2}{n-1}} = \sqrt{\frac{2(1-\alpha)^2 \sum_{1\le i<j\le n}(d_{ij})^2 + \alpha^2 \sum_{i=1}^{n} Tr_i^2 - \frac{4\alpha^2 W^2(G)}{n}}{n}}.$$

Thus, we have $\partial_1 = P + \frac{2\alpha W(G)}{n}$ and $\partial_i = \frac{2\alpha W(G)}{n} + P$ or $\partial_i = \frac{2\alpha W(G)}{n} - P$, where $P = \sqrt{\frac{2(1-\alpha)^2 \sum_{1\le i<j\le n}(d_{ij})^2 + \alpha^2 \sum_{i=1}^{n} Tr_i^2 - \frac{4\alpha^2 W^2(G)}{n}}{n}}$. In the first case G has one distinct D_α-eigenvalue, which is so if $G = K_1$. In the second case G has two distinct eigenvalues namely, $P + \frac{2\alpha W(G)}{n}$ and $\frac{2\alpha W(G)}{n} - P$ with multiplicity 1 and $n-1$, respectively and so by Lemma 2.1, we have $G = K_n$. Conversely, it is easy to see that if $G = K_n$, the equality occurs. This completes the proof. □

Corollary 2.6. *Suppose G is a connected graph over n vertices. Assume the diameter is d. We have*

$$E^{D_\alpha}(G) \le n\sqrt{(n-1)\left((1-\alpha)^2 d^2 + \frac{\alpha^2 n^2 (n-1)}{4} - \alpha^2 (n-1)\right)}. \tag{3}$$

The above equality holds if and only if $G \cong K_2$.

Proof. Since $d_{ij} \le d$ for $i \ne j$, and there are $\frac{n(n-1)}{2}$ pairs of vertices in G, from the upper bound part in Theorem 2.5, and also using the inequality $Tr_i \le \frac{n(n-1)}{2}$, we conclude

$$\begin{aligned} E^{D_\alpha}(G) &\le \sqrt{n\left(2(1-\alpha)^2 \sum_{1\le i<j\le n}(d_{ij})^2 + \alpha^2 \sum_{i=1}^{n} Tr_i^2 - \frac{4\alpha^2 W^2(G)}{n}\right)} \\ &\le \sqrt{n\left(2(1-\alpha)^2 d^2 \frac{n(n-1)}{2} + \frac{\alpha^2 n^3 (n-1)^2}{4} - \alpha^2 n (n-1)^2\right)} \\ &= n\sqrt{(n-1)\left((1-\alpha)^2 d^2 + \frac{\alpha^2 n^2 (n-1)}{4} - \alpha^2 (n-1)\right)}. \end{aligned}$$

Equality occurs in the inequality $Tr_i \le \frac{n(n-1)}{2}$ if and only if $G \cong P_n$. This together with Theorem 2.5, gives that equality occurs in the inequality (3) if and only if $G \cong K_2$. □

The following result provides an upper bound for $E^{D_\alpha}(G)$ via transmission degrees together with Θ_1 and Θ_n.

Theorem 2.7. *Let G be a connected graph of order n. Then*

$$E^{D_\alpha}(G) \le \sqrt{n\left(2(1-\alpha)^2 \sum_{1\le i<j\le n}(d_{ij})^2 + \alpha^2 \sum_{i=1}^{n} Tr_i^2 - \frac{4\alpha^2 W^2(G)}{n}\right) - \frac{n}{2}\left(|\Theta_1| - |\Theta_n|\right)^2}. \tag{4}$$

Proof. According to $n\sum_{i=1}^{n}|\Theta_i|^2-\left(\sum_{i=1}^{n}|\Theta_i|\right)^2=\sum_{1\le i<j\le n}(|\Theta_i|-|\Theta_j|)^2$, we derive that

$$n\sum_{i=1}^{n}|\Theta_i|^2-\left(\sum_{i=1}^{n}|\Theta_i|\right)^2\ge\sum_{i=2}^{n-1}\left(\left(|\Theta_1|-|\Theta_i|\right)^2+\left(|\Theta_i|-|\Theta_n|\right)^2\right)+\left(|\Theta_1|-|\Theta_n|\right)^2. \tag{5}$$

Thanks to Jensen's inequality (see [19]), we obtain

$$\sum_{i=2}^{n-1}\left(\left(|\Theta_1|-|\Theta_i|\right)^2+\left(|\Theta_i|-|\Theta_n|\right)^2\right)\ge\frac{n-2}{2}\left(|\Theta_1|-|\Theta_n|\right)^2. \tag{6}$$

By inequalities (5) and (6), we arrive at the inequality (4). $\square$

Remark 2.8. *Since for every connected graph with the property $|\partial_1|\neq|\partial_n|$, we have*

$$\sqrt{n\left(2(1-\alpha)^2\sum_{1\le i<j\le n}(d_{ij})^2+\alpha^2\sum_{i=1}^{n}Tr_i^2-\frac{4\alpha^2W^2(G)}{n}\right)-\frac{n}{2}(|\Theta_1|-|\Theta_n|)^2}$$
$$\le\sqrt{n\left(2(1-\alpha)^2\sum_{1\le i<j\le n}(d_{ij})^2+\alpha^2\sum_{i=1}^{n}Tr_i^2-\frac{4\alpha^2W^2(G)}{n}\right)},$$

it follows that the given upper bound in (4) is better than the given bound in (1).

Corollary 2.9. *Let G be a connected graph satisfying $|\partial_1|\neq|\partial_n|$. Then*

$$E^{D_\alpha}(G)\le\sqrt{\frac{n}{2}}\left(\frac{n\left(2(1-\alpha)^2\sum_{1\le i<j\le n}(d_{ij})^2+\alpha^2\sum_{i=1}^{n}Tr_i^2-\frac{4\alpha^2W^2(G)}{n}\right)}{|\Theta_1|-|\Theta_n|}\right). \tag{7}$$

Proof. Inequality (4) can be rewritten as

$$(E^{D_\alpha}(G))^2+\frac{n}{2}(|\Theta_1|-|\Theta_n|)^2\le n\left(2(1-\alpha)^2\sum_{1\le i<j\le n}(d_{ij})^2+\alpha^2\sum_{i=1}^{n}Tr_i^2-\frac{4\alpha^2W^2(G)}{n}\right).$$

From the inequality between arithmetic and geometric means [19] we obtain

$$2\sqrt{\frac{n}{2}(E^{D_\alpha}(G))^2(|\Theta_1|-|\Theta_n|)^2}\le n\left(2(1-\alpha)^2\sum_{1\le i<j\le n}(d_{ij})^2+\alpha^2\sum_{i=1}^{n}Tr_i^2-\frac{4\alpha^2W^2(G)}{n}\right),$$

where from the inequality (7) follows. $\square$

Our next result presents a sharp upper bound on the generalized distance energy. Invariants like transmission degrees and α are involved.

Theorem 2.10. *Let G be an n-vertex connected graph with $n\ge3$. We have*

$$\begin{aligned}E^{D_\alpha}(G)\ &\le\ (1-\alpha)\sqrt{\frac{\sum_{i=1}^{n}Tr_i^2}{n}}+\frac{(1-\alpha)\sqrt{\sum_{i=1}^{n}Tr_i^2}}{n}\\&\quad+\ \sqrt{(n-2)\left(2\zeta-\frac{(1-\alpha)^2\sum_{i=1}^{n}Tr_i^2}{n}-\frac{(1-\alpha)^2\sum_{i=1}^{n}Tr_i^2}{n^2}\right)}.\end{aligned} \tag{8}$$

If equality holds in (8) then G must have exactly three or exactly four distinct D_α-eigenvalues.

Proof. Using the Cauchy-Schwarz inequality on two $(n-2)$-vectors $(\underbrace{1,1,\ldots,1}_{\text{(n-2) times}})$ and

$$\left(\left|\partial_2-\frac{2\alpha W(G)}{n}\right|,\left|\partial_3-\frac{2\alpha W(G)}{n}\right|,\ldots,\left|\partial_{n-1}-\frac{2\alpha W(G)}{n}\right|\right),$$

we have

$$\left(\sum_{i=2}^{n-1}\left|\partial_i-\frac{2\alpha W(G)}{n}\right|\right)^2\leq\left(\sum_{i=2}^{n-1}1\right)\left(\sum_{i=2}^{n-1}\left|\partial_i-\frac{2\alpha W(G)}{n}\right|^2\right), \tag{9}$$

and then

$$\begin{aligned}E^{D_\alpha}(G) \;\leq\; & \left|\partial_1-\frac{2\alpha W(G)}{n}\right|+\left|\partial_n-\frac{2\alpha W(G)}{n}\right| \\ & + \sqrt{(n-2)\left(2\zeta-\left(\partial_1-\frac{2\alpha W(G)}{n}\right)^2-\left(\partial_n-\frac{2\alpha W(G)}{n}\right)^2\right)},\end{aligned}$$

where $2\zeta = 2(1-\alpha)^2\sum_{1\leq i<j\leq n}(d_{ij})^2+\alpha^2\sum_{i=1}^{n}Tr_i^2-\frac{4\alpha^2W^2(G)}{n}$. Let $x=\left|\partial_1-\frac{2\alpha W(G)}{n}\right|$ and $y=\left|\partial_n-\frac{2\alpha W(G)}{n}\right|$. Define a function

$$f(x,y)=x+y+\sqrt{(n-2)(2\zeta-x^2-y^2)}.$$

Differentiating $f(x,y)$ with respect to x and y, we have

$$f_x=1-\frac{x(n-2)}{\sqrt{(n-2)\left(2\zeta-x^2-y^2\right)}},\quad f_y=1-\frac{y(n-2)}{\sqrt{(n-2)\left(2\zeta-x^2-y^2\right)}},$$

$$f_{xx}=-\frac{\left(2\zeta-y^2\right)\sqrt{n-2}}{\sqrt{\left(2\zeta-x^2-y^2\right)^3}},\quad f_{yy}=-\frac{\left(2\zeta-x^2\right)\sqrt{n-2}}{\sqrt{\left(2\zeta-x^2-y^2\right)^3}}$$

and

$$f_{xy}=-\frac{xy\sqrt{n-2}}{\sqrt{\left(2\zeta-x^2-y^2\right)^3}}.$$

For maxima or minima, $f_x=0$ and $f_y=0$ which implies $x=y=\sqrt{\frac{2\zeta}{n}}$. At this point the values of f_{xx}, f_{yy}, f_{xy} and $\Delta=f_{xx}f_{yy}-f_{xy}^2$ are

$$f_{xx}=-\frac{(n-1)\sqrt{n-2}}{\sqrt{\frac{2(n-2)^3\zeta}{n}}}\leq 0,\quad f_{yy}=-\frac{(n-1)\sqrt{n-2}}{\sqrt{\frac{2(n-2)^3\zeta}{n}}}\leq 0$$

$$f_{xy}=-\frac{\sqrt{n-2}}{\sqrt{\frac{2(n-2)^3\zeta}{n}}}\leq 0,\quad\text{and}\quad \Delta=\frac{n(n^2-3n+3)}{2(n-2)^2\zeta}\geq 0.$$

Therefore $f(x,y)$ attains its maximum value at this point, hence $f\left(\sqrt{\frac{2\zeta}{n}},\sqrt{\frac{2\zeta}{n}}\right)=\sqrt{2n\zeta}$. However $f(x,y)$ decreases in the intervals $\sqrt{\frac{2\zeta}{n}}\leq x\leq\sqrt{\zeta}$ and $0\leq y\leq\sqrt{\frac{2\zeta}{n}}$. Since $2\sum_{1\leq i<j\leq n}(d_{ij})^2<\frac{\left(\sum_{i=1}^{n}Tr_i\right)^2}{n}$ (see

[10]), and by the Cauchy-Schwartz inequality we have $\left(\sum_{i=1}^n Tr_i\right)^2 \leq n\sum_{i=1}^n Tr_i^2$, hence for $0 \leq \alpha \leq \frac{1}{2}$ we get

$$\begin{aligned}
&\sqrt{\frac{2(1-\alpha)^2\sum_{1\leq i<j\leq n}(d_{ij})^2 + \alpha^2\sum_{i=1}^n Tr_i^2 - \frac{4\alpha^2 W^2(G)}{n}}{n}}\\
&< \sqrt{\frac{\frac{(1-\alpha)^2\left(\sum_{i=1}^n Tr_i\right)^2}{n} + \alpha^2\sum_{i=1}^n Tr_i^2 - \frac{\alpha^2\left(\sum_{i=1}^n Tr_i\right)^2}{n}}{n}}\\
&= \sqrt{\frac{(1-2\alpha)\frac{\left(\sum_{i=1}^n Tr_i\right)^2}{n} + \alpha^2\sum_{i=1}^n Tr_i^2}{n}}\\
&\leq \sqrt{\frac{(1-2\alpha)\sum_{i=1}^n Tr_i^2 + \alpha^2\sum_{i=1}^n Tr_i^2}{n}} = (1-\alpha)\sqrt{\frac{\sum_{i=1}^n Tr_i^2}{n}},
\end{aligned}$$

then $\sqrt{\frac{2\zeta}{n}} \leq (1-\alpha)\sqrt{\frac{\sum_{i=1}^n Tr_i^2}{n}} \leq x \leq \sqrt{\zeta}$.

Now, by Cauchy-Schwarz inequality, we have $Tr_i^2 = \left(\sum_{j=1}^n d_{ij}\right)^2 \leq n\sum_{j=1}^n d_{ij}^2$. Hence $\sum_{i=1}^n Tr_i^2 \leq n\sum_{i=1}^n\sum_{j=1}^n d_{ij}^2$, and then we get $\sum_{1\leq i<j\leq n} d_{ij}^2 \geq \frac{1}{2n}\sum_{i=1}^n Tr_i^2$. Then we have

$$\begin{aligned}
&\sqrt{\frac{2(1-\alpha)^2\sum_{1\leq i<j\leq n}(d_{ij})^2 + \alpha^2\sum_{i=1}^n Tr_i^2 - \frac{4\alpha^2 W^2(G)}{n}}{n}}\\
&\geq \sqrt{\frac{2(1-\alpha)^2\sum_{1\leq i<j\leq n}(d_{ij})^2 + \frac{\alpha^2\left(\sum_{i=1}^n Tr_i\right)^2}{n} - \frac{\alpha^2\left(\sum_{i=1}^n Tr_i\right)^2}{n}}{n}}\\
&= \sqrt{\frac{2(1-\alpha)^2\sum_{1\leq i<j\leq n}(d_{ij})^2}{n}} \geq \frac{(1-\alpha)\sqrt{\sum_{i=1}^n Tr_i^2}}{n}.
\end{aligned}$$

Hence $0 \leq y \leq \frac{(1-\alpha)\sqrt{\sum_{i=1}^n Tr_i^2}}{n} \leq \sqrt{\frac{2\zeta}{n}}$. Therefore

$$f(x,y) \leq f\left((1-\alpha)\sqrt{\frac{\sum_{i=1}^n Tr_i^2}{n}}, \frac{(1-\alpha)\sqrt{\sum_{i=1}^n Tr_i^2}}{n}\right) \leq f\left(\sqrt{\frac{2\zeta}{n}}, \sqrt{\frac{2\zeta}{n}}\right).$$

Consequently,

$$\begin{aligned}
E^{D_\alpha}(G) &\leq (1-\alpha)\sqrt{\frac{\sum_{i=1}^n Tr_i^2}{n}} + \frac{(1-\alpha)\sqrt{\sum_{i=1}^n Tr_i^2}}{n}\\
&\quad + \sqrt{(n-2)\left(2\zeta - \frac{(1-\alpha)^2\sum_{i=1}^n Tr_i^2}{n} - \frac{(1-\alpha)^2\sum_{i=1}^n Tr_i^2}{n^2}\right)}.
\end{aligned}$$

The first part is complete.

Suppose the equality holds in (8). From equality in (9), we get

$$\sqrt{\left|\partial_2 - \frac{2\alpha W(G)}{n}\right|} = \sqrt{\left|\partial_3 - \frac{2\alpha W(G)}{n}\right|} = \cdots = \sqrt{\left|\partial_{n-1} - \frac{2\alpha W(G)}{n}\right|},$$

and hence

$$\sum_{i=2}^{n-1}\left(\partial_i - \frac{2\alpha W(G)}{n}\right)^2 = \left(2\zeta - \left(\partial_1 - \frac{2\alpha W(G)}{n}\right)^2 - \left(\partial_n - \frac{2\alpha W(G)}{n}\right)^2\right).$$

That is,

$$(n-2)\left(\partial_i - \frac{2\alpha W(G)}{n}\right)^2 = \frac{2\zeta - \left(\partial_1 - \frac{2\alpha W(G)}{n}\right)^2 - \left(\partial_n - \frac{2\alpha W(G)}{n}\right)^2}{n-2}, \quad i = 2, \ldots, n-1.$$

Therefore,

$$\left|\partial_i - \frac{2\alpha W(G)}{n}\right| = \sqrt{\frac{2\zeta - \left(\partial_1 - \frac{2\alpha W(G)}{n}\right)^2 - \left(\partial_n - \frac{2\alpha W(G)}{n}\right)^2}{n-2}}, \quad i = 2, \ldots, n-1.$$

Hence $\left|\partial_i - \frac{2\alpha W(G)}{n}\right|$ can have at most two distinct values. Thus, if equality holds in (8), then G must be a connected graph with exactly three or exactly four distinct D_α-eigenvalues. □

Next, we consider some lower bounds for the generalized distance energy of graphs. The following theorem describes a lower bound based on the Wiener index $W(G)$ and the parameter α.

Theorem 2.11. *Suppose G is an n-vertex connected graph. Its Wiener index is $W(G)$. We have*

$$E^{D_\alpha}(G) > \frac{4\alpha W(G)}{n} - 2\sqrt{\frac{2(1-\alpha)^2 \sum_{1\le i<j\le n}(d_{ij})^2 + \alpha^2 \sum_{i=1}^n Tr_i^2 - \partial_1^2}{n-1}}. \tag{10}$$

Proof. Let $\partial_1 \ge \partial_2 \ge \ldots \ge \partial_n$ be the generalized distance eigenvalues and t is the number of generalized distance eigenvalues of G which are greater than or equal to $\frac{2\alpha W(G)}{n}$. Note that

$$2(1-\alpha)^2 \sum_{1\le i<j\le n}(d_{ij})^2 + \alpha^2 \sum_{i=1}^n Tr_i^2 = \sum_{i=1}^n \partial_i^2 \ge \partial_1^2 + (n-1)\partial_n^2, \tag{11}$$

hence

$$\partial_n \le \sqrt{\frac{2(1-\alpha)^2 \sum_{1\le i<j\le n}(d_{ij})^2 + \alpha^2 \sum_{i=1}^n Tr_i^2 - \partial_1^2}{n-1}}.$$

In view of generalized distance energy, we see that

$$\begin{aligned} E^{D_\alpha}(G) &= 2\max_{1\le j\le n}\left(\sum_{i=1}^j \partial_i - \frac{2\alpha j W(G)}{n}\right) \ge 2\left(\sum_{i=1}^{n-1}\partial_i - \frac{2\alpha(n-1)W(G)}{n}\right) \\ &= 2\left(2\alpha W(G) - \partial_n - \frac{2\alpha(n-1)W(G)}{n}\right) = \frac{4\alpha W(G)}{n} - 2\partial_n \\ &\ge \frac{4\alpha W(G)}{n} - 2\sqrt{\frac{2(1-\alpha)^2 \sum_{1\le i<j\le n}(d_{ij})^2 + \alpha^2 \sum_{i=1}^n Tr_i^2 - \partial_1^2}{n-1}}. \end{aligned}$$

By contradiction, we will prove that the inequality in (10) is strict. For this we assume that the equality in (10) holds true. Therefore, all inequalities above are essentially equalities. Thus we arrive at

$$\max_{1\le j\le n}\left(\sum_{i=1}^j \partial_i - \frac{2\alpha j W(G)}{n}\right) = \sum_{i=1}^{n-1}\partial_i - \frac{2\alpha(n-1)W(G)}{n}. \tag{12}$$

The equality in (12) holds if and only if $t = n-1$ and equality occurs in (11) if and only if $\partial_2 = \cdots = \partial_n$. In other words, this happens if and only if G has exactly two distinct eigenvalues. Since $t = n-1$, we see that $\partial_n < \frac{2\alpha W(G)}{n} \le \partial_{n-1} = \partial_n$, which is clearly a contradiction. The proof is complete. □

The following observation is immediate from Lemma 2.2 and Theorem 2.11.

Corollary 2.12. *Suppose G is an n-vertex connected graph with minimum transmission degree $Tr_{\min}$ and Wiener index $W(G)$. We have*

$$E^{D_\alpha}(G) > \frac{4\alpha W(G)}{n} - 2\sqrt{\frac{n^2(2(1-\alpha)^2\sum_{1\le i<j\le n}(d_{ij})^2 + \alpha^2\sum_{i=1}^n Tr_i^2) - 4W^2(G)}{n-1}}.$$

Next, we establish the lower bound for $E^{D_\alpha}(G)$ involving Wiener index $W(G)$, transmission degrees as well as generalized distance spectral spread $D_\alpha S(G)$.

Theorem 2.13. *Suppose that G is an n-vertex connected graph. We have*

$$E^{D_\alpha}(G) \ge \frac{2\left(2(1-\alpha)^2\sum_{1\le i<j\le n}(d_{ij})^2 + \alpha^2\sum_{i=1}^n Tr_i^2 - \frac{4\alpha^2W^2(G)}{n}\right)}{D_\alpha S(G)}. \tag{13}$$

Proof. Let $\partial_1 \ge \partial_2 \ge \ldots \ge \partial_n$ be the generalized distance eigenvalues of G. Assume $x = (x_i)$ and $a = (a_i), i = 1, 2, \ldots, n$, be real number sequences satisfying

$$\sum_{i=1}^n |x_i| = 1 \quad \text{and} \quad \sum_{i=1}^n x_i = 0.$$

It is proved in [20] that

$$\left|\sum_{i=1}^n a_i x_i\right| \le \frac{1}{2}\left(\max_{1\le i\le n} a_i - \min_{1\le i\le n} a_i\right). \tag{14}$$

Let $a_i = \partial_i - \frac{2\alpha W(G)}{n}$ and $x_i = \frac{\partial_i - \frac{2\alpha W(G)}{n}}{\sum_{i=1}^n\left|\partial_i - \frac{2\alpha W(G)}{n}\right|}$, $i = 1, 2, \ldots, n$. Since

$$\sum_{i=1}^n x_i = \frac{\sum_{i=1}^n\left(\partial_i - \frac{2\alpha W(G)}{n}\right)}{\sum_{i=1}^n\left|\partial_i - \frac{2\alpha W(G)}{n}\right|} = 0 \quad \text{and} \quad \sum_{i=1}^n |x_i| = \frac{\sum_{i=1}^n\left|\partial_i - \frac{2\alpha W(G)}{n}\right|}{\sum_{i=1}^n\left|\partial_i - \frac{2\alpha W(G)}{n}\right|} = 1,$$

the conditions for inequality (14) are satisfied. Hence, we have

$$\left|\frac{\sum_{i=1}^n\left(\partial_i - \frac{2\alpha W(G)}{n}\right)^2}{\sum_{i=1}^n\left|\partial_i - \frac{2\alpha W(G)}{n}\right|}\right| \le \frac{1}{2}\left(\max_{1\le i\le n}\left(\partial_i - \frac{2\alpha W(G)}{n}\right) - \min_{1\le i\le n}\left(\partial_i - \frac{2\alpha W(G)}{n}\right)\right),$$

that is

$$\frac{2(1-\alpha)^2\sum_{1\le i<j\le n}(d_{ij})^2 + \alpha^2\sum_{i=1}^n Tr_i^2 - \frac{4\alpha^2W^2(G)}{n}}{E^{D_\alpha}(G)} \le \frac{1}{2}\left(\partial_1 - \frac{2\alpha W(G)}{n} - \partial_n + \frac{2\alpha W(G)}{n}\right),$$

then, we get

$$E^{D_\alpha}(G) \ge \frac{2\left(2(1-\alpha)^2\sum_{1\le i<j\le n}(d_{ij})^2 + \alpha^2\sum_{i=1}^n Tr_i^2 - \frac{4\alpha^2W^2(G)}{n}\right)}{D_\alpha S(G)}.$$

$\square$

The following bound for $D_\alpha S(G)$ was established in [22]:

$$D_\alpha S(G) \leq \sqrt{2\left(2(1-\alpha)^2 \sum_{1\leq i<j\leq n}(d_{ij})^2 + \alpha^2\sum_{i=1}^{n} Tr_i^2 - \frac{4\alpha^2 W^2(G)}{n}\right)}. \tag{15}$$

Hence, using (15) and (13), we obtain the following lower bound for the generalized distance energy.

Corollary 2.14. *Suppose that G is an n-vertex connected graph. We have*

$$E^{D_\alpha}(G) \geq \sqrt{2\left(2(1-\alpha)^2 \sum_{1\leq i<j\leq n}(d_{ij})^2 + \alpha^2\sum_{i=1}^{n} Tr_i^2 - \frac{4\alpha^2 W^2(G)}{n}\right)}.$$

Corollary 2.15. *Assume G is connected and has n vertices. We obtain*

$$E^{D_\alpha}(G) \geq (1-\alpha)\sqrt{2n(n-1)}.$$

Proof. Since $d_{ij} \geq 1$ for $i \neq j$ and there are $\frac{n(n-1)}{2}$ pairs of vertices in G, from the lower bound of Corollary 2.14, we get

$$\begin{aligned} E^{D_\alpha}(G) &\geq \sqrt{2\left(2(1-\alpha)^2 \sum_{1\leq i<j\leq n}(d_{ij})^2 + \alpha^2\sum_{i=1}^{n} Tr_i^2 - \alpha^2\sum_{i=1}^{n} Tr_i^2\right)} \geq \sqrt{4(1-\alpha)^2\frac{n(n-1)}{2}} \\ &= (1-\alpha)\sqrt{2n(n-1)}. \end{aligned}$$

□

We conclude by giving the following lower bound for the generalized distance energy $E^{D_\alpha}(G)$ involving some graph-theoretic invariants such as $W(G)$, $Tr_{\max}$ and $Tr_{\min}$.

Theorem 2.16. *Suppose that G is an n-vertex graph. Let $Tr_{\max}$ and $Tr_{\min}$ be respectively the largest and the least transmissions of G. Let $\Gamma = \left|det\left(D_\alpha(G) - \frac{2\alpha W(G)}{n}I\right)\right|$. We have*

$$E^{D_\alpha}(G) \geq Tr_{\min} - \frac{2\alpha W(G)}{n} + \sqrt{2\zeta - \omega^2 + (n-1)(n-2)\left(\frac{\Gamma}{\omega}\right)^{\frac{2}{n-1}}}, \tag{16}$$

where $2\zeta = 2(1-\alpha)^2\sum_{1\leq i<j\leq n}(d_{ij})^2 + \alpha^2\sum_{i=1}^{n} Tr_i^2 - \frac{4\alpha^2W^2(G)}{n}$ and $\omega = Tr_{\max} - \frac{2\alpha W(G)}{n}$. Equality holds if and only if either G is a complete graph or a k-transmission regular graph with three distinct D_α-eigenvalues

$$\left(k, \sqrt{\frac{M-k^2(1-\alpha)^2}{n-1}} + \alpha k, -\sqrt{\frac{M-k^2(1-\alpha)^2}{n-1}} + \alpha k\right),$$

where $M = 2(1-\alpha)^2\sum_{1\leq i<j\leq n}(d_{ij})^2$.

Proof. It is not difficult to see that

$$\begin{aligned} \left(E^{D_\alpha}(G)\right)^2 &= \sum_{i=1}^{n}\left(\partial_i - \frac{2\alpha W(G)}{n}\right)^2 + 2\sum_{1\leq i<j\leq n}\left|\partial_i - \frac{2\alpha W(G)}{n}\right|\left|\partial_j - \frac{2\alpha W(G)}{n}\right| \\ &= 2\zeta + 2\left(\partial_1 - \frac{2\alpha W(G)}{n}\right)\sum_{i=2}^{n}\left|\partial_i - \frac{2\alpha W(G)}{n}\right| \\ &+ 2\sum_{2\leq i<j\leq n}\left|\partial_i - \frac{2\alpha W(G)}{n}\right|\left|\partial_j - \frac{2\alpha W(G)}{n}\right|, \end{aligned} \tag{17}$$

where $2\zeta = 2(1-\alpha)^2 \sum_{1\le i<j\le n}(d_{ij})^2 + \alpha^2 \sum_{i=1}^{n} Tr_i^2 - \frac{4\alpha^2 W^2(G)}{n}$. It follows from the arithmetic-geometric mean inequality that

$$\sum_{2\le i<j\le n} \left|\partial_i - \frac{2\alpha W(G)}{n}\right|\left|\partial_j - \frac{2\alpha W(G)}{n}\right| \ge \frac{(n-1)(n-2)}{2}\left(\prod_{i=2}^{n}\left|\partial_i - \frac{2\alpha W(G)}{n}\right|\right)^{\frac{2}{n-1}} \tag{18}$$

$$= \frac{(n-1)(n-2)}{2}\left(\frac{\Gamma}{\partial_1 - \frac{2\alpha W(G)}{n}}\right)^{\frac{2}{n-1}}. \tag{19}$$

Applying the inequality (19), we get from (17) that

$$\begin{aligned}\left(E^{D_\alpha}(G)\right)^2 &\ge 2\zeta + 2\left(\partial_1 - \frac{2\alpha W(G)}{n}\right)\left(E^{D_\alpha}(G) - \left(\partial_1 - \frac{2\alpha W(G)}{n}\right)\right)\\ &+ (n-1)(n-2)\left(\frac{\Gamma}{\partial_1 - \frac{2\alpha W(G)}{n}}\right)^{\frac{2}{n-1}},\end{aligned}$$

which implies that

$$\begin{aligned}&\left(E^{D_\alpha}(G)\right)^2 - 2\left(\partial_1 - \frac{2\alpha W(G)}{n}\right)E^{D_\alpha}(G) - \left[2\zeta - 2\left(\partial_1 - \frac{2\alpha W(G)}{n}\right)^2\right.\\ &\left.+(n-1)(n-2)\left(\frac{\Gamma}{\partial_1 - \frac{2\alpha W(G)}{n}}\right)^{\frac{2}{n-1}}\right] \ge 0.\end{aligned}$$

By solving the above inequality, we get

$$E^{D_\alpha}(G) \ge \partial_1 - \frac{2\alpha W(G)}{n} + \sqrt{2\zeta - \left(\partial_1 - \frac{2\alpha W(G)}{n}\right)^2 + (n-1)(n-2)\left(\frac{\Gamma}{\partial_1 - \frac{2\alpha W(G)}{n}}\right)^{\frac{2}{n-1}}}.$$

Applying Corollary 2.4, we have

$$\begin{aligned}E^{D_\alpha}(G) &\ge Tr_{\min} - \frac{2\alpha W(G)}{n}\\ &+\sqrt{2\zeta - \left(Tr_{\max} - \frac{2\alpha W(G)}{n}\right)^2 + (n-1)(n-2)\left(\frac{\Gamma}{Tr_{\max} - \frac{2\alpha W(G)}{n}}\right)^{\frac{2}{n-1}}}.\end{aligned} \tag{20}$$

Hence we get the required result in (16). The first part is complete.

Assume that the equality in (16) holds. The rest inequalities above are essentially equalities. Hence, $\partial_1 = Tr_{\max} = Tr_{\min}$, and G is a transmission regular graph. It follows from the equality in (18), we see that $\left|\partial_2 - \frac{2\alpha W(G)}{n}\right| = \left|\partial_3 - \frac{2\alpha W(G)}{n}\right| = \cdots = \left|\partial_n - \frac{2\alpha W(G)}{n}\right|$, and hence

$$\sum_{i=2}^{n}\left(\partial_i - \frac{2\alpha W(G)}{n}\right)^2 = \left(2\zeta - \left(\partial_1 - \frac{2\alpha W(G)}{n}\right)^2\right),$$

that is,

$$(n-1)\left(\partial_i - \frac{2\alpha W(G)}{n}\right)^2 = \left(2\zeta - \left(\partial_1 - \frac{2\alpha W(G)}{n}\right)^2\right),$$

then for each $i = 2, \dots, n$, we have

$$\left|\partial_i - \frac{2\alpha W(G)}{n}\right| = \sqrt{\frac{2\zeta - \left(\partial_1 - \frac{2\alpha W(G)}{n}\right)^2}{n-1}}.$$

Consequently, $\left|\partial_i - \frac{2\alpha W(G)}{n}\right|$ can have at most two distinct values and we arrive at the following:
(i) G has exactly one distinct D_α-eigenvalue. Thus, $G = K_1$.
(ii) G has exactly two distinct D_α-eigenvalues. Using Lemma 2.1, $G = K_n$.
(iii) G has exactly three distinct D_α-eigenvalues. We have $\partial_1 = Tr_{\max}$, and $\left|\partial_i - \frac{2\alpha W(G)}{n}\right| = \sqrt{\frac{2\zeta - \left(Tr_{\max} - \frac{2\alpha W(G)}{n}\right)^2}{n-1}}$, $i = 2, \dots, n$. Moreover, G is k-transmission regular graph. We have G as a graph with three distinct D_α-eigenvalues

$$\left(k, \sqrt{\frac{M - k^2(1-\alpha)^2}{n-1}} + \alpha k, -\sqrt{\frac{M - k^2(1-\alpha)^2}{n-1}} + \alpha k\right),$$

in which $M = 2(1-\alpha)^2 \sum_{1 \le i < j \le n} (d_{ij})^2$. Hence the result follows. □

Since for any i, we have $n - 1 \le Tr_i \le \frac{n(n-1)}{2}$, hence from Theorem 2.16, the following observation is immediate.

Corollary 2.17. *Suppose that G is an n-vertex graph and $\Gamma = \left|det\left(D_\alpha(G) - \frac{2\alpha W(G)}{n} I\right)\right|$. We have*

$$E^{D_\alpha}(G) \ge n - 1 - \frac{2\alpha W(G)}{n} + \sqrt{2\zeta - h^2 + (n-1)(n-2)\left(\frac{\Gamma}{h}\right)^{\frac{2}{n-1}}},$$

where $2\zeta = 2(1-\alpha)^2 \sum_{1 \le i < j \le n} (d_{ij})^2 + \alpha^2 \sum_{i=1}^n Tr_i^2 - \frac{4\alpha^2 W^2(G)}{n}$ *and* $h = \frac{n(n-1)}{2} - \frac{2\alpha W(G)}{n}$.

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