TRANSIENT BEHAVIOR OF STRAIGHT FINS

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ABSTRACT
The transient behavior of straight fins is considered in this work. Kantrovich variational method is used to reduce the partial differential equation which is solved analytically to determine the temperature distribution and the fin efficiency. Four different cross-section shapes are examined; rectangular, triangular, hyperbolic and parabolic fin shapes. The steady state fin efficiency and the time required to reach 99% of such values were determined for three different cases of design.

INTRODUCTION
Fins design is of great importance in engineering applications of heat transfer. Straight fins are commonly used with flat surfaces to increase the rate of removal of heat generated by the source.

Almost the shape and the dimensions of the fin for a specific process are chosen based on the steady state analysis of the problem. Many tables and charts are still in use to determine the efficiency and the optimum fin dimensions. Gardner[1], determined the efficiency of various straight fins and spines. His figures appeared in Eckert and Drake[2], and in some other books.

The optimum dimensions of the fin was solved by Schmidt[3], and was confirmed by Duffin and Mclain[4].

The disturbance in the environment temperature or the base temperature of the fin and the fluctuation of the flow rate of the fluid surrounding the fin, have an important influence on the performance of the fin and its efficiency. This may lead to unexpected changes in the rate of the heat removal. This is mainly occurred during the starting period of the thermal process.

Champan[5], studied transient heat conduction in an annular fin by using the separation of variables method. Yang[6], obtained temperature distributions at large time for rectangular fin subject to periodic base temperature. Chang Yi-Hsu et al[7], used the perturbation technique and an averaging method to reduce the problem of transient conduction in a 2-D rectangular fin to that of a 1-D problem which solved by the linear operator method.

In this study a variational method is used to solve the transient heat conduction in different shapes of straight fin. The Kantrovich method is applied to reduce the partial differential equation, which is solved to determine the fin efficiency and the transient behavior of different shapes of straight fins.

PROBLEM FORMULATION

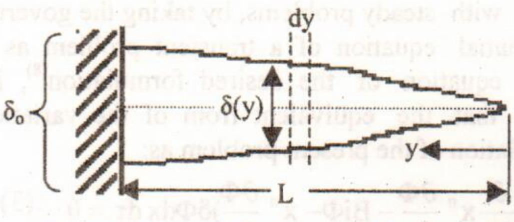


Fig. (1) Schematic Diagram of Straight Fin

Consider a straight fin as shown schematically in Fig.(1). The analysis of the problem is based on the following assumptions:

- The properties of the fin are constant.
- The heat transfer coefficient is constant.
- The fin thickness is smaller than its length and the thermal conductivity is much larger than the heat transfer coefficient. This makes the temperature gradients are only in one dimension and the temperature gradients may be neglected at the thickness of the fin.
- The heat transferred through the edge of the fin is neglected compared to the heat removed from the entire surface of the fin.
- The fin is basically at equilibrium with the environment and suddenly subjected to change in environment without changing base of the fin.

Under the above assumptions, the energy balance for a control volume of length, dy, is given by:

$$\rho C p \delta(y)\frac{\partial \theta}{\partial t} = k\frac{\partial}{\partial y}\left(\delta(y)\frac{\partial \theta}{\partial y}\right) - 2h\theta \qquad \cdots(1)$$

and the boundary and the initial conditions

$$\theta(0,y) = \theta_o, \quad \theta(t,L) = \theta_o, \frac{\partial\theta}{\partial x}\Big|_{(t,0)} = 0$$

Introducing the following dimensionless variables

$$x=\frac{y}{L},\quad \Phi=\frac{\theta}{\theta_o},\quad Bi=Bi'\frac{2L}{\delta_o},\quad Bi'=\frac{hL}{k}$$

$$\bar{\delta}=\frac{\delta(y)}{\delta_o}=x^n,\quad \tau=Fo=\alpha t/L^2,\quad \alpha=\frac{k}{\rho\, Cp}$$

into Equation (1) results in

$$\frac{\partial}{\partial x}\left(x^n\frac{\partial\Phi}{\partial x}\right)-Bi\Phi-x^n\frac{\partial\Phi}{dt}=\Phi \qquad \cdots(2)$$

$$\Phi(0,x)=1,\quad \Phi(t,1)=1,\quad \frac{\partial\Phi}{\partial x}(t,0)=0$$

In Equation (2), the shape of the fin is represented by the power, n, the variation of the fin thickness along the fin. Four common fin shapes are used in this work, they are, rectangular; $n=\phi$, hyperbolic; n = 1/2 , triangular; n = 1 and parabolic; n = 2.

Variational Formulation and Solution

As with steady problems, by taking the governing differential equation of a transient problem as the Euler equation of the desired formulation[8], it is found that the equivalent from of the variational formulation of the present problem as:

$$\int_{\tau_1}^{\tau_2}\int_x\left(\frac{\partial}{\partial x}x^n\frac{\partial\Phi}{\partial x}-Bi\Phi-x^n\frac{\partial\Phi}{\partial\tau}\right)\delta\Phi dx\,d\tau=0\cdots(3)$$

According to Kantorvich method[9], the temperature profile was assumed as:

$$\Phi(x,\tau)=\Phi(x,\Im_o(\tau),\Im_1(\tau),\Im_2(\tau),\cdots,\Im_m(\tau))$$
...(4)

Which is specified in x-direction, but depends on the unknown functions $\Im o(\tau),\cdot,\Im m(\tau)$ in time. Inserting Equation (4) into Equation (3) gives:

$$\int_{\tau_1}^{\tau_2}\int_x\left(\frac{\partial}{\partial x}(x^n\frac{\partial\Phi}{\partial x})-Bi\Phi-x^n\frac{\partial\Phi}{\partial\tau}\right)\times \qquad \ldots(5)$$
$$\left(\frac{\partial\Phi}{\partial\Im_o}-\delta\Im_1+\frac{\partial\Phi}{\partial\Im_2}\delta\Im_1\cdots\frac{\partial\Phi}{\partial\Im_m}-\delta\Im_m\right)dx\,d\tau=0$$

Since Equation (5) is true for an arbitrary time interval $(\tau 1,\tau 2)$, the integrated relative to time integration must vanish every where in this interval. Thus it will be

$$\int_x\left(\frac{\partial}{\partial x}(x^n\frac{\partial\Phi}{\partial x})-Bi\Phi-x^n\frac{\partial\Phi}{\partial\tau}\right)\times \qquad \ldots(6)$$
$$\left(\frac{\partial\Phi}{\partial\Im_o}-\delta\Im_1+\frac{\partial\Phi}{\partial\Im_2}+\cdots\frac{\partial\Phi}{\partial\Im_m}-\delta\Im_m\right)dx=0$$

Furthermore, since the variations $\delta\Im o,\delta\Im 1,\cdots,\delta\Im m$ are arbitrary their coefficient must vanish in Equation (6) giving the equations

$$\int_x\left(\frac{\partial}{\partial x}(x^n\frac{\partial\Phi}{\partial x})-Bi\Phi-x^n\frac{\partial\Phi}{\partial\tau}\right)\frac{\partial\Phi}{\partial\Im_i}dx=0 \qquad \ldots(7)$$
$i=1,2,..,m$

After x-direction integration, Equation (7) yields (n + 1) simultaneous ordinary differential equations in term of the unknown functions

$\Im o(\tau),\Im 1(\tau),\Im 2(\tau)\cdot,\Im m(\tau)$.

In this work the following approximation is used to represent the temperature profile of the fin which satisfy the initial and boundary conditions of the problem.

$$\Phi(x,\tau)=1-(1-x^2)(\Im_o(\tau)+x^2\Im_1(\tau))\ldots(8)$$

Inserting Equation (8) into Equation (7) and then solving the resulting equations for the time-dependent functions, $\Im o(\tau)$ and $\Im 1(\tau)$, gives

$$\Im_i(\tau)=K_{0i}+\sum_{j=1}^{2}K_{ji}e^{\lambda_j\tau} \qquad i=1,2\cdots(9)$$

Where K_{ji} and λ_j are the eigenvectors and the eigenvalues of the characteristic equation associated with Equation (7), listed in Table (1).

Fin Efficiency

The fin efficiency is defined as the ratio of the actual to a hypothetical heat transfer

$$\eta=\frac{\text{Actual heat transfer from the fin}}{\text{Heat transfer from the fin at base temp}}$$

$$\eta=\frac{hP\int_0^L\theta\,dy}{hP\int_0^L\theta_o\,dy}=\int_0^1\Phi\,dx \qquad \ldots(10)$$

Where, P, is the perimeter of the fin, which is constant for straight fins ($P\cong 2$ width).

Introducing the temperature profile, Equation (8), in Equation (10), the fin efficiency becomes

$$\eta=1-2\left(\frac{\Im_o(\tau)}{3}+\frac{\Im_1(\tau)}{15}\right)\ldots(11)$$

Equation (11) describe the transient fin efficiency which is depending upon transient profile for any type of the fins considered in this work.

A comparison of the efficiency for straight fins is made here, it is being on the basis of the following:

1. Case (1) equal length, L, and thickness, δo.
2. Case (2) equal profile area and thickness.
3. Case (3) equal profile area and length.

For the first case the fin parameter, ζ_1, used to analyze fin parameter, ζ_2, will be as follows

$$\zeta_2=\frac{2hA^2}{k\delta_o^3}=\frac{Bi}{(1+n)^2}\ldots(12)$$

where A is the profile area determined as follows

$$A = \int_0^L \delta(y)\,dy = \frac{L\delta_o}{1+n} \quad \ldots(13)$$

and for the third case, when the area and the length are identical for each type of fins, the fin parameter, $\zeta 3$, is

$$\zeta_3 = \frac{2hL^2}{k\,A} = (1+n)Bi \quad \ldots(14)$$

RESULTS AND DISCUSSION

First of all, the accuracy of the variational method used in this work was examined against available exact solutions. The exact solutions of the steady state fin efficiency for different shapes of a straight fin (listed in Table (2)), were compared with that resulted from Equation (11) as the time approach infinity $(\eta_{\tau\to\infty})$. Fig.(2) shows that the procedure suggested in this work presents the fin efficiency for different shapes with acceptable error. As shown in Fig.(3), there is a maximum error between the exact and the approximated solution for each type of fins. The magnitude of these values depends on the shape of the fin and the value of Bi. The results indicate also, that the error decreases for Bi >I which is reduced to about 0.5% for Bi = 3, independent of the fin shape. In general the error between the exact solution and the approximated one, for parabolic fin (n = 2) is always higher than other shapes.

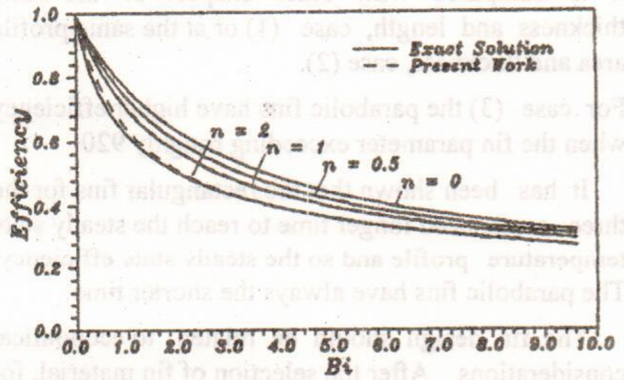


Fig.(2) Comparison Between the Exact and Approximated Solutions of Steady State Fin Efficiency.

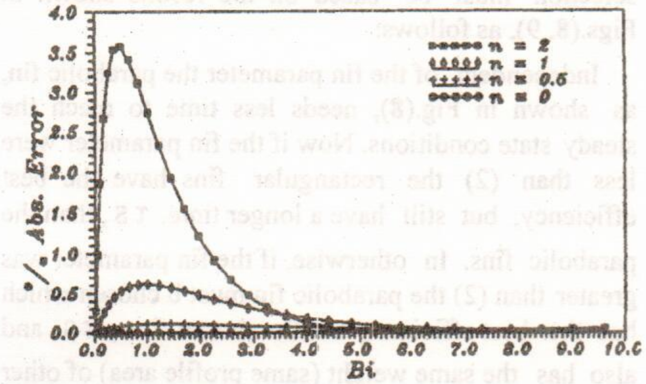


Fig.(3) Absolute Error Between Exact and Approximated Solutions of Steady State Fin Efficiency.

As it is mentioned earlier, the main purpose of this work is to investigate the effect of fin shape on the transient behavior of the fin when it is subjected to a sudden change in wall temperature and then to determine the period, τS, required to each the steady state temperature profile and so the steady state fin efficiency. This can be achieved by estimated the time at which the transient efficiency approach 99% of the steady state value.

These values of τS, were determined in a similar way of that presented in Fig.(4); for rectangular fin. For a certain value of Bi, the transient efficiency were calculated from equation (11) and compared with the steady state value of efficiency for a same value of Bi, then evaluated the time τS at which the efficiency approach 0.99 of the steady state value.

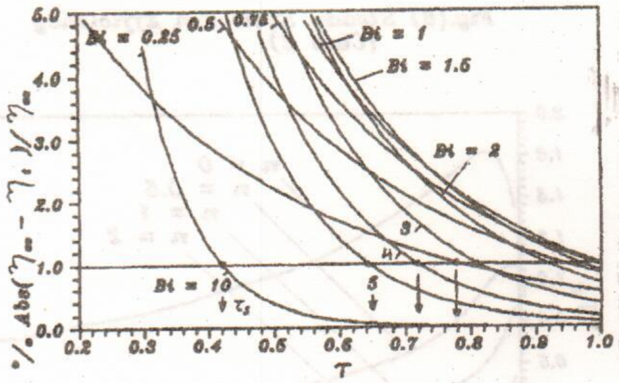


Fig.(4) Determination of τ_s for Rectangular Fin

Fig.(5) illustrated the effect of the shape of the fin on the value of τS. In this case it is assumed that the length and the thickness of the fins are similar, i.e., constant Bi, case (1). As shown in Fig.(5), the rectangular fin which has a best fin efficiency, needs the longest time to reach the steady state efficiency, whereas for the parabolic fin which has the lowest efficiency, the values of τS were less than other fins.

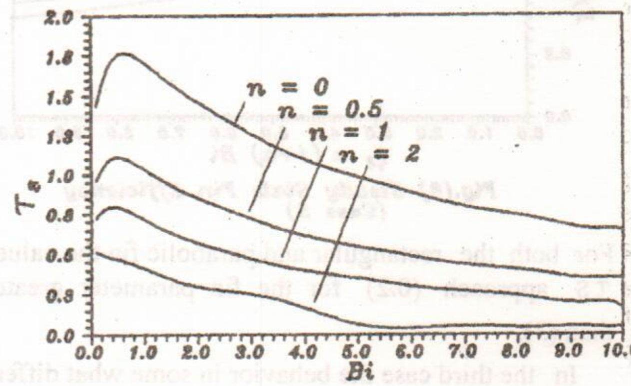


Fig.(5) Effect of Bi on Values of τ_s (Case 1)

For case (2); Figs.(6, 7), the results did not differ from that of case (1) except that the differences among the efficiencies of each fin became larger and the values of τS were smaller than that the former case for all shapes but the rectangular one where the efficiency and τS were identical for the three cases.

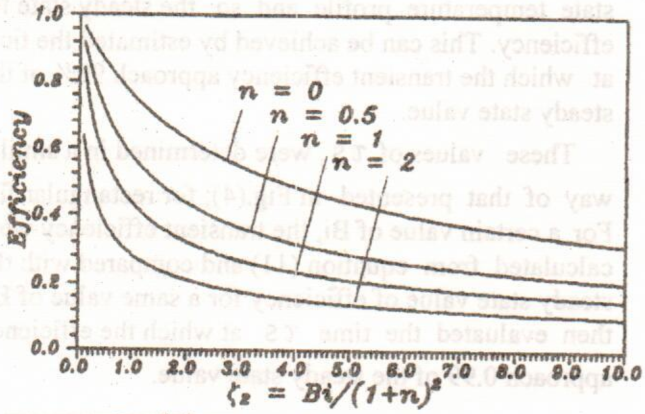


Fig.(6) Steady State Fin Efficiency (Case 2)

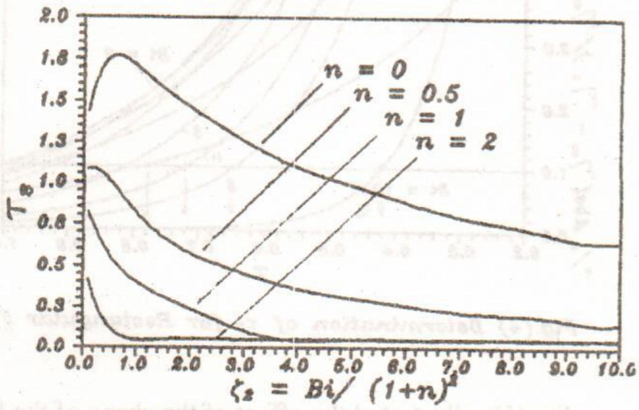


Fig.(7) Effect of ζ_2 on Values of τ_s (Case 2)

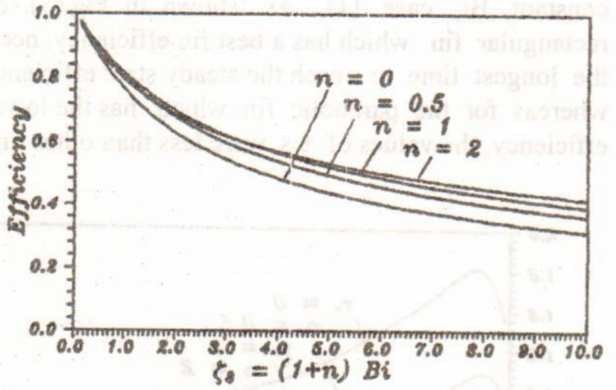


Fig.(8) Steady State Fin Efficiency (Case 3)

For both the rectangular and parabolic fin the values τS approach (0.2) for the fin parameter greater than (3).

In the third case the behavior in some what differs from above cases. For fin parameter greater than (2) the efficiency of the parabolic fin is higher than other shapes which was in reverse with the former cases. Parabolic fin has the lowest values of τS, the magnitude of these values were found to be between the magnitude of case (1) and case (2).

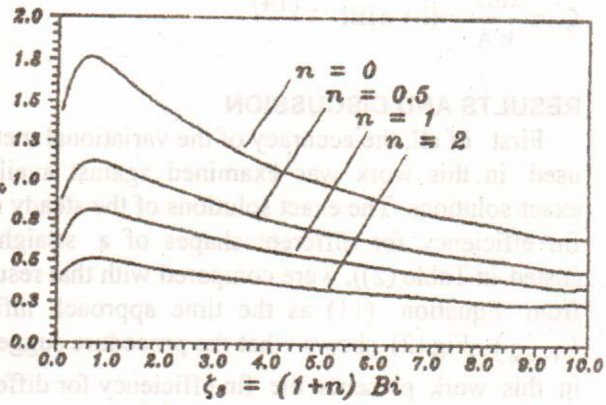


Fig.(9) Effect of ζ_3 on Values of τ_s (Case 3)

CONCLUSIONS

The partial differential equation derived from the transient state heat balance of straight fins has been solved using Kantrovich variational method. With acceptable accuracy of approximation, the results indicate that the parabolic fin have lower efficiency if it is compared with other shapes, at the same thickness and length, case (1) or at the same profile area and thickness, case (2).

For case (3) the parabolic fins have higher efficiency when the fin parameter exceeding roughly 920.

It has been shown that the rectangular fins for the three cases, need longer time to reach the steady state temperature profile and so the steady state efficiency. The parabolic fins have always the shorter time.

The fin design should be related to economical considerations. After the selection of fin material, for a given environment, the designer should choose the most suitable fin. For example, under some geometric requirements, the fin length may be limited, the selection must be based on the results shown in Figs.(8, 9), as follows:

Independent of the fin parameter the parabolic fin, as shown in Fig.(8), needs less time to reach the steady state conditions. Now if the fin parameter were less than (2) the rectangular fins have the best efficiency, but still have a longer time, τS, than the parabolic fins. In otherwise, if the fin parameter was greater than (2) the parabolic fin must b chosen which has the best efficiency and the shorter time, τS, and also has the same weight (same profile area) of other fins.

NOMENCLATURE

Bi'	Biot number, hL/k	[-]
Bi	Modified Biot number $Bi' 2L/\delta_o$	[-]
C_p	Heat capacity of fin	J/kg K
Fo	Fourier number, $\alpha_t / L^2 = \tau$	[-]
h	Convection heat transfer coefficient	$W/m^2 K$
k	Thermal conductivity	W/m K
L	Fin length	m
n	Constant for fin shape definition	[-]
P	Fin perimeter	m
t	Time	s
x	Dimensionless distance	[-]
y	Distance from fin edge	m
$\delta(x)$	Fin thickness	m
$\bar{\delta}$	Dimensionless fin thickness	[-]
δ_o	Fin thickness at the base	m
η	Fin efficiency	[-]
θ	Temperature excess of fin over environment	K
ρ	Fin density	kg/m^3
τ	Dimensionless time	[-]
$\Im_i$	Time dependent functions, Eq. (4)	[-]
ζ	Fin parameter	[-]

REFRERENCES

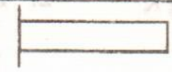
1. Gardener, K. A., "Efficiency of Extended Surface" Trans. ASME, 67, 621-631, 1945.
2. Eckert, E. R. G. and Dark, R. M., "Analysis of Heat and Mass Transfer" 94-95. Mcgrow-Hill, New Tork, 1972.
3. Schmidt, E. "Die Warmeubertraung Durch Rippen", Z. Ver. Dt. Ing. 70, 885-889, 947-951, cited in Reference (4), 1926.
4. Duffin, J. and Mclain, D. K., "Optimum Shape of a Cooling Fin on Convex Cylinder", J. Math. Mech, 17, 769-784, 1968.
5. Champan, A. J., "Transient Heat Conduction in Annular Fins of Uniform Thickness", Chem. Eng. Symp., 55, 195-201, 1989.
6. Jang, J. W., "Periodic Heat Transfer in Straight Fins", Trans, ASME, J. Heat Transfer, 94, 310-314, 1972.
7. Hsu, C. Y., Shyong, C. Y. and Chun, H. C., "A New Approach to the Transient Conduction in a 2-D Rectangular Fin", Int. J. Heat & Mass Transfer, 32, 1657-1661, 1989.
8. Kantorovich, L. V. and Kryiov, V. I., "Approximate Methods of Higher Analysis", NY-Groningen (Holland), Interscience-Noordhoff, 1958.

APPENDIX

Table (1) Eigenvectors and Eigenvalues of Equation (9)

$K_{01} = \varepsilon_1 \gamma / \alpha$	$K_{02} = \varepsilon_2 \gamma / \alpha$
$\Delta K = K_{01} - K_{02}$	
$K_{11} = (-K_{01} + \Delta K / (1 + U_2)) / (1 - (1 + U_1)/(1 + U_2))$	
$K_{12} = (-\Delta K + K_{11}(1 + U_1)) / (1 + U_2)$	
$K_{21} = -U_1 K_{11}$	$K_{22} = -U_2 K_{12}$
$\lambda_i = -\beta/2 \pm \sqrt{\beta^2 - 4\alpha}$	
$\beta = (B_1 C_2 + D_1 A_2 - B_2 C_1 - D_2 A_1)/\gamma$	
$\alpha = B_1 A_2 - B_2 A_1$	$\gamma = D_1 C_2 - D_2 C_1$
$\varepsilon_1 = E_2 B_1 - E_1 B_2$	$\varepsilon_1 = E_2 B_1 - E_1 B_2$
$A_1 = 4/M_3 + (8/15)Bi$	$A_2 = 4M_1/M_{35} + (8/105)Bi$
$B_1 = 4M_1/M_{35} + (8/105)Bi$	$B_2 = 4(M^2 + 4M + 1)/M_{357} + (8/105)Bi$
$C_1 = 8/M_{135}$	$C_2 = 8/M_{357}$
$D_2 = 8/M_{579}$	$D_1 = 8/M_{357}$
$E_1 = (2/3)Bi$	$E_1 = (2/15)Bi$
$U_i = -(A_1 C_1 \lambda_i)/(B_1 + D_1 \lambda_i)$	
$M_i = n + 1$	$M_{ijk} = M_i M_j M_k$

Table (2) Exact Solutions of Steady State Fin Efficiency [2,8]

Fin Type, n	Fin Shape	Fin Efficiency, η
Rectangular, 0		$\frac{1}{\sqrt{Bi}}\tanh\sqrt{Bi}$
Hyperbolic, 0.5		$\frac{1}{\sqrt{Bi}}\frac{I_{2/3}\left(\frac{4}{3}\sqrt{Bi}\right)}{I_{-1/3}\left(\frac{4}{3}\sqrt{Bi}\right)}$
Triangular, 1		$\frac{1}{\sqrt{Bi}}\frac{I_1\left(2\sqrt{Bi}\right)}{I_0\left(2\sqrt{Bi}\right)}$
Parabolic, 2		$\frac{2}{1+\sqrt{1+4Bi}}$