(Short communications)

Iraqi Journal of Chemical and Petroleum Engineering
Vol.12 No.2 (June 2011) 18 – 20
ISSN: 1997-4884

IJCPE
Iraqi Journal of Chemical and Petroleum Engineering

University of Baghdad
College of Engineering

A PARTICULAR SOLUTION OF THE TWO AND THREE DIMENSIONAL TRANSIENT DIFFUSION EQUATIONS

Adel Al-Hemiri

Chemical Engineering Department – University of Baghdad

ABSTRACT

A particular solution of the two and three dimensional unsteady state thermal or mass diffusion equation is obtained by introducing a combination of variables of the form,

$\eta = (x+y) / \sqrt{ct}$, and $\eta = (x+y+z) / \sqrt{ct}$, for two and three dimensional equations respectively. And the corresponding solutions are,

$$\theta(t,x,y) = \theta_0 \ \text{erfc} \frac{x+y}{\sqrt{8ct}} \quad \text{and} \quad \theta(t,x,y,z) = \theta_0 \ \text{erfc} \left(\frac{x+y+z}{\sqrt{12ct}}\right)$$

Keywords: Two and three dimensional equations, Particular solution.

INTRODUCTION

The unsteady state two and three dimensional diffusion equations may be solved by the method of Fourier transform (separation of variables) or by numerical methods to obtain a general solution. However both these methods leads to a double or triple series of the characteristic function and two or three separate expansion problems- depending, of course, on whether the equation is two or three dimensional- which is a difficult task that requires rigorous calculations[1,2,3]. Thus in this work we present a solution easily obtained using the method of combination of variables in the same manner it was used to solve the one dimensional equation:

$\partial\theta/\partial t = c\ (\partial 2\theta/\partial x^2)$,

Where, the parameter $\eta = x/\sqrt{ct}$, giving the solution,

$\theta = \theta_0\ \text{erf}\ (x/\sqrt{4ct})$.

MATHEMATICAL TREATMENT

Consider the two dimensional transient equation,

$$\frac{\partial\theta}{\partial t} = c\left(\frac{\partial 2\theta}{\partial x^2} + \frac{\partial 2\theta}{\partial y^2}\right) \qquad (1)$$

will be solved for an initial value of the function θ equal to zero, i.e.

$$\theta\ (0,x,y) = 0 \tag{2}$$

Introducing the combined variable,

$$\eta = \frac{x+y}{\sqrt{ct}} \tag{3}$$

Differentiating η with respect to t, x and y to find the equivalent forms of $\frac{\partial\theta}{\partial t}$, $\frac{\partial 2\theta}{\partial x^2}$ and $\frac{\partial 2\theta}{\partial y^2}$ in terms of η, thus:

$$\frac{d\eta}{dt} = -\frac{1}{2}\frac{x+y}{\sqrt{ct}} = -\eta/2t \tag{4}$$

$$\therefore\ \frac{\partial\theta}{\partial t} = \frac{d\theta}{d\eta}.\frac{d\eta}{dt} = -\frac{\eta}{2t}\frac{d\theta}{d\eta} \tag{5}$$

$$\text{And } \frac{d\eta}{dx} = \frac{1}{\sqrt{ct}}\ ;\ \frac{d\eta}{dy} = \frac{1}{\sqrt{ct}} \tag{6}$$

Therefore,

$$\frac{\partial 2\theta}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial\theta}{\partial x}\right) = \frac{d}{d\eta}.\frac{d\eta}{dx}\left(\frac{d\theta}{d\eta}.\frac{d\eta}{dx}\right)$$

$$= \frac{1}{ct}\ \frac{d2\theta}{d\eta^{\ 2}} \tag{7}$$

Similarly,

$$\frac{\partial 2\theta}{\partial y^2} = \frac{1}{ct}\ \frac{d2\theta}{d\eta^{\ 2}} \tag{8}$$

Substituting equations 5, 7 and 8 into equation 1 gives,

$$\frac{d2\theta}{d\eta^{\ 2}} + \frac{\eta}{4}\ \frac{d\theta}{d\eta} = 0 \tag{9}$$

By putting $P = \frac{d\theta}{d\eta}$ and hence

$$\frac{dP}{d\eta} = \frac{d2\theta}{d\eta^{\ 2}}$$

This yields,

$$\frac{dP}{d\eta} + \eta\ P = 0 \tag{10}$$

Therefore,

$$P = A\ \exp\left(\frac{-\eta^2}{8}\right) = \frac{d\theta}{d\eta} \tag{11}$$

Where A is constant of integration. Integrating equation (11) gives,

$$\theta = A\,\mathrm{erf}\left(\frac{\eta}{\sqrt{8}}\right) + B \tag{12}$$

Applying the initial conditions, [equation (2)] leads to,

$$0 = A\ \mathrm{erf}\ (\infty) + B$$

But, erf (∞) = 1

Therefore, A = -B. Then the solution becomes,

$$\theta = B\ \left[1 - \mathrm{erf}\left(\frac{\eta}{\sqrt{8}}\right)\right]$$

$$\theta = B\ \mathrm{erfc}\left(\frac{\eta}{\sqrt{8}}\right) \tag{13}$$

The constant B may be found for a specified boundary condition. But, for convenience it is assigned a value of constant distribution, θ_0 . Then the final solution after substituting for η is,

$$\theta(t,x,y) = \theta_0\ \mathrm{erfc}\frac{x+y}{\sqrt{8ct}} \tag{14}$$

Similar procedure is applied to the three dimensional equation,

$$\frac{\partial\theta}{\partial t} = c\left(\frac{\partial 2\theta}{\partial x^2} + \frac{\partial 2\theta}{\partial y^2} + \frac{\partial 2\theta}{\partial z^2}\right) \tag{15}$$

with,

$$\eta = \frac{(x+y+z)}{\sqrt{ct}} \tag{16}$$

to give the solution,

$$\theta(t,x,y,z) = \theta_0\ \mathrm{erfc}\ \left(\frac{x+y+z}{\sqrt{12ct}}\right) \tag{17}$$

CONCLUSIONS

- The particular solution obtained, here, is useful for in problems of transient heat and mass transfer in multi-dimensions.

- The solution may be used for problems of heat and mass diffusion in a flowing fluid through a conduit, described by the equation,

$$\nu_z \ \frac{\partial\theta}{\partial t} = c\left(\frac{\partial 2\theta}{\partial x^2} + \frac{\partial 2\theta}{\partial y^2}\right)$$

Where ν_z is the average velocity along the length of the conduit.

- The solution is also useful for statistical equations and in problems of stochastic nature such as Brownian motion.

- In applying the suggested solution the dimensions x, y and z should be normalized to vary between zero and unity and where the second derivative exist.

NOMENCLATURE

A, B: Constant of integration.

c: Diffusion parameter.

erf : Error function.

erfc: Complimentary error function.

P: $\frac{d\theta}{d\eta}$

t: Independent variable (time).

ν_z: Velocity in z-direction.

x,y,z: Independent variables (linear dimensions).

η: Combined variable, defined by eq. 3.

θ: Dependant variable.

REFERENCES

1- Wylie, C.R. and L.C. Barrett, (1995), "Advanced Engineering Mathematics", 6th Edition, McGraw-Hill.

2- Jenson, V.G. and G.V. Jeffreys, (2001), "Mathematical Methods in Chemical Engineering", 4th Edition, Academic Press.

3- Bird, R.B.; W.E. Stewart and E.N. Lightfoot, (2002), "Transport Phenomena", 2nd Edition, John Wiley.

4- Kreyszig, E., (1998), "Advanced Engineering Mathematics", 8th Edition, John Wiley.

5- Dennis, G.Z. and M.R. Cullen, (1999), "Advanced Engineering Mathematics", 3rd Edition, Jones and Bartlett