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Smallest Number of Sensors for k-Covering

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Abstract:
This paper presents some theoretical results on the smaller number $N_k(a,b)$ of sensors to achieve k coverage for the rectangular area $[0,a]\times[0,b]$. The first properties show the numbers $N_k(a,b)$ are sub-additive and increasing on each variable. Based on these results, some lower and upper bounds for $N_k(a,b)$ are introduced. The main result of the article proves that the minimal density of sensors to achieve k-coverage is $\lambda(k)\leq k/2$ improving a previous result of Ammari and Das [2]. Finally, the numbers $N_1(a,b)$ are tabled for some small values of a,b.
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1 Introduction

Wireless Sensor Networks (WSN) consist of a large number numbers of sensors distributed uniformly in a target area, which monitor in a cooperative manner the physical word. Sensors are in fact small devices capable of sensing variations in temperature, light, gas, motion *etc*, computing and storing information and communicating with the neighbour sensors. The technology has nowadays made it possible to produce these devices at a cheap price so that WSN networks are now involved in various applications from agriculture, surveillance, asset tracking, health care to building safety and evacuation.

An important aspect in WSN applications is the coverage problem, which investigates how well the target area is monitored by sensors. If each point of the area is covered by at least k sensors then the WSN network is said to be k-covered, where k is the degree of the coverage. For example, tracking WSN networks are at least 3-covered as they use triangulation. Moreover, most WSN networks must be at least 2-covered to assure the robustness property.

It is clear that the bigger the coverage degree is, the more sensors must be used in WSN networks. So far, all research works on the WSN coverage problem have assumed that a large number of sensors is distributed in the target area to assure the network is k-covered. However, no information has been given about the number of sensors to use for the target area or equivalently about the sensor density. This article comes to investigate some properties of the minimal number of sensors and to provide un upper bound for the minimal density of sensors to achieve k-coverage in a rectangular area.

1.1 Problem Statement

Consider a set of sensors $S=\{s_1,s_2,s_3,...,s_n\}$ in the 2D plane with the same sensing range r. The position of each sensor s_i is known and given by the coordinates (x_i,y_i). The target area A to monitor can be of any shape but for simplicity it is considered to be rectangular $A=[0,w]\times[0,h]$ of width w and height h.

Definition 1. A point $(x, y) \in A$ is covered by a sensor s_i if $\sqrt{(x_i - x)^2 + (y_i - y)^2} \leq r$. The target area A is k-covered by the sensors S if each point $(x, y) \in A$ is covered by at least k different sensors.

It is clear that the number of sensors n to achieve k-coverage increases directly with k. Hence, the bigger is k the more sensors are needed for k-coverage. So far, the main assumption has been that sensors are cheap devices and the numbers of them to deploy is not important. Consequently, it has been considered that the number of sensors to deploy is big enough to achieve k-coverage on the target area.

1.2 Related Works

Investigating 1-coverage or circle packing has been researched as geometrical combinatorial since 1939. Researchers have tried to find mathematical equations for the optimal 1-covering or even to prove that some configurations are optimal. For example Kershner [4] investigated the problem of covering any 2D set of points with similar circles based on some geometrical combinatorial techniques. This early work proved that the minimal number of circles $N(\varepsilon)$ of radius ε to cover a close set of point M satisfies

$$\lim_{\varepsilon \to 0} N(\varepsilon) = \frac{2\sqrt{3}}{9 \cdot \varepsilon^2} \cdot |M|,$$

where $|M|$ denotes the area of closed by M. The result was proven by using a double inequality for the quantity $\pi\varepsilon^2 N(\varepsilon)$ representing the total area covered by the circles. An important consequence of this result is that the proportion of unavoidable overlapping can be approximated by $\frac{2\pi\sqrt{3}}{9} \simeq 1.209$. We can also mention the early work of Verblunsky [10] who proved that the minimum number $N(l)$ of circles of radius 1 to cover a square of length l should satisfy

$$N(l) \geq \frac{2l^2 + l}{3\sqrt{3}}.$$

These two results come to suggest that the sensor density for 1-coverage can be estimated by $\frac{2\sqrt{3}}{9} \approx 0.384$.

However, these early works [4], [10] do not provide any information about the pattern of circles used to achieve the minimal coverage. Recently, several articles on circle packing problems investigated efficient ways to cover a rectangle with similar circles (see [6], [8] or [9] amongst others). These geometrical combinatorics researches confirmed that optimal packing is difficult to be achieve even for small number of circles. Furthermore, no pattern was detected for the packing configuration that gives optimality.

Recently, several papers have dealt with the k-coverage problems in the context of sensor networks studying conditions when this is achieved or algorithms to detected when this happens. Some of these contributions have made marginal reference to the minimum number or equivalently to the minimum density of sensors that assures k-covering of a given area. Generally, all these works have considered that the number of sensors to use is big enough to k-cover the target area. Adlakha and Srivasyava [1] developed an exposure-based model to find the sensor density required to achieve full coverage of a given area. They proved that the number of sensors to achieve 1-covering is in the order of $O(A/r^2)$, where A is the area to cover however they did not provide any constant for the magnitude of A/r^2. Ammari and Das [2] investigated the problem of k-coverage proposing a condition to achieve it. They considered the target area divided in "Reuleaux" triangles which are formed by the intersection of 3 circles. The main result of their work states that the target area is k-covered if and only if each "Reuleaux" triangle contains at least k-active sensors. Another important results proposed by Ammari and Das gives that the minimal density of sensors to guarantee k-coverage is $\lambda(r,k) = \frac{2k}{(\pi-\sqrt{3})\cdot r^2} = \frac{1.4188\cdot k}{r^2}$, where r is the radius of the sensing disks.

Most covering problems present huge difficulties to solve or to derive a polynomial algorithm even in particular cases like regular or simple shapes and lower dimensional space. The 2D problem of covering a bounded domain with arbitrary shaped objects was proven to be exponential on the size of the packing space [7]. The particular case of covering any polygon with n similar disks is known to be NP-hard [5]. Consequently, the problem of finding the least number of disks to k-cover a rectangle is NP-hard.

All these works have shown that calculating the minimal number of circles to pack a rectangle is a hard problem and there is no patter associated with this covering. Moreover, the results concerning the minimal number of circles for k-coverage are all either asymptotic based on some limits or approximative based on some inequalities.

2 Minimal Number of Sensors

Consider the following problem *"Find the smallest number of sensors $N_k(a,b)$ that should be used to achieve k-coverage for a rectangular area of sizes a and b with sensors of the same radius"*. We can suppose that all the sensors have the coverage radius of 1 unit. By convention $N_k(a,b) = 0$ when $a \leq 0$ or $b \leq 0$. It is clear that a k-coverage with n sensors satisfies

$$n \geq N_k(a,b). \tag{1}$$

The following results can be directly obtained based on Equation 1 and on the definition of $N_k(a,b)$.

Lemma 1. The function $N_k(a,b)$ is symmetrical on a, b

$$N_k(a,b) = N_k(b,a), \quad \forall a,b > 0.$$

Lemma 2. The function $N_k(a,b)$ is monotonically on each variable:

$$a_1 \leq a_2 \Rightarrow N_k(a_1,b) \leq N_k(a_2,b).$$

$$b_1 \leq b_2 \Rightarrow N_k(a,b_1) \leq N_k(a,b_2).$$

$$k_1 \leq k_2 \Rightarrow N_{k_1}(a,b) \leq N_{k_2}(a,b).$$

Lemma 3. The function $N_k(a,b)$ is sub-additive on each variable:

$$N_k(a_1+a_2,b) \leq N_k(a_1,b) + N_k(a_2,b).$$

$$N_k(a,b_1+b_2) \leq N_k(a,b_1) + N_k(a,b_2).$$

$$N_{k_1+k_2}(a,b) \leq N_{k_1}(a,b) + N_{k_2}(a,b).$$

Proposition 1. $N_1(\sqrt{2}\cdot n, \sqrt{2}\cdot m) \leq n\cdot m$ when $n,m \in N$.

Proof: Consider that the rectangular area of sizes $a = \sqrt{2}\cdot n,\ b = \sqrt{2}\cdot m$ is divided into a grid of $n\times m$ squares of size $\sqrt{2}$. Each square can be 1-covered by its circle as Figure 2 shows. Hence, $N_1(\sqrt{2}\cdot n, \sqrt{2}\cdot m) \leq m\cdot m$ since there is a 1-coverage with $n\cdot m$ circles. □

Evidence shows that $N_1(\sqrt{2}\cdot n, \sqrt{2}\cdot m) = m\cdot m$ for many values of $m,\ n$. However, we have not been able to produce a coherent proof of the fact that $N_1(\sqrt{2}\cdot n, \sqrt{2}\cdot m) \geq n\cdot m$ nor a counterexample to show that $N_1(\sqrt{2}\cdot n, \sqrt{2}\cdot m) < n\cdot m$ for some values of m,n.

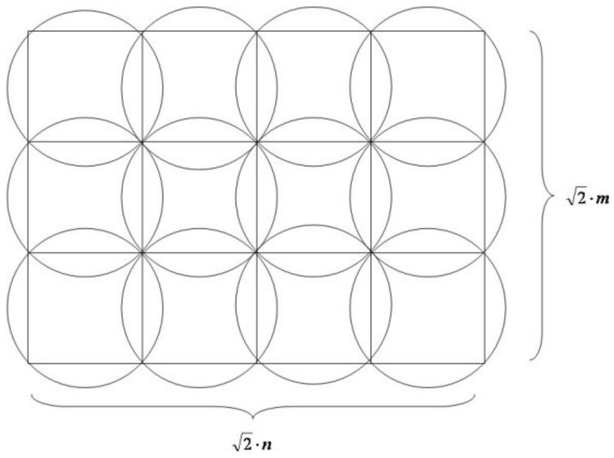

Figure 1: 1-Covering of the Rectangle $w = \sqrt{2} \cdot n, h = \sqrt{2} \cdot m$.

Proposition 2. The numbers $N_1(a,b)$ satisfy the following inequality

$$N_1(a,b) \leq \lceil \frac{a}{\sqrt{2}} \rceil \cdot \lceil \frac{b}{\sqrt{2}} \rceil, \quad \forall a,b \in R \tag{2}$$

where $\lceil x \rceil$ is the ceiling function.

Proof: Consider $n = \lceil \frac{a}{\sqrt{2}} \rceil \in N$ so that we have $\frac{a}{\sqrt{2}} \leq n$ or $a \leq n \cdot \sqrt{2}$. Similarly, if $m = \lceil \frac{b}{\sqrt{2}} \rceil \in N$ we obtain $b \leq m \cdot \sqrt{2}$. Now, the following inequality can be derived based on Lemma 1

$$N_1(a,b) \leq N_1 \left(n \cdot \sqrt{2}, m \cdot \sqrt{2} \right) \Rightarrow$$

$$N_1(a,b) \leq n \cdot m \Rightarrow N_1(a,b) \leq \lceil \frac{a}{\sqrt{2}} \rceil \cdot \lceil \frac{b}{\sqrt{2}} \rceil,$$

which it proves the theorem. □

The result above gives only an upper bound of values in which the number $N_1(a,b)$ can be located.

Theorem 1. For k-coverage problem, the numbers $N_k(a,b)$ satisfy the following inequality

$$\frac{k \cdot a \cdot b}{\pi} \leq N_k(a,b) \leq k \cdot \lceil \frac{a}{\sqrt{2}} \rceil \cdot \lceil \frac{b}{\sqrt{2}} \rceil, \quad \forall a,b \in R \tag{3}$$

Proof: The sub-additivity property is used as follows

$$N_k(a,b) = N_{1+...+1}(a,b) \leq N_1(a,b) + ... + N_1(a,b) =$$

$$= k \cdot N_1(a,b) \leq k \cdot \lceil \frac{a}{\sqrt{2}} \rceil \cdot \lceil \frac{b}{\sqrt{2}} \rceil,$$

which proves the right hand side inequality. For the left hand side we considered that each point of the rectangle is covered by at least k circles. Hence, the $N_k(a,b)$ circles cover the whole rectangle surface by k times. Hence, the area of the circles is greater than k times the area of the rectangle.

$$N_k(a,b) \cdot \pi \cdot 1^2 \geq k \cdot a \cdot b \Rightarrow N_k(a,b) \geq \frac{k \cdot a \cdot b}{\pi}.$$

□

Assume that the minimal density of sensors $\frac{N_k(a,b)}{a \cdot b}$ does not depend on a, b for large values of a, b, so that we can write $\lambda(k) \simeq \frac{N_k(a,b)}{a \cdot b}, \forall a, b$. In this case the minimal density of sensors can be evaluated by the following result.

Theorem 2. The minimum density of sensors to achieve k-covering for rectangular areas satisfies

$$\frac{k}{\pi} \leq \lambda(k) \leq \frac{k}{2}. \tag{4}$$

Proof: Each member of Equation 3 is divided by $a \cdot b$ to obtain

$$\frac{k}{\pi} \leq \frac{N_k(a,b)}{a \cdot b} \leq \frac{k \cdot \lceil \frac{a}{\sqrt{2}} \rceil \cdot \lceil \frac{b}{\sqrt{2}} \rceil}{a \cdot b} \Rightarrow \frac{k}{\pi} \leq \frac{N_k(a,b)}{a \cdot b} \leq k \cdot \frac{\lceil \frac{a}{\sqrt{2}} \rceil}{a} \cdot \frac{\lceil \frac{b}{\sqrt{2}} \rceil}{b}.$$

Consider that the minimum density to achieve k-covering is independent of the area to cover hence it can be denoted by $\lambda(k)$. So that we have

$$\frac{k}{\pi} \leq \lambda(k) \leq k \cdot \frac{\lceil \frac{a}{\sqrt{2}} \rceil}{a} \cdot \frac{\lceil \frac{b}{\sqrt{2}} \rceil}{b}, \quad \forall a, b > 0.$$

If $a, b \to \infty$ are big then the fractions become $\lim_{a \to \infty} \frac{\lceil \frac{a}{\sqrt{2}} \rceil}{a} = \frac{1}{\sqrt{2}}$ and $\lim_{b \to \infty} \frac{\lceil \frac{b}{\sqrt{2}} \rceil}{b} = \frac{1}{\sqrt{2}}$ so that $\frac{k}{\pi} \leq \lambda(k) \leq \frac{k}{2}$. □

Theorem 2 shows that the minimal density to achieve k-coverage with sensors of radius 1 is between $0.318109 \cdot k$ and $0.5 \cdot k$. The first conclusion we can extract is that this number is far smaller than the density proposed in [2] which is $1.4188 \cdot k$. This huge difference would raise serious question marks on the "Reuleaux" triangulation approach developed by Ammari and Das and hence on the results they proposed. The second conclusion is that, in particular for $k = 1$, this result states that the density for 1-covering is between 0.318 and 0.5, which is in concordance with the early results of Kershner and Verblunsky.

On the other hand, the minimal density of sensors $\frac{N_k(a,b)}{a \cdot b}$ can also have the following upper bound for any $a, b \geq 2$.

$$\frac{N_k(a,b)}{a \cdot b} \leq k \cdot \frac{\lceil \frac{a}{\sqrt{2}} \rceil}{a} \cdot \frac{\lceil \frac{b}{\sqrt{2}} \rceil}{b} \leq k \cdot \frac{\frac{a}{\sqrt{2}} + 1}{a} \cdot \frac{\frac{b}{\sqrt{2}} + 1}{b} \Rightarrow$$

$$\frac{N_k(a,b)}{a \cdot b} \leq k \cdot \left(\frac{1}{\sqrt{2}} + \frac{1}{a} \right) \cdot \left(\frac{1}{\sqrt{2}} + \frac{1}{b} \right) = k \cdot \left[\frac{1}{2} + \frac{1}{\sqrt{2}} \cdot \left(\frac{1}{a} + \frac{1}{b} \right) + \frac{1}{a \cdot b} \right] \Rightarrow$$

$$\frac{N_k(a,b)}{a \cdot b} \leq \frac{k}{2} \cdot \left[1 + \frac{P + 2\sqrt{2}}{A} \right],$$

where P and A are the perimeter and the area of the target region respectively. This provides an upper bound for the density based on the perimeter and the surface of the target area.

3 Some Computational Results

This section is to find directly or using some computation some of the numbers $N_k(a,b)$. Firstly, we start with the numbers $N_1(a,b)$, which can be calculated for several small values of a, b. For example, $N_1(\sqrt{2}, \sqrt{2}) = 1$ and furthermore $N_1(a,b) = 1, \ \forall a, b \leq \sqrt{2}$. This simple case can be extended to the situation where we have a row of sensors to achieve minimal 1-coverage (see Figure 3).

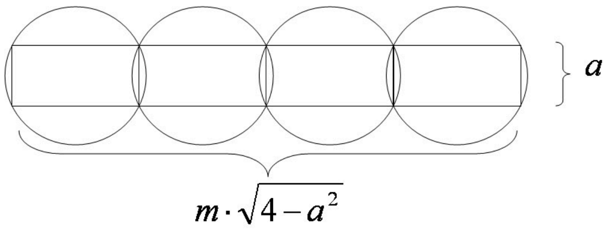

Figure 2: Optimal 1-Covering of the Rectangle with $a \leq \sqrt{2}$.

Theorem 3. The following two results can apply for the situation when the rectangle has either the width or the height less than $\sqrt{2}$ (see Figure 3):

$$N_1(a, \sqrt{4-a^2} \cdot m) = m, \forall m \in N, \quad a \leq \sqrt{2}$$

$$N_1(\sqrt{4-b^2} \cdot n, b) = n, \forall n \in N, \quad b \leq \sqrt{2}.$$

Proof: The proof only considers the case when $a \leq \sqrt{2}$ as the second one is similar. Figure 3) shows a 1-covering of the rectangle with m sensors so that $N_1(a, \sqrt{4-a^2} \cdot m) \leq m$. Lets start from an 1-covering with $N_1(a, \sqrt{4-a^2} \cdot m)$ sensors of a target area with the sizes $w = a, h = m \cdot \sqrt{4-a^2}$. Consider the rectangle is divided into m small rectangles $R_1, R_2, ..., R_m$ each of sizes $w = a, h = \sqrt{4-a^2}$. The focus is now on R_1 which is fully covered with some circles from which there is one with the smallest x coordinate for the centre. This circle is then translated so that it will fit into the whole rectangle. It is clear that the area of R_1, previously covered by some circles, is now covered by one circle. Hence, this new configuration is still a 1-coverage with the same number of sensors. Now, R_2 must have at least one circle to cover the nodes in common with R_1 so that we can use it to repeat the same type of transformation. After m steps we find that there are m circles amongst the $N_1(a, \sqrt{4-a^2} \cdot m)$ circles that can be positioned as in Figure 3. Hence, $N_1(a, \sqrt{4-a^2} \cdot m) \geq m$. $\square$

This theorem provides directly the following two consequences.

Remark 3.1. $N_1(1, m) = \lceil \frac{m}{\sqrt{3}} \rceil, \forall m \in N$.

Remark 3.2. $N_1(2, m) \leq \lceil \frac{m}{\sqrt{2}} \rceil + \lceil \frac{m}{4\sqrt{2}-2} \rceil, \forall m \in N$.

For the second remark it is clear that $N_1(2, m) = N_1(\sqrt{2} + 2 - \sqrt{2}, m) \leq N_1(\sqrt{2}, m) + N_1(2 - \sqrt{2}, m) = \lceil \frac{m}{\sqrt{2}} \rceil + \lceil \frac{m}{4\sqrt{2}-2} \rceil$. It seems that $\lceil \frac{m}{\sqrt{2}} \rceil + \lceil \frac{m}{4\sqrt{2}-2} \rceil$ represents the value of $N_1(2, m)$ for several small values of $m = 1, 2, 3, 4$. However, there is no proof to show that $N_1(2, m) = \lceil \frac{m}{\sqrt{2}} \rceil + \lceil \frac{m}{4\sqrt{2}-2} \rceil$. These simple results help in tabling the numbers $N_1(n, m)$ for some small values of $n, m \in N$ when one of the indices is 1 or 2 (see Table 1).

n,m	m=1	m=2	m=3	m=4	m=5
n=1	1	2	2	3	3
n=2	2	4	4	6	7
n=3	2	4	6	7	11
n=4	3	6	7	9	11
n=5	3	7	11	11	14

Table 1: Table with the Values $N_1(n, m), n, m = 1, 2, 3, 4, 5$

The values of $N_k(n,m)$ for big n,m are not simple to calculate since generating a k-coverage is NP-complete. A simple generic computation for $N_k(n,m)$ has to go firstly through all the values between $nr = n \cdot m \cdot k/\pi, n \cdot m \cdot k/2$ in order to generate all the possible configurations of nr circles. Secondly, the k-coverage property should be tested for each configuration of nr circles. If the property holds for a particular configuration of nr circles then $N_k(n,m) = nr$ (see Algorithm 1). Testing whether a set of sensors or configuration of circles achieves k-coverage is a well studied problem with few polynomial solutions (see [3] for an $O(nr \log nr)$ solution). However, the problem of generating all the configurations with nr circles within the target area $[0,n] \times [0,m]$ is computationally hard and it can be solved only by using searching methods like backtracking. The running time to search exhaustively for the optimal configuration can be in this case very big but the algorithm provides the correct value for $N_k(n,m)$.

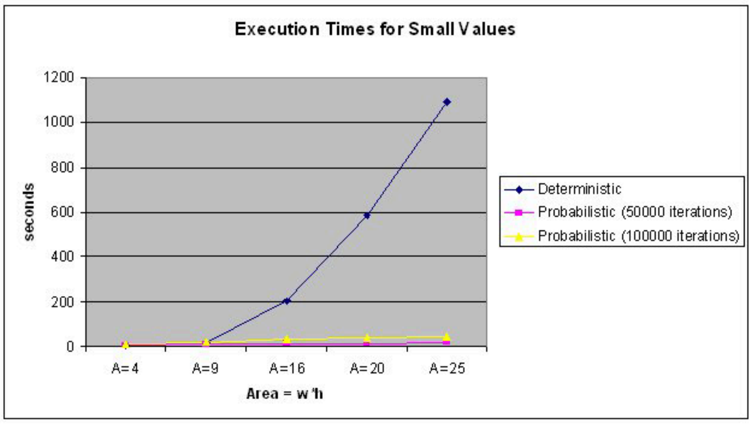

Figure 3: Execution Times for Deterministic and Probabilistic Approaches.

Algorithm 1 Generic Scheme to Calculate $N_k(n,m)$.

function(n,m,k)
for $nr = \frac{n \cdot m \cdot k}{\pi}$ to $\frac{n \cdot m \cdot k}{2}$ **do**
  // generate all the possible nr circles
  repeat
    generate a new configuration with nr circles
    test k-coverage for the circles
    if k-coverage holds **then**
      return nr;
    end if
  until possible
end for

The alternative to this approach is to generate a very large number of random configurations hoping that the k-coverage with $N_k(n,m)$ circles is reached by one of them. In this case the number $N_k(n,m)$ is not accurately computed but the execution time can substantially be reduced. The following simulations give a good illustration of the trade off between accuracy and running time. Figure 4 presents the execution times for the deterministic algorithm against the probabilistic approach with 50000 and 100000 iterations. One can see that the execution times of the deterministic algorithm grew at an exponential rate when $w \cdot h$ increases. The execution for the small area of $w = 5, h = 5$ took more than 18 minutes. On the other hand, the execution times for the probabilistic approaches have a slow increasing rate far smaller than in the deterministic case. Moreover, the probabilistic version with 100000 iterations provided the same results as the deterministic solution.

4 Conclusions

This article has investigated some theoretical properties related with the minimum number of sensors $N_k(a,b)$ to achieve k-covering of a rectangular area. Firstly, the numbers $N_k(a,b)$ have been proven to be sub-additive on each variable. Secondly, we have found a interval of possible values for $N_k(a,b)$ numbers between $\frac{k \cdot a \cdot b}{\pi}$ and $k \cdot \lceil \frac{a}{\sqrt{2}} \rceil \cdot \lceil \frac{b}{\sqrt{2}} \rceil$. Based on that the minimal density of sensors to achieve k-coverage has been proven to be less than $k/2$ improving a result of [2]. Some computation has been used to generate the numbers $N_1(a,b)$ for small values of a, b.

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