INTERNATIONAL JOURNAL OF COMPUTERS COMMUNICATIONS & CONTROL
ISSN 1841-9836, 11(6):845-859, December 2016.

# Stochastic Stability Analysis of Power Control in Wireless Networks via a Norm-inequality-based Approach

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**Abstract:** Owing to the requirements from realistic wireless networks, the stochastic stability analysis for discrete-time power control, which concerns the randomness brought by the fading channels and noise of wireless systems, is of practical significance. By developing a norm-inequality-based framework of analyzing the stochastic stability of linear systems with random parameters, we show that a typical power-control law with linear system model is stable in the sense of the $p$th-moment stability. Several conditions of achieving the $p$th-moment stability for the considered power-control law are obtained, which can easily applied to realistic wireless networks. Besides, within this study, the stability analysis of power control for the first time takes into account the effect of multiple-access methods.
**Keywords:** wireless networks, power control, stochastic stability.

## 1 Introduction

There has been a great deal of research over the past several decades on the power control of wireless networks. These studies span multiple disciplines and include information theory [1, 8], communication [2, 6, 7, 9, 13], and control theory [3–5, 21]. It has already been recognized that the stability analysis of power-control laws can efficiently investigate the intrinsic properties of various power-control algorithms [3–5, 21].

The power-control laws with two-sided scalable iterative (interference) functions are convergent under the condition that an equilibrium exists [1], and the laws with contractive interference functions guarantee the existence and uniqueness of equilibriums along with linear convergence of iterates [2]. The global asymptotic stability of power control laws involving two-sided scalable interference functions and the exponentially scalability of laws with contractive interference functions are seen even under bounded time-varying delays [3]. A general class of power-control laws whose interference functions are monotonic and scalable are considered in [4, 5]. By employing appropriately constructed Lyapunov functions, [5] shows that any bounded power distribution obtained from these laws is uniformly asymptotically stable. Further, in [5] Lyapunov-Razumikhin functions are used to show that, even when the system incorporates time-varying delays, any solution along which the generalized system nonlinearity is bounded must also be uniformly asymptotically stable. In both of above cases the stability is shown to be global. Most of current wireless networks are digital communication systems, in which the link gains are random (stochastic) variables fluctuating as wireless channels of the underlying networks are experiencing fading at all time and the noises are also random variables. It is therefore important for the power-control laws to be designed and verified when considering the impact of randomness in discrete-time wireless systems with fading channels and random noise.

Stochastic power-control algorithm that uses noisy interference estimates (observations) is first studied in [6]. With conventional matched filter receivers, the stochastic power control is shown by [6] to converge to the optimal power vector in the mean square error sense. These results are later extended to the cases when a linear receiver or a decision feedback receiver is used [7, 22]. In [9], a stochastic-approximation based power-control algorithm is proposed to handle both measurement errors and randomness in the channel gain matrix, which is proved to converge to the optimal solution in the mean-squared sense.

Treading the elegant footsteps of recent works [3]- [5], one can gain a deep insight into the stability theory of typical power-control laws in wireless networks. Rather than being concerned with the stability analysis of power control, the studies [6]- [9] focus on developing extra techniques of reducing the impact of randomness encountered by power control in wireless networks, where these techniques includes matched filter receiver, decision feedback receiver, and stochastic approximation. To the best of our knowledge, it still lacks of systemic study on stability analysis of typical power-control laws taking into account the randomness existed in practical wireless networks while the power control does not use extra randomness reducing techniques proposed by [6]- [9]. We shall emphasize here that, this kind of study is indispensable, because, on the one hand, it could attentively reveal the inherent attributes of the typical power-control laws when the randomness exists (but no randomness reducing technique is involved) so as to make the stability theory of typical power-control laws complete, on the other hand, extra technique of reducing randomness may not necessarily be available in practical systems due to the objective factors such as realtime processing demands so that engineers have to be aware of the stability of power control in randomness environments without any randomness reducing technique. Hence, the aim of our work is to perform this study; specifically, we will develop a framework of stochastic stability analysis for discrete-time power control, which takes the randomness brought by the fading channels and noise of wireless systems into consideration.

Our main works are: (i) developing a norm-inequality-based framework of analyzing the stochastic stability (to be specific, the $p$th-moment stability) for linear systems with random parameters so as to investigate the stochastic stability of the power control in consideration of the randomness caused by the fading channels and noise; (ii) clarifying the conditions of achieving the stochastic stability for the considered linear systems and power-control law; and (iii) investigating the effect of multiple-access methods to stochastic stability of power control.

## 2 Notation and preliminaries

### 2.1 Notation

Throughout, the interval $[0\ +\infty)$ is denoted by $\mathbb{R}_+$, and the set of positive integers by $\mathbb{Z}_+$. The non-negative orthant of the $N$-dimensional real space is represented by $\mathbb{R}_+^N$. The vectors are written in **bold** lower case letters and matrices in **bold** capital letters, e.g., $\mathbf{a}$ and $\mathbf{A}$. The $i$th component of a vector $\mathbf{a}$ is denoted by $a_i$, and the $ij$th entry of a matrix $\mathbf{A}$ is denoted by $A_{ij}$ such that $A_{ij} = [\mathbf{A}]_{ij}$ and $\mathbf{A} = [A_{ij}]$. The notation $\mathbf{A} \geq 0$ means that all of the components of $\mathbf{A}$ are greater than or equal to zero. The inequality $\mathbf{A} \geq \mathbf{B}$ implies that $a_{ij} \geq b_{ij}$ for all components $ij$. We let $(\cdot)^T$ denote the transpose of a vector or a matrix. If $\mathbf{a}$ is a vector with components $a_1, a_2, \cdots, a_N$, then its $p$-norm is defined by $\|\mathbf{a}\|_p = \left(\sum_{i=1}^N |a_i|^p\right)^{1/p}$ and its Euclidean norm by $\|\mathbf{a}\|_2 = \left(\sum_{i=1}^N |a_i|^2\right)^{1/2}$ that is actually $p$-norm with $p = 2$. For a square matrix $\mathbf{A}$, the induced norm corresponding to the $p$-norm of vectors is defined as

$$\|\mathbf{A}\|_p = \max_{\|\mathbf{x}\|_p \neq 0} \frac{\|\mathbf{A}\mathbf{x}\|_p}{\|\mathbf{x}\|_p} = \max_{\|\mathbf{x}\|_p = 1} \frac{\|\mathbf{A}\mathbf{x}\|_p}{\|\mathbf{x}\|_p}, \tag{1}$$

where $\|\mathbf{A}\|_1 = \max_j \sum_{i=1}^N |A_{ij}|$ is also known as the maximum column sum matrix norm, and $\|\mathbf{A}\|_\infty = \max_i \sum_{j=1}^N |A_{ij}|$ is the maximum row sum matrix norm.

A probability space is a triple $(\Omega, \mathcal{F}, P)$ where $\Omega$ is a set of "outcomes", $\mathcal{F}$ is a set of "events", and $P : \mathcal{F} \to [0\ \ 1]$ is a function that assigns probabilities to events. If $x$ is a random variable on $(\Omega, \mathcal{F}, P)$ then we define the expected value operator to be $Ex = \int x dP$. If $Ex^2 < +\infty$ then the variance of $x$ is defined to be $\mathrm{var}(x) = E(x - Ex)^2$. We let $\{\boldsymbol{x}[k], k \in \mathbb{Z}_+\}$ denote a stochastic process with random values in a set of vectors, and $\{\boldsymbol{X}[k], k \in \mathbb{Z}_+\}$ denote a stochastic process with random values in a set of matrices, by writing $\boldsymbol{x}[k]$ and $\boldsymbol{X}[k]$ in *italic* and **bold** letters. If $\boldsymbol{x}$ is a $N \times 1$ random vector then we define its expected value as $\bar{\mathbf{x}} = E\boldsymbol{x} \triangleq [Ex_1\ Ex_2 \cdots Ex_N]^T$. Analogously, for a $N \times N$ random matrix $\boldsymbol{X}$, we define its expected value as $\bar{\mathbf{X}} = E\boldsymbol{X} \triangleq [EX_{ij}]$ and its variance as $\mathrm{var}(\boldsymbol{X}) \triangleq [\mathrm{var}(X_{ij})]$.

Let $L^p(\Omega, \mathcal{F}, P)$ be the set of measurable function $f$ on $\Omega$ such that $\int_\Omega |f|^p d\mu < +\infty$, we introduce an operator $E_{L^p}$ as

$$E_{L^p} f \triangleq (E|f|^p)^{1/p}. \tag{2}$$

From [15, 2.2.5 Example], one shall find that $L^p(\Omega, \mathcal{F}, P)$ is a linear space and $E_{L^p}$ is a semi-norm.

## 2.2 Preliminaries

In this part, we collect basic properties and definitions of matrix theory, algebra theory, probability theory, and stochastic stability theory, which will be used in the following analysis. For more details, see, e.g., [10, 11, 23–25].

*Basic norm inequalities [23, 25]:* The $p$-norm of vectors and the corresponding induced norm of square matrices are nonnegative numbers have the properties that

1. $\|\mathbf{x} + \mathbf{y}\|_p \le \|\mathbf{x}\|_p + \|\mathbf{y}\|_p$ and $\|\mathbf{A} + \mathbf{B}\|_p \le \|\mathbf{A}\|_p + \|\mathbf{B}\|_p$;
2. $\|\mathbf{A}\mathbf{x}\|_p \le \|\mathbf{A}\|_p \|\mathbf{x}\|_p$, which is derived from the definition of $\|\mathbf{A}\|_p$;
3. $\|\mathbf{A}\mathbf{B}\|_p \le \|\mathbf{A}\|_p \|\mathbf{B}\|_p$, since $\|\mathbf{A}\mathbf{B}\mathbf{x}\|_p \le \|\mathbf{A}\|_p \|\mathbf{B}\mathbf{x}\|_p \le \|\mathbf{A}\|_p \|\mathbf{B}\|_p \|\mathbf{x}\|_p$ and
$$\max_{\|\mathbf{x}\|_p = 1} \|\mathbf{A}\mathbf{B}\mathbf{x}\|_p = \|\mathbf{A}\mathbf{B}\|_p.$$

*Upper bound of induced matrix norm [17]:* For any $N \times N$ matrix $\mathbf{A}$, the induced norm $\|\mathbf{A}\|_p$ has no explicit representation unless $p = 1, 2$ or $\infty$. However, one can have the below inequalities [17, (1.8),(1.11)]

$$\|\mathbf{A}\|_p \le N^{1-1/p} \|\mathbf{A}\|_1, \tag{3}$$

and

$$\|\mathbf{A}\|_p \le \|\mathbf{A}\|_1^{1/p} \|\mathbf{A}\|_\infty^{1-1/p}, \tag{4}$$

provide two closed-form upper bounds of $\|\mathbf{A}\|_p$ with $p$ other than $1, 2$ and $\infty$.

*Cauchy-Schwarz-Buniakowsky inequality involving real numbers [25]:* Let $a_1, a_2, \cdots, a_N$ and $b_1, b_2, \cdots, b_N$ be any two arbitrary sets of real numbers, then

$$\left( \sum_{i=1}^{N} a_i b_i \right)^2 \le \left( \sum_{i=1}^{N} a_i^2 \right) \left( \sum_{i=1}^{N} b_i^2 \right). \tag{5}$$

This inequality can be expressed in the vector form as $\mathbf{a}^T \mathbf{b} \le \|\mathbf{a}\|_2 \|\mathbf{b}\|_2$, where $\mathbf{a} = [a_1\ a_2\ \cdots\ a_N]^T$ and $\mathbf{b} = [b_1\ b_2\ \cdots\ b_N]^T$.

*Definition 1 [24]:* Let $(\mathcal{S}, \mathcal{S})$ be a measurable space. A stochastic process $\{\mathbf{\Phi}[k], k \in \mathbb{Z}_+\}$ taking values in $\mathcal{S}$ is said to be a Markov chain with respect to a filtration $\mathcal{F}_k$, if $\mathbf{\Phi}[k] \in \mathcal{F}_k$ and for all $B \in \mathcal{S}$, $P(\mathbf{\Phi}[k+1] \in B|\mathcal{F}_k) = P(\mathbf{\Phi}[k+1] \in B|\mathbf{\Phi}[k])$. In words, given the present, the rest of the past is irrelevant for predicting the value of $\mathbf{\Phi}[k+1]$.

Stability properties of stochastic systems need to be established in the context of stochastic stability, in which a variety of inter-related definitions exist [10, 11]. This study concerns the so-called $p$th-moment stability which is borrowed from [11] with trivial differences and defined as follows.

*Definition 2 [11]:* The $p$th-moment stability can be stated as, for each initial distribution, there exists $\lim_{k\to+\infty} E\left(\|\boldsymbol{x}[k]\|_p^p\right) < +\infty$, where $p \in \mathbb{Z}_+$, and it shall hold that $\left(E\left(\|\boldsymbol{x}[k]\|_p^p\right)\right)^{1/p} = E_{L^p}\|\boldsymbol{x}[k]\|_p$.

This study will seek for an analytical framework of deriving the upper bound of $E\left(\|\boldsymbol{x}[k]\|_p^p\right)$ so as to prove the $p$th-moment stability for linear systems with random parameters.

# 3 System model and problem statement

## 3.1 System model of power control

We consider a wireless network with $N$ wireless nodes, which employs a discrete-time power control algorithm given by $\mathbf{x}[k+1] = \boldsymbol{I}(\mathbf{x}[k])$, where $\mathbf{x}[k] = [x_1[k]\ x_2[k]\ \cdots\ x_N[k]]^T$ and $x_j[k] \in \mathbb{R}_+$ is the transmitted power of node $j$ at the $k$th iteration, $\boldsymbol{I}(\mathbf{x}) = [I_1(\mathbf{x})\ I_2(\mathbf{x})\ \cdots\ I_N(\mathbf{x})]^T$ and $I_j : \mathbb{R}_+^N \to \mathbb{R}_+$ is the interference function modeling the interference together with noise measured at the receivers for node $j$ that mainly comes from other nodes and local noise source. Denote the link gain between the transmitter of node $j$ and the receiver for node $i$ by $G_{ij}$.

To perform the study in a systematic fashion, we proceed from a simple but considerably typical law of power control that has a linear system model, helping us avoid any entanglement due to nonlinear effects. This power-control law is given by

$$\boldsymbol{x}[k+1] = \boldsymbol{I}(\mathbf{x}[k]), \tag{6}$$

where $\boldsymbol{I}(\mathbf{x}[k]) = \boldsymbol{D}[k]\left(\boldsymbol{C}[k]\boldsymbol{x}[k] + \boldsymbol{n}[k]\right) = \boldsymbol{D}[k]\boldsymbol{C}[k]\boldsymbol{x}[k] + \boldsymbol{D}[k]\boldsymbol{n}[k]$, $\boldsymbol{D}[k]$ is a $N \times N$ diagonal matrix whose diagonal elements are $\left\{\frac{\gamma_1[k]}{G_{11}[k]}, \frac{\gamma_2[k]}{G_{22}[k]}, \cdots, \frac{\gamma_N[k]}{G_{NN}[k]}\right\}$, in which $\gamma_j[k]$ is the target Signal-to-Interference-and-Noise Ratio (SINR) of node $j$ at the $k$th iteration, and $G_{ij}[k]$ is the link gain $G_{ij}$ at the $k$th iteration. In this study, we set $\gamma_j[k] = \gamma_j$ where $\gamma_j$ is fixed target SINR value for node $j$. $\boldsymbol{C}[k] = [C_{ij}[k]]$ is a $N \times N$ matrix whose entries are either zero or positive depending on whether the entry is diagonal or off-diagonal, i.e.,

$$\boldsymbol{C}[k] = \begin{bmatrix} 0 & G_{12}[k] & \cdots & G_{1N}[k] \\ G_{21}[k] & 0 & \cdots & G_{2N}[k] \\ \vdots & \vdots & \ddots & \vdots \\ G_{N1}[k] & G_{N2}[k] & \cdots & 0 \end{bmatrix}. \tag{7}$$

$\boldsymbol{n}[k] = [n_1[k]\ n_1[k]\ \cdots\ n_N[k]]^T$ denotes the vector of noise power at the receivers for all $N$ nodes. Here, note that $\boldsymbol{x}[k], \boldsymbol{n}[k] \geq 0$ and $\boldsymbol{D}[k], \boldsymbol{C}[k] \geq 0$ because the powers and link gains are all positive values.

The model (6) is thought to be typical because it covers the well-known Foschini-Miljanic algorithm [12] and can be extended (in future) to describe the power-control algorithms of *opportunistic* communications e.g., the utility-based power control (UBPC) algorithm [13].

In wireless channels, fading is deviation of the attenuation affecting a signal over certain propagation media. The fading can vary with time or geographical position, and is often modeled as a stochastic process. If let $\boldsymbol{G}[k] = [G_{ij}[k]]$, now one should bear in mind that, $\{\boldsymbol{G}[k], k \in \mathbb{Z}_+\}$ and $\{\boldsymbol{n}[k], k \in \mathbb{Z}_+\}$ are two stochastic processes when the fading channels and random noise appear in the wireless networks. As a consequences, $\{\boldsymbol{D}[k], k \in \mathbb{Z}_+\}$ and $\{\boldsymbol{C}[k], k \in \mathbb{Z}_+\}$ will also be stochastic processes. To define the randomness behavior of the wireless networks with fading channels and noises, the below assumptions are always employed.

*Assumption 1 (Additive White Gaussian Noise).* The noises existed in the wireless networks are i.i.d. additive white Gaussian noises with average power $\delta_n^2 > 0$ such that the noise power vector $\boldsymbol{n}[k]$ satisfies $E(\|\boldsymbol{n}[k]\|_1{}^1) = N\delta_n^2$. The noises are independent with the link gains. In words, if $n_q[k_2]$ has distribution $\mu^n_{q,k_2}$ and $G_{ij}[k_1]$ has $\mu^G_{i,j,k_1}$, then $(n_q[k_2], G_{ij}[k_1])$ has distribution $\mu^n_{q,k_2} \times \mu^G_{i,j,k_1}$ [24], for $1 \le i, j, q \le N$ and $k_1, k_2 \in \mathbb{Z}_+$.

*Assumption 2 (Temporal Independency of Link Gains).* The link gains at different iterations are independent. In essence this implies, if $G_{ij}[k_1]$ has distribution $\mu^G_{i,j,k_1}$ and $G_{pq}[k_2]$ has $\mu^G_{p,q,k_2}$, then $(G_{ij}[k_1], G_{pq}[k_2])$ has distribution $\mu^G_{i,j,k_1} \times \mu^G_{p,q,k_2}$ [24], for $1 \le i, j, p, q \le N$ whenever $k_1 \ne k_2$.

*Assumption 3 (Stationarity).* The distributions of $G_{ij}[k]$ and $n_q[k]$ are unrelated to $k$, i.e., whatever $k$ is, $G_{ij}[k]$ and $n_q[k]$ has distribution $\mu^G_{i,j}$ and $\mu^n_q$, respectively, for $1 \le i, j, q \le N$.

This assumption stipulates that $\{\boldsymbol{G}[k], k \in \mathbb{Z}_+\}$, $\{\boldsymbol{D}[k], k \in \mathbb{Z}_+\}$, $\{\boldsymbol{C}[k], k \in \mathbb{Z}_+\}$, and $\{\boldsymbol{n}[k], k \in \mathbb{Z}_+\}$ are *stationary* stochastic processes which do not change their statistical properties with $k$.

*Assumption 4 (Deployment of Multiple Access Methods).* In wireless networks, multiple access methods can suppress the leakage of signal power from one node to the receivers for other nodes under certain level such that $G_{ij}[k] \le \beta_{ij} G_{jj}[k]$ with constant values $\beta_{ij} \ll 1$ for any $i \ne j$. Moreover, $G_{jj}[k], 1 \le j \le N$ are $N$ stationary and i.i.d. random variables with $E(G_{jj}[k]) = \mu_G$, $E(1/G_{jj}[k]) = \mu_{1/G}$, $E(G^2_{jj}[k]) = \mu_{G^2}$, and $E(1/G^2_{jj}[k]) = \mu_{1/G^2}$.

This assumption holds in case of the power control of wireless networks with multiple access methods, which implies

$$\boldsymbol{C}[k] \le \begin{bmatrix} 0 & \beta_{12}G_{22}[k] & \cdots & \beta_{1N}G_{NN}[k] \\ \beta_{21}G_{11}[k] & 0 & \cdots & \beta_{2N}G_{NN}[k] \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{N1}G_{11}[k] & \beta_{N2}G_{22}[k] & \cdots & 0 \end{bmatrix}.$$

In summary, the power-control system (6) has a linear system model with random parameters. This study will analyze the $p$th-moment stability for such a system.

### 3.2 Problem statement

In what follows, we begin by considering the linear systems with random parameters as

$$\boldsymbol{x}[k+1] = \boldsymbol{A}[k]\boldsymbol{x}[k] + \boldsymbol{B}[k]\boldsymbol{n}[k], \tag{8}$$

where $\{\boldsymbol{A}[k], k \in \mathbb{Z}_+\}$ and $\{\boldsymbol{B}[k], k \in \mathbb{Z}_+\}$ be two stationary stochastic processes. $\{\boldsymbol{n}[k], k \in \mathbb{Z}_+\}$ is a stationary stochastic process of additive white Gaussian noise, and independent with $\{\boldsymbol{A}[k], k \in \mathbb{Z}_+\}$ and $\{\boldsymbol{B}[k], k \in \mathbb{Z}_+\}$. Clearly, the model of system (8) is a generalization of the model (6), which is not same but closely related to the models appeared in several existed works [10] [14] [16].

**Main problem of this work.** How can we estimate whether the linear system (8) is $p$th-moment stable or not?

**Theorem.** The stochastic process $\{\boldsymbol{x}[k], k \in \mathbb{Z}_+\}$ that corresponds to the state vector $\boldsymbol{x}[k]$ in (8), is a Markov chain.

**Proof** The conclusion directly follows from the definition of Markov chain. □

The result above is a trivial but fundamental understanding of the system (8). We then turn to taking a closer look at this system. To state the further results, we need to rewrite (8) as

$$\boldsymbol{x}[k+1] \quad = \quad \underbrace{\left(\prod_{u=1}^{k} \boldsymbol{A}[u]\right) \boldsymbol{x}[1]}_{\text{non-noise term}} + \underbrace{\sum_{i=1}^{k} \left(\prod_{u=i+1}^{k} \boldsymbol{A}[u]\right) \boldsymbol{B}[i]\boldsymbol{n}[i]}_{\text{noise term}}. \tag{9}$$

If the noise term does not exist in (9), the Furstenberg-Kesten theorem [14] and the analytical framework developed by Feng *et al.* [10] would be available to study the stochastic stability properties for the associated system. However now, due to the existence of the noise term, we have to seek for a new proper framework to analyze such a system. In [16], Koning provided an analytical framework which is usable to investigate (8), but such a framework can only reflect the first- and second-order statistics of $\boldsymbol{x}[k]$. In this work, we will develop a norm-inequality-based framework that is capable of analyzing higher-order as well as the first- and second-order statistics of $\boldsymbol{x}[k]$.

## 4 Results

In this section, the $p$th-moment stability of the linear system (8) is analyzed through a norm-inequality-based approach, and then the analysis is applied to the power-control system (6). To attain the main results of this study, we need to derive and use several lemmas; however, we prefer not to introduce them in the main text but, rather, in Appendix, so as not to interrupt the presentation. For details, please refer to Lemmas A.1 to A.5.

### 4.1 A Norm-inequality-based approach of the $p$th-moment stability analysis

We are now ready to perform the $p$th-moment stability analysis of the linear system (8) via a norm-inequality-based approach.

**Theorem 1.** A sufficient condition for the first-moment stability of the system (8) is $\lim_{k\to+\infty} \bar{\mathbf{A}}^k = \mathbf{0}_{N\times N}$, where $\mathbf{0}_{N\times N}$ is the $N \times N$ zero matrix.

**Proof** Since $\boldsymbol{x}[k+1] \geq 0$,

$E\left(\|\boldsymbol{x}[k+1])\|_1^{\,1}\right) = \|E(\boldsymbol{x}[k+1])\|_1^{\,1} = \|\bar{\mathbf{x}}[k+1]\|_1^{\,1} = \left\|\bar{\mathbf{A}}^k\bar{\mathbf{x}}[1] + \sum_{i=1}^{k} \bar{\mathbf{A}}^{k-i}\,\bar{\mathbf{B}}\bar{\mathbf{n}}\right\|_1^{\,1}$. If the matrix $\bar{\mathbf{A}}$ has the property that $\lim_{k\to+\infty} \bar{\mathbf{A}}^k = \mathbf{0}_{N\times N}$, $\mathbf{I} - \bar{\mathbf{A}}$ will be nonsingular and its inverse can be expressed by [23, Corollary 5.6.16]: $(\mathbf{I} - \bar{\mathbf{A}})^{-1} = \sum_{k=0}^{+\infty} \bar{\mathbf{A}}^k$, and then we shall have $\lim_{k\to+\infty} E\left(\|\boldsymbol{x}[k+1])\|_1^{\,1}\right) = \left\|(\mathbf{I} - \bar{\mathbf{A}})^{-1}\bar{\mathbf{B}}\bar{\mathbf{n}}\right\|_1^{\,1} < +\infty$ as long as $\lim_{k\to+\infty} \bar{\mathbf{A}}^k = \mathbf{0}_{N\times N}$ holds. □

The model (8) has an alternative formulation as

$$\boldsymbol{x}[k+1] = \boldsymbol{s}_x[k] + \sum_{i=1}^{k} \boldsymbol{s}_n^k[i], \tag{10}$$

where, for notational simplicity, $\boldsymbol{s}_x[k] = \left(\prod_{u=1}^{k} \boldsymbol{A}[u]\right) \boldsymbol{x}[1]$, $\boldsymbol{s}_n^k[i] = \left(\prod_{u=i+1}^{k} \boldsymbol{A}[u]\right) \boldsymbol{B}[i]\boldsymbol{n}[i]$.

The forthcoming analysis will involve applying the operator $E_{L^p}$ to p-norm of some random vector $\boldsymbol{x}$, i.e., substituting $f = \|\boldsymbol{x}\|_p$ into $E_{L^p}f$, or to induced matrix norm of some random matrix $\boldsymbol{X}$, i.e., substituting $f = \|\boldsymbol{X}\|_p$ into $E_{L^p} f$. There is one important issue herein that must be mentioned:

*Remark 2:* Computing the induced matrix norm is a nonlinear optimization problem[1], and the induced norm $\|\boldsymbol{X}\|_p$ has no explicit representation unless $p = 1, 2$ or $\infty$. If $p \neq 1, 2$, or $\infty$, the integral $\int_\Omega \|\boldsymbol{X}\|_p^p d\mu < +\infty$ might not exist, in which case we can not take $E_{L^p}$ to $\|\boldsymbol{X}\|_p$. Therefore, in this study when it is needed to apply the operator $E_{L^p}$ to $\|\boldsymbol{X}\|_p$, we will seek for an integrable upper-bound $\varphi(\mathbf{X})$ of $\|\boldsymbol{X}\|_p$ and use $E_{L^p}\varphi(\mathbf{X})$ for theoretical analysis.

**Theorem 2.** Assume that $\|\mathbf{X}\|_p$ has an upper-bound $\varphi(\mathbf{X}) > 0$, i.e., $\|\mathbf{X}\|_p \leq \varphi(\mathbf{X})$, where $\int_\Omega |\varphi(\mathbf{X})|^p d\mu$ is integrable. A sufficient condition for the $p$th-moment stability of the system (8) is $E_{L^p}\varphi(\boldsymbol{A}) < 1$, $E_{L^p}\varphi(\boldsymbol{B}) < +\infty$, and $E_{L^p}\|\boldsymbol{n}\|_p < +\infty$.[2]

**Proof** By (10), we get

$$\begin{aligned} E_{L^p} \left\|\boldsymbol{x}[k+1])\right\|_p &= E_{L^p} \left\|\boldsymbol{s}_x[k] + \sum_{i=1}^{k} \boldsymbol{s}_n^k[i]\right\|_p \leq E_{L^p} \left\|\boldsymbol{s}_x[k]\right\|_p + \sum_{i=1}^{k} E_{L^p} \left\|\boldsymbol{s}_n^k[i]\right\|_p \\ &\leq \prod_{u=1}^{k} E_{L^p}\varphi(\boldsymbol{A}[u]) E_{L^p}\|\boldsymbol{x}[1]\|_p + \sum_{i=1}^{k} \prod_{u=i+1}^{k} E_{L^p}\varphi(\boldsymbol{A}[u]) E_{L^p}\varphi(\boldsymbol{B}[i]) E_{L^p}\|\boldsymbol{n}[i]\|_p, \end{aligned}$$

where the first inequality follows from Lemma A.4 and the second one is from Lemma A.5.

Under the assumption of stationarity, $E_{L^p}\varphi(\boldsymbol{A}[k])$, $E_{L^p}\varphi(\boldsymbol{B}[k])$, and $E_{L^p}\|\boldsymbol{n}[k]\|_p$ shall not change with $k$ such that we can drop $k$ for notational simplicity. Thus,

$$\begin{aligned} E_{L^p} \left\|\boldsymbol{x}[k+1])\right\|_p &\leq \left[E_{L^p}\varphi(\boldsymbol{A})\right]^k E_{L^p}\|\boldsymbol{x}[1]\|_p + \sum_{i=1}^{k} \left[E_{L^p}\varphi(\boldsymbol{A})\right]^{k-i} E_{L^p}\varphi(\boldsymbol{B}) E_{L^p}\|\boldsymbol{n}\|_p \\ &= \left[E_{L^p}\varphi(\boldsymbol{A})\right]^k E_{L^p}\|\boldsymbol{x}[1]\|_p + \frac{1 - \left[E_{L^p}\varphi(\boldsymbol{A})\right]^k}{1 - E_{L^p}\varphi(\boldsymbol{A})} E_{L^p}\varphi(\boldsymbol{B}) E_{L^p}\|\boldsymbol{n}\|_p. \end{aligned}$$

If $E_{L^p}\varphi(\boldsymbol{A}) < 1$, $E_{L^p} \left\|\boldsymbol{B}\right\|_p < +\infty$, and $E_{L^p}\|\boldsymbol{n}\|_p < +\infty$, then

$$\lim_{k\to+\infty} E\left(\left\|\boldsymbol{x}[k+1]\right\|_p^p\right) \leq \left(\frac{E_{L^p}\varphi(\boldsymbol{B}) E_{L^p}\|\boldsymbol{n}\|_p}{1 - E_{L^p}\varphi(\boldsymbol{A})}\right)^p < +\infty.$$

Therefore, Theorem 2 is justified. □

Theorem 2 holds for all $p \in \mathbb{Z}_+$, thus it can reveal any $p$th-order statistics of $\boldsymbol{x}[k]$. It's a progress made by the proposed analytical framework of this study, compared to the framework developed by [16] that can only investigate the first- and second-order statistics of $\boldsymbol{x}[k]$. Since the derivation methods of Theorems 1 and 2 are not exactly same, the sufficient condition of the first-moment stability obtained by Theorem 1 is not necessarily identical as that by Theorem 2 with $p = 1$.

**Theorem 3.** If $E_{L^p}\varphi(\boldsymbol{A}) < 1$, $E_{L^p} \left\|\boldsymbol{B}\right\|_p < +\infty$, and $E_{L^p}\|\boldsymbol{n}\|_p < +\infty$, there exists $\alpha < +\infty$, such that $\lim_{k\to+\infty} E\left(\left\|\boldsymbol{x}[k])\right\|_p^p\right) = \alpha$.

**Proof** Let us begin by assuming that $k > j > J$. We get $E_{L^p} \left\|\boldsymbol{x}[j+1])\right\|_p \geq E_{L^p} \left\|\sum_{i=1}^{j} \boldsymbol{s}_n^j[i]\right\|_p$,

[1]An already known approach is to make use of the algorithm of estimating the induced matrix norm as well as the Matlab routines provided by Higham [17].

[2]Note that $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{n}$ are short for $\boldsymbol{A}[k]$, $\boldsymbol{B}[k]$, and $\boldsymbol{n}[k]$ by dropping the index $k$, since the statistics of $\boldsymbol{A}[k]$, $\boldsymbol{B}[k]$, and $\boldsymbol{n}[k]$ are irrelevant with $k$ under the stationarity assumption.

because $\boldsymbol{s}_x[j]$ and $\boldsymbol{s}_n^j[i]$ ($i, j \in \mathbb{Z}_+$, $1 \le i \le j$) are all positive vectors. By (10), we can have

$$E_{L^p} \left\| \boldsymbol{x}[k+1]) \right\|_p \le E_{L^p} \left\| \boldsymbol{s}_x[k] \right\|_p + E_{L^p} \left\| \sum_{i=k-j+1}^{k} \boldsymbol{s}_n^k[i] \right\|_p + E_{L^p} \left\| \sum_{i=1}^{k-j} \boldsymbol{s}_n^k[i] \right\|_p . \tag{11}$$

Due to the stationarity property, it holds true that $E_{L^p} \left\| \sum_{i=1}^{j} \boldsymbol{s}_n^j[i] \right\|_p = E_{L^p} \left\| \sum_{i=k-j+1}^{k} \boldsymbol{s}_n^k[i] \right\|_p$. Then, with $E_{L^p}\varphi(\boldsymbol{A}) < 1$, $E_{L^p} \left\| \boldsymbol{B} \right\|_p < +\infty$, and $E_{L^p}\|\boldsymbol{n}\|_p < +\infty$, it implies that

$$\begin{aligned} E_{L^p} \left\| \boldsymbol{x}[k+1]) \right\|_p - E_{L^p} \left\| \boldsymbol{x}[j+1]) \right\|_p &\le E_{L^p} \left\| \boldsymbol{s}_x[k] \right\|_p + E_{L^p} \left\| \sum_{i=1}^{k-j} \boldsymbol{s}_n^k[i] \right\|_p \\ &\le \left[ E_{L^p}\varphi(\boldsymbol{A}) \right]^k E_{L^p}\|\boldsymbol{x}[1]\|_p + \frac{[E_{L^p}\varphi(\boldsymbol{A})]^j}{1 - E_{L^p}\varphi(\boldsymbol{A})} E_{L^p}\varphi(\boldsymbol{B}) E_{L^p}\|\boldsymbol{n}\|_p. \end{aligned} \tag{12}$$

Now we can conclude that, $\forall \varepsilon > 0$, $\exists J > 0$, such that $E_{L^p} \left\| \boldsymbol{x}[k+1]) \right\|_p - E_{L^p} \left\| \boldsymbol{x}[j+1]) \right\|_p < \varepsilon$ as long as $k, j \ge J$. It states that $E_{L^p} \left\| \boldsymbol{x}[j+1]) \right\|_p$ has a limit value as $k \to +\infty$, and thus finishes the proof. □

**Theorem 4.** For the system (8), if let $\varphi(\boldsymbol{A}) = N^{1-1/p} \left\| \boldsymbol{A} \right\|_1$ or $\varphi(\boldsymbol{A}) = \left\| \boldsymbol{A} \right\|_1^{1/p} \left\| \boldsymbol{A} \right\|_\infty^{1-1/p}$, where $\boldsymbol{A}$ is short for $\boldsymbol{A}[k]$, $\int_\Omega |\varphi(\boldsymbol{A})|^p d\mu$ would exist which means $E_{L^p}\varphi(\boldsymbol{A})$ exists.

**Proof** Both $\left\| \boldsymbol{A} \right\|_1$ and $\left\| \boldsymbol{A} \right\|_\infty$ are continues measurable functions. Then $\int_\Omega N^{p-1} \left\| \boldsymbol{A} \right\|_1^p d\mu$ and $\int_\Omega \left\| \boldsymbol{A} \right\|_1 \left\| \boldsymbol{A} \right\|_\infty^{p-1} d\mu$ exist. This leads to the results of Theorem 4. □

*Remark 3:* Theorem 4 yields two sufficient conditions for the $p$th-moment stability of the system (8), i.e., $E_{L^p}(N^{1-1/p} \left\| \boldsymbol{A} \right\|_1) < 1$ and $E_{L^p}(\left\| \boldsymbol{A} \right\|_1^{1/p} \left\| \boldsymbol{A} \right\|_\infty^{1-1/p}) < 1$. Although there might exist certain conservation, these two conditions are convenient for practical operations, because both $\left\| \boldsymbol{A} \right\|_1$ and $\left\| \boldsymbol{A} \right\|_\infty$ have the explicit representations.

*Remark 4:* Taniguchi [18] provided stochastic stability theorems of the nonlinear difference equations through using norm inequalities; however, the theorems obtained in [18] can not assist us to achieve the results with practical significance for the system (8). While by employing the norm-inequality-based framework, this study dedicates to derive the results for the system (8). One could also find the moment stability studies attract many interests recently, e.g., the $p$th-moment exponential ultimate boundedness is investigated for impulsive stochastic differential systems [19], and the $p$th-moment asymptotic stability is analyzed for stochastic delayed hybrid systems with Levy noise [20].

### 4.2 The $p$th-moment stability of power control

Going back to the power-control system (6), we can obtain many useful results without too much efforts based on the previous analysis.

*Remark 5:* By letting $\boldsymbol{A}[k] = \boldsymbol{D}[k]\boldsymbol{C}[k]$ and $\boldsymbol{B}[k] = \boldsymbol{D}[k]$, one can directly apply Theorems 1 to 4 to the power-control system (6).

One important novelty of this study is not only to assess the stability of power-control system (6) but also to acquire more knowledge of relations between the stochastic stability and power control together with other wireless communication technologies. We will show that the proposed norm-inequality-based approach allows us to recognize the effect of multiple-access methods to the $p$th-moment stability of power control.

The sufficient conditions for the $p$th-moment stability given by Theorems 1, 2, 3, and 4 are only related to $\boldsymbol{A}[k]$ ($= \boldsymbol{D}[k]\boldsymbol{C}[k]$), while $\boldsymbol{D}[k]\boldsymbol{C}[k]$ is partly determined by the target SINRs and link gains according to (6)-(7). This fact inspires us to investigate the $p$th-moment stability by thinking over the power control together with the effect of multiple access technique.

Consider the power-control system (6) with a multiple access method, under Assumption 4, we have an upper bound of $\boldsymbol{A}[k]$ as

$$\boldsymbol{A}[k] = \boldsymbol{D}[k]\boldsymbol{C}[k] \leq \begin{bmatrix} 0 & \gamma_1\beta_{12}\frac{G_{22}[k]}{G_{11}[k]} & \cdots & \gamma_1\beta_{1N}\frac{G_{NN}[k]}{G_{11}[k]} \\ \gamma_2\beta_{21}\frac{G_{11}[k]}{G_{22}[k]} & 0 & \cdots & \gamma_2\beta_{2N}\frac{G_{NN}[k]}{G_{22}[k]} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_N\beta_{N1}\frac{G_{11}[k]}{G_{NN}[k]} & \gamma_N\beta_{N2}\frac{G_{22}[k]}{G_{NN}[k]} & \cdots & 0 \end{bmatrix}.$$

In the reminder of this section, the index $k$ will be dropped from $\boldsymbol{A}[k]$, $\boldsymbol{D}[k]$, $\boldsymbol{C}[k]$, and $G_{ii}[k]$ such that $\boldsymbol{A}$, $\boldsymbol{D}$, $\boldsymbol{C}$, and $G_{ii}$ are used.

From above, we get the following results.

**Theorem 5.** If the power-control system (6) employs a multiple access method so that Assumptions 1 to 4 are satisfied, it will hold that $\bar{\mathbf{A}} = E\boldsymbol{A} \leq \mu_G \cdot \mu_{1/G} \cdot \Theta_{\bar{\gamma},\beta}$, where

$$\Theta_{\gamma,\beta} = \begin{bmatrix} 0 & \gamma_1\beta_{12} & \cdots & \gamma_1\beta_{1N} \\ \gamma_2\beta_{21} & 0 & \cdots & \gamma_2\beta_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_N\beta_{N1} & \gamma_N\beta_{N2} & \cdots & 0 \end{bmatrix}.$$

Then if the values of $\gamma_i$ $(1 \leq i \leq N)$ and $\beta_{ij}$ $(1 \leq i, j \leq N)$ are properly chosen such that

$$\min\left\{\max_i \sum_{j\neq i}^{N} \gamma_i\beta_{ij}, \max_j \sum_{i\neq j}^{N} \gamma_i\beta_{ij}\right\} < \frac{1}{\mu_G \cdot \mu_{1/G}}, \tag{13}$$

the system will be first-moment stable.

**Proof** The upper bound of $\bar{\mathbf{A}}$ can be obtained by taking expectation to the upper bound of $\boldsymbol{A}$ given above.

Let $\rho(\cdot)$ denote the spectral radius. Using [23, Theorem 8.1.22] to show that

$$\max\left\{\min_i \sum_{j=1}^{N} \bar{A}_{ij}, \min_j \sum_{i=1}^{N} \bar{A}_{ij}\right\} \leq \rho\left(\bar{\mathbf{A}}\right) \leq \min\left\{\max_i \sum_{j=1}^{N} \bar{A}_{ij}, \max_j \sum_{i=1}^{N} \bar{A}_{ij}\right\}.$$

i.e., the smallest row sum of a nonnegative matrix is a lower bound on its spectral radius, and the largest row sum is an upper bound. Then, by applying $\bar{A}_{ij} \leq \mu_G \cdot \mu_{1/G} \cdot \gamma_i\beta_{ij}$, we have

$$\rho\left(\bar{\mathbf{A}}\right) \leq \mu_G \cdot \mu_{1/G} \cdot \min\left\{\max_i \sum_{j\neq i}^{N} \gamma_i\beta_{ij}, \max_j \sum_{i\neq j}^{N} \gamma_i\beta_{ij}\right\}. \tag{14}$$

Combing this result with [23, Theorem 5.6.12] which says $\lim_{k\to+\infty} \bar{\mathbf{A}}^k = \mathbf{0}$ if and only if $\rho(\bar{\mathbf{A}}) < 1$, we see that letting the right-hand side of (14) be less than 1 is a sufficient condition for $\lim_{k\to+\infty} \bar{\mathbf{A}}^k = \mathbf{0}$, which therefore makes the power-control system to be first-moment stable (see Theorem 1 for reference). □

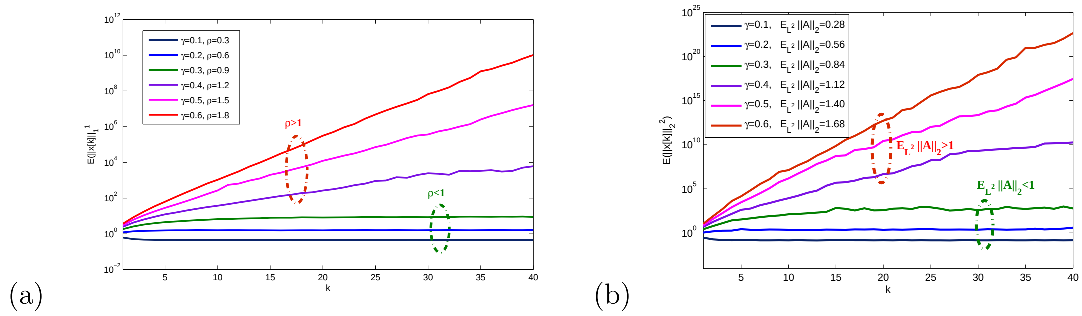



Figure 1: (a): $E(\|\boldsymbol{x}[k]\|_1^{\,1})$ versus $k$, where $\rho$ is short for $\rho\left(\bar{\mathbf{A}}\right)$. The curves with $\rho < 1$ tend to finite values, while those with $\rho > 1$ increase ceaselessly; (b): $E(\|\boldsymbol{x}[k]\|_2^{\,2})$ versus $k$. The curves with $E_{L^2}\,\|\boldsymbol{A}\|_2 < 1$ tend to finite values, while those with $E_{L^2}\,\|\boldsymbol{A}\|_2 > 1$ are progressively growing

**Theorem 6.** Consider the power-control system (6) with a multiple access method such that Assumptions 1 to 4 are satisfied, if the values of $\gamma_i$ $(1 \le i \le N)$ and $\beta_{ij}$ $(1 \le i, j \le N)$ are properly chosen such that

$$\sum_{j=1}^{N}\sum_{i\neq j}^{N} \gamma_j^2 \beta_{ij}^2 < \frac{1}{\mu_{G^2}\cdot \mu_{1/G^2}}, \tag{15}$$

the system will be second-moment stable.

**Proof** Since $\|\boldsymbol{A}\|_2 = \sqrt{\mathrm{tr}(\boldsymbol{A}^T\boldsymbol{A})}$ [23], where $\mathrm{tr}(\cdot)$ is the trace operation, by setting $\varphi(\boldsymbol{A}) = \sqrt{\mu_{1/G^2}\cdot \mathrm{tr}\left(\Theta_{\gamma,\beta}^T \Theta_{\gamma,\beta}\right)}$, it follows that $E_{L^2}\varphi(\boldsymbol{A}) = \mu_{G^2}\cdot\mu_{1/G^2}\cdot \mathrm{tr}\left(\Theta_{\gamma,\beta}^T\Theta_{\gamma,\beta}\right) = \mu_{G^2}\cdot\mu_{1/G^2}\cdot \left(\sum_{j=1}^N\sum_{i\neq j}^N \gamma_j^2\beta_{ij}^2\right)$. Observe that $E_{L^2}\varphi(\boldsymbol{A}) < 1$ as long as (15) holds. Then, recalling Theorem 2 completes the proof. □

*Remark 6:* The Cauchy-Schwarz inequality [24] leads to $\frac{1}{\mu_G\cdot\mu_{1/G}} < 1$ and $\frac{1}{\mu_{G^2}\cdot\mu_{1/G^2}} < 1$. So we can have more conservative but simpler conditions than (13) and (15) to achieve the first- and second-moment stability, respectively, which are $\min\left\{\max_i \sum_{j\neq i}^N \gamma_i\beta_{ij}, \max_j \sum_{i\neq j}^N \gamma_i\beta_{ij}\right\} < 1$, and $\sum_{j=1}^N\sum_{i\neq j}^N \gamma_j^2\beta_{ij}^2 < 1$.

Furthermore, let us extend Theorems 5 and 6 to a generalized case, i.e., the $p$th-moment stability with any $p \in \mathbb{Z}_+$.

**Theorem 7.** Suppose that the power-control system (6) employs a multiple access method so that Assumptions 1 to 4 are established, the system will be $p$th-moment stable if $E_{L^p}\varphi(\boldsymbol{A}) < 1$ with $\varphi(\boldsymbol{A}) = N^{1-1/p}\left(\max_j \sum_{i\neq j}^N \gamma_i\beta_{ij}\frac{G_{jj}}{G_{ii}}\right)$, or

$$\varphi(\boldsymbol{A}) = \left(\max_j \sum_{i\neq j}^{N} \gamma_i\beta_{ij}\frac{G_{jj}}{G_{ii}}\right)^{1/p}\left(\max_i \sum_{j\neq i}^{N} \gamma_i\beta_{ij}\frac{G_{jj}}{G_{ii}}\right)^{1-1/p}.$$

**Proof** Through replacing $\|\boldsymbol{A}\|_1$ and $\|\boldsymbol{A}\|_\infty$ in Theorem 4 with the maximum column sum and maximum row sum of $\boldsymbol{A}[k]$, respectively, Theorem 7 can be validated. □

The importance of Theorems 5, 6, 7, and Remark 6 lies in that they can guide system designers to assess and select suitable target SINR schemes and multiple access methods for wireless-network systems and also to pick out the proper system parameters for them, from the perspective of power-control stability.

Table 1: Numerical Values of Examples 2 and 3.

| $\gamma$ | 73.7 | 86.0 | 98.3 | 110.6 |
|---|---|---|---|---|
| $\min\left\{\max_i \sum_{j\neq i}^N \gamma\beta_{ij}, \max_j \sum_{i\neq j}^N \gamma\beta_{ij}\right\}$ | 0.90 | 1.05 | 1.20 | 1.35 |
| $\rho\left(\bar{\mathbf{A}}\right)$ | 0.66 | 0.77 | 0.87 | 1.01 |
| $\sum_{j=1}^N \sum_{i\neq j}^N \gamma^2\beta_{ij}^2$ | 0.85 | 1.16 | 1.51 | 1.91 |
| $E_{L^2}\left\|\boldsymbol{A}\right\|_2$ | 0.53 | 0.72 | 0.94 | 1.19 |
| $\left(\int_\Omega \left\|\boldsymbol{A}\right\|_5^5 d\mu\right)^{1/5}$ | 2.05 | 2.40 | 2.70 | 3.13 |
| $E_{L^5}\left(\left\|\boldsymbol{A}\right\|_1^{1/5}\left\|\boldsymbol{A}\right\|_\infty^{4/5}\right)$ | 2.26 | 2.64 | 2.98 | 3.45 |

# 5 Numerical examples

*Example 1*: We consider the power-control system (8) in main text with i.i.d. Rayleigh fading link gains (that is, all $G_{ij}$ are Rayleigh distributed with unit variance) and fixed target SINRs $\gamma_1[k] = \gamma_2[k] = \cdots = \gamma_N[k] = \gamma$. There are four nodes in the network.

We let $\boldsymbol{n}[k]$ be the power vector of Gaussian noise with unit variance, and initially set $\boldsymbol{x}[1] = [1\ 0\ 0\ 0]^T$. In case of $\gamma = 0.1, 0.2, \cdots, 0.6$, Figs. 1(a) and 1(b) illustrate how $E(\|\boldsymbol{x}[k]\|_1^1)$ and $E(\|\boldsymbol{x}[k]\|_2^2)$ grow with $k$, where $\rho\left(\bar{\mathbf{A}}\right)$ and $E_{L^2}\left\|\boldsymbol{A}\right\|_2$ are estimated during the simulations. Fig. 1(a) shows that the curves of $E(\|\boldsymbol{x}[k]\|_1^1)$ with $\gamma = 0.3, 0.6, 0.9$ tend to finite values (in other words, the system is first-moment stable), while others increase ceaselessly. This result is in accordance with Theorem 1 because $\lim_{k\to+\infty} \bar{\mathbf{A}}^k = \mathbf{0}$ if and only if $\rho(\bar{\mathbf{A}}) < 1$ [23, Theorem 5.6.12]. From Fig. 1(b), it is observed that the curves of $E(\|\boldsymbol{x}[k]\|_2^2)$ with $\gamma = 0.28, 0.56, 0.84$ tend to finite values (or, equivalently, the system is second-moment stable), while others are progressively growing. This observation is a consequence of Theorems 2, 3.

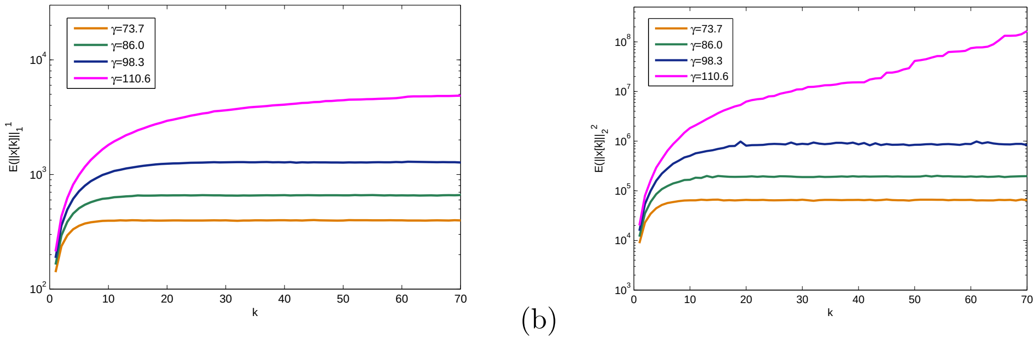
*Example 2*: We consider the power-control system (8) in main text with multiple access method and fixed target SINRs $\gamma_1[k] = \gamma_2[k] = \cdots = \gamma_N[k] = \gamma$. All $G_{jj} = 1 + G\prime_j$ where $G\prime_j$ is the Rayleigh distributed with unit variance. Let $G_{ij}[k] = \beta_{ij}G_{jj}[k]$ without loss of generality. It can be obtained that $\mu_G \cdot \mu_{1/G} = 1.20$ and $\mu_{G^2} \cdot \mu_{1/G^2} = 2.03$. There are four nodes in the network which employs a multiple access method such that

$$\begin{bmatrix} 0 & \beta_{12} & \beta_{13} & \beta_{14} \\ \beta_{21} & 0 & \beta_{23} & \beta_{24} \\ \beta_{31} & \beta_{32} & 0 & \beta_{34} \\ \beta_{41} & \beta_{42} & \beta_{43} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{100} & \frac{1}{200} & \frac{1}{300} \\ \frac{1}{400} & 0 & \frac{1}{500} & \frac{1}{600} \\ \frac{1}{700} & \frac{1}{800} & 0 & \frac{1}{900} \\ \frac{1}{1000} & \frac{1}{1100} & \frac{1}{1200} & 0 \end{bmatrix}.$$

For above,

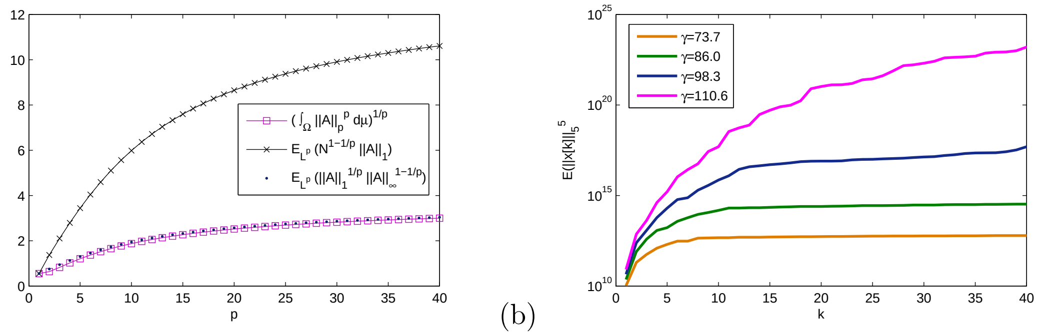
$$\min\left\{\max_i \sum_{j\neq i}^N \beta_{ij}, \max_j \sum_{i\neq j}^N \beta_{ij}\right\} = 1.22\times 10^{-2}, \qquad \sum_{j=1}^N \sum_{i\neq j}^N \beta_{ij}^2 = 1.565\times 10^{-4}. \tag{16}$$

Again, let $\boldsymbol{n}[k]$ be the power vector of Gaussian noise with unit variance, and initially set $\boldsymbol{x}[1] = [1\ 0\ 0\ 0]^T$. In case of $\gamma = 73.7, 86.0, 98.3, 110.6$, we build Table 1 by recalling two previous equalities, i.e., (16), and performing simulations. From Table 1, it is clear that $\min\left\{\max_i \sum_{j\neq i}^N \gamma\beta_{ij}, \max_j \sum_{i\neq j}^N \gamma\beta_{ij}\right\}$ is less than $\rho\left(\bar{\mathbf{A}}\right)$, and $\sum_{j=1}^N \sum_{i\neq j}^N \gamma^2\beta_{ij}^2$ is less than $E_{L^2}\left\|\boldsymbol{A}\right\|_2$, as we could expect from the relations between Theorems 1, 2, 5, and 6. The numerical results of Figs. 2(a) and 2(b) are in agreement with Theorems 1, 2, 5, and 6.




(a) (b)

Figure 2: (a): $E(\|\boldsymbol{x}[k]\|_1^1)$ versus $k$; (b): $E(\|\boldsymbol{x}[k]\|_2^2)$ versus $k$.




(a) (b)

Figure 3: (a): Comparison among $\left(\int_\Omega \|\boldsymbol{A}\|_p^p \, d\mu\right)^{1/p}$, $E_{L^p}(N^{1-1/p}\|\boldsymbol{A}\|_1)$, and $E_{L^p}\left(\|\boldsymbol{A}\|_1^{1/p}\|\boldsymbol{A}\|_\infty^{1-1/p}\right)$ for different $p$. Note that, the Matlab routines provided by Higham [17] which can directly estimate $\|\boldsymbol{A}\|_p$ is used for computing $\left(\int_\Omega \|\boldsymbol{A}\|_p^p \, d\mu\right)^{1/p}$; (b): $E(\|\boldsymbol{x}[k]\|_5^5)$ versus $k$

*Example 3*: The same system model and parameters as Example 2 are used.

In Fig. 3(a), we have compared $\left(\int_\Omega \|\boldsymbol{A}\|_p^p \, d\mu\right)^{1/p}$ with its two upper bounds as given by Remark 3, i.e., $E_{L^p}(N^{1-1/p}\|\boldsymbol{A}\|_1)$ and $E_{L^p}\left(\|\boldsymbol{A}\|_1^{1/p}\|\boldsymbol{A}\|_\infty^{1-1/p}\right)$, which can also be refereed to Theorem 4. It is seen that, $E_{L^p}\left(\|\boldsymbol{A}\|_1^{1/p}\|\boldsymbol{A}\|_\infty^{1-1/p}\right)$ stays quite close to $\left(\int_\Omega \|\boldsymbol{A}\|_p^p \, d\mu\right)^{1/p}$, however, there is an evident gap between $E_{L^p}(N^{1-1/p}\|\boldsymbol{A}\|_1)$ and $\left(\int_\Omega \|\boldsymbol{A}\|_p^p \, d\mu\right)^{1/p}$, which is amplified as $p$ increases. Therefore, being the upper bound of $\left(\int_\Omega \|\boldsymbol{A}\|_p^p \, d\mu\right)^{1/p}$, $E_{L^p}\left(\|\boldsymbol{A}\|_1^{1/p}\|\boldsymbol{A}\|_\infty^{1-1/p}\right)$ is more tight than $E_{L^p}(N^{1-1/p}\|\boldsymbol{A}\|_1)$. As a consequence, we propose to use $E_{L^p}\left(\|\boldsymbol{A}\|_1^{1/p}\|\boldsymbol{A}\|_\infty^{1-1/p}\right)$ when the upper bound of $\left(\int_\Omega \|\boldsymbol{A}\|_p^p \, d\mu\right)^{1/p}$ is needed. For $\gamma = 73.7, 86.0, 98.3, 110.6$, Fig. 3(b) illustrates how $E(\|\boldsymbol{x}[k]\|_5^5)$ evolves with $k$, and Table 1 presents the data of $\left(\int_\Omega \|\boldsymbol{A}\|_5^5 \, d\mu\right)^{1/5}$ and $E_{L^5}\left(\|\boldsymbol{A}\|_1^{1/5}\|\boldsymbol{A}\|_\infty^{4/5}\right)$. When $\gamma = 73.7, 86.0$, $E(\|\boldsymbol{x}[k]\|_5^5)$ tends to finite values as long as $k$ is sufficiently large. Then if $\gamma = 98.3, 110.6$, $E(\|\boldsymbol{x}[k]\|_5^5)$ will be found to grow infinitely. The numerical result is in accordance with Theorem 3.

## Conclusion

This study develops a norm-inequality-based framework of analyzing the $p$th-moment stability of linear systems with random parameters, so as to show that a typical power control law with linear system model is stable in the sense of the $p$th-moment stability. It is the first time to recognize the effect of multiple-access methods to stability analysis of power control.

### Acknowledgment

The authors are grateful to the anonymous reviewers for many valuable comments concerning the presentation. This work was supported by the National Natural Science Foundation of China under Grant 61501043. We would like to thank Dr. Miao Diao and Prof. Duan Zhisheng for all their help and support while writing this paper.

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## Appendix: Several lemmas

This Appendix is devoted to present the lemmas (and their proofs) which are required to derive the main results of our work.

*Lemma A.1:* Let the nonnegative numbers $a_1, a_2, \cdots, a_N$ and the positive $p_1, p_2, \cdots, p_N$ be given. Set $\sum_{j=1}^{N} \frac{1}{p_j} = 1$ then the inequality $\prod_{j=1}^{N} a_j \leq \sum_{j=1}^{N} \frac{1}{p_j} a_j^{p_j}$ holds with equality if and only if all $a_k$ with $p_k > 0$ are equal.

*Proof:* $\prod_{j=1}^{N} a_j \leq \sum_{j=1}^{N} \frac{1}{p_j} a_j^{p_j}$ in Lemma A.1 is an inequality of the weighted arithmetic mean and geometric mean, which can be proved by using the finite form of Jensen's inequality [24] for the natural logarithm. □

*Lemma A.2:* Let $x_1, x_2, \cdots, x_N$ be $N$ random variables and $p_1, p_2, \cdots, p_N$ be nonnegative numbers numbers. If $\sum_{j=1}^{N} \frac{1}{p_j} = 1$ and $E|x_j|^{p_j} < +\infty$ for $1 \le j \le N$, then $E\left(\prod_{j=1}^{N} |x_j|\right) \le \prod_{j=1}^{N} (E|x_j|^{p_j})^{\frac{1}{p_j}}$, where $(E|x_j|^{p_j})^{\frac{1}{p_j}} = E_{L^p}||x_j||$ with $p = p_j$.

*Proof:* By using Lemma A.1, we get $\frac{\prod_{j=1}^{N} |x_j|}{\prod_{j=1}^{N} (E|x_j|^{p_j})^{\frac{1}{p_j}}} = \prod_{j=1}^{N} \frac{|x_j|}{(E|x_j|^{p_j})^{\frac{1}{p_j}}} \le \sum_{j=1}^{N} \frac{|x_j|^{p_j}}{p_j E|x_j|^{p_j}}$. Then applying the expectation to above inequality, $\frac{E(\prod_{j=1}^{N} |x_j|)}{\prod_{j=1}^{N} (E|x_j|^{r_j})^{\frac{1}{p_j}}} \le \sum_{j=1}^{N} \frac{E|x_j|^{p_j}}{p_j E|x_j|^{p_j}} = \sum_{j=1}^{N} \frac{1}{p_j} = 1$. Thus, Lemma A.2 is verified. □

*Lemma A.3:* Suppose $\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_K$ are random matrices with size $S_1 \times S_2$, $S_2 \times S_3$, $\cdots$, $S_K \times S_{K+1}$, where $S_1, S_2, \cdots, S_K, S_{K+1}$ are all simply positive integers and the subscripts are labels corresponding to the matrices. If the entries of $\boldsymbol{X}_k$ are independent with those of $\boldsymbol{X}_l$ for any $k \neq l$ then $E\left(\prod_{k=1}^{K} \boldsymbol{X}_k\right) = \prod_{k=1}^{K} E(\boldsymbol{X}_k)$.

*Proof:* The product of $K$ matrices can be expressed in the index notation as

$$\left[\prod_{k=1}^{K} \boldsymbol{X}_k\right]_{ij} = \sum_{i_1=1}^{S_1} \sum_{i_2=1}^{S_2} \cdots \sum_{i_{K-1}=1}^{S_{K-1}} [\boldsymbol{X}_1]_{ii_1} [\boldsymbol{X}_2]_{i_1 i_2} [\boldsymbol{X}_3]_{i_2 i_3} \cdots [\boldsymbol{X}_{n-1}]_{i_{n-2} i_{n-1}} [\boldsymbol{X}_n]_{i_{n-1} j}.$$

This implies every entry of the resultant matrix after matrix product is a linear function of the entries of all $\boldsymbol{A}_k$ matrices. The independence condition can further yield [24]

$$E\left(\left[\prod_{k=1}^{K} \boldsymbol{X}_k\right]_{ij}\right) = \sum_{i_1=1}^{S_1} \sum_{i_2=1}^{S_2} \cdots \sum_{i_{K-1}=1}^{S_{K-1}} E([\boldsymbol{X}_1]_{ii_1}) E([\boldsymbol{X}_2]_{i_1 i_2}) \cdots E([\boldsymbol{X}_{n-1}]_{i_{n-2} i_{n-1}}) E([\boldsymbol{X}_n]_{i_{n-1} j}).$$

Therefore, Lemma A.3 is proved. □

*Lemma A.4:* Suppose that $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_I$ are random matrices with size $S \times 1$, one can have $E_{L^p} \left\|\sum_{i=1}^{I} \boldsymbol{x}_i\right\|_p \le \sum_{i=1}^{I} E_{L^p} \|\boldsymbol{x}_i\|_p$, where $p \in \mathbb{Z}_+$.

*Proof:* The norm inequalities and Lemma A.2 combine to provide

$$\begin{aligned}
\left(E_{L^p} \left\|\sum_{i=1}^{I} \boldsymbol{x}_i\right\|_p\right)^p &= E\left(\left\|\sum_{i=1}^{I} \boldsymbol{x}_i\right\|_p^p\right) \le E\left(\left(\sum_{i=1}^{I} \|\boldsymbol{x}_i\|_p\right)^p\right) \\
&= \sum_{i_1=1}^{I} \sum_{i_2=1}^{I} \cdots \sum_{i_p=1}^{I} E\left(\|\boldsymbol{x}_{i_1}\|_p \|\boldsymbol{x}_{i_2}\|_r \cdots \|\boldsymbol{x}_{i_p}\|_p\right) \\
&\le \sum_{i_1=1}^{I} \cdots \sum_{i_p=1}^{I} \left(E\left(\|\boldsymbol{x}_{i_1}\|_p^p\right) \cdots E\left(\|\boldsymbol{x}_{i_p}\|_p^p\right)\right)^{1/p} = \left(\sum_{i=1}^{I} E_{L^p} \|\boldsymbol{x}_i\|_p\right)^p.
\end{aligned}$$

This gives the desired result. □

*Lemma A.5:* Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \cdots, \boldsymbol{X}_K$ be random matrices with size $S_1 \times S_2$, $S_2 \times S_3$, $\cdots$, $S_K \times S_{K+1}$ and $\boldsymbol{y}$ be a $S_{K+1} \times 1$ random vector. If the entries of $\boldsymbol{X}_k$ are independent with those of $\boldsymbol{X}_l$ for any $k \neq l$ and $\boldsymbol{y}$ for any $1 \le k \neq K$, then $E_{L^p} \left\|\left(\prod_{k=1}^{K} \boldsymbol{X}_k\right) \boldsymbol{y}\right\|_p \le \left(\prod_{k=1}^{K} E_{L^p} \varphi(\boldsymbol{X}_k)\right) \left(E_{L^p} \|\boldsymbol{y}\|_p\right)$, where $p \in \mathbb{Z}_+$.

*Proof:* The norm inequalities implies $\left\|\left(\prod_{k=1}^{K} \boldsymbol{X}_k\right) \boldsymbol{y}\right\|_p \le \left(\prod_{k=1}^{K} \|\boldsymbol{X}_k\|_p\right) \|\boldsymbol{y}\|_p$, and thus we get $E\left(\left\|\left(\prod_{k=1}^{K} \boldsymbol{X}_k\right) \boldsymbol{y}\right\|_p^p\right) \le \left(\prod_{k=1}^{K} E(|\varphi(\boldsymbol{X}_k)|^{\,p})\right) E(\|\boldsymbol{y}\|_p^p)$. Such an inequality directly yields the result. □