

# Meta-Rationality in Normal Form Games

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**Abstract:** A new generative relation for Nash equilibrium is proposed. Different types of equilibria are considered in order to incorporate players different rationality types for finite non cooperative generalized games with perfect information. Proposed equilibria are characterized by use of several generative relations with respect to players rationality. An evolutionary technique for detecting approximations for equilibria is used. Numerical experiments show the potential of the method.

**Keywords:** non-cooperative games, evolutionary equilibrium detection, generative relations, Nash-Pareto, meta-strategy.

## 1 Introduction

The most common solutions proposed in Game Theory are the equilibrium concepts. Within the present day approaches each equilibrium concept is addressed separately, meaning that in a particular game players interact accordingly to a unique equilibrium concept. This restriction induces unrealistic results. For example, the concept of Nash equilibrium, alone, sometimes can lead to deceptive results so we need to cope with more complex situations.

In real life players can be more or less cooperative, more or less competitive and more or less rational, therefore agents guided by the different kind of equilibrium concepts should be allowed to interact.

We consider a generalized game where players are allowed to have different behaviours according to their rationality type. Players can have different behaviours/rationality types resulting in an adequate meta-strategy concept.

Game equilibria can be characterized using appropriate generative relations [4]. Thus Nash equilibrium is characterized by the ascendancy relation [6] and Pareto equilibrium by the Pareto domination. Combining the two relations may lead to different types of joined Nash–Pareto equilibria.

We introduce a new generative relation for Nash equilibrium and we use it to compose a new joined Nash-Pareto equilibria.

An evolutionary technique for detecting the two joined Nash–Pareto equilibria for generalized games is used.

## 2 Generalized games

In order to cope with different rationality types the concept of generalized game is defined [4].

**Definition 1.** A finite strategic *generalized game* is defined as a system by  $G = (N, M, U)$  where:

- $N = \{1, \dots, n\}$ , represents the set of players,  $n$  is the number of players;

- for each player  $i \in N$ ,  $S_i$  represents the set of actions available to him,  $S_i = \{s_{i_1}, s_{i_2}, \dots, s_{i_{m_i}}\}$ ;  $S = S_1 \times S_2 \times \dots \times S_n$  is the set of all possible situations of the game;
- for each player  $i \in N$ ,  $M_i$  represents the set of available meta-strategies, a meta-strategy is a system  $(s_i|r_i)$  where  $s_i \in S_i$  and  $r_i$  is the  $i^{\text{th}}$  player rationality type;
- $M = M_1 \times M_2 \times \dots \times M_N$  is the set of all possible situations of the generalized game and  $(s_1|r_1, s_2|r_2, \dots, s_n|r_n) \in M$  is a meta-strategy profile.
- for each player  $i \in N$ ,  $u_i : S \rightarrow \mathbf{R}$  represents the payoff function.

$$U = \{u_1, \dots, u_n\}.$$

*Remark 2.* In a generalized game the set of all possible meta-strategies represents the meta-strategy search space.

The rationality type of a player usually represents the player bias towards a certain equilibrium.

### 3 Generative relations for generalized games

Three generative relations are considered in this section. Two of them correspond to Pareto and Nash equilibria. The third induces a new type of joined Nash–Pareto equilibrium.

#### 3.1 $n_p$ -strict Pareto domination

We consider the  $n_p$ -strict Pareto domination in order to be able to combine several concepts of Nash and Pareto domination.

In a finite strategic generalized game consider the set of players Pareto biased

$$I_p = \{j \in \{1, \dots, n\} | r_j = \text{Pareto}\}$$

and  $n_p = \text{card}I_p$ .

Let us consider two meta strategy profiles  $x$  and  $y$  from  $M$ .

**Definition 3.** The meta strategy profile  $x$   $n_p$ -strict Pareto dominates the meta strategy profile  $y$  if the payoff of each Pareto biased player from  $I_p$  using meta strategy  $x$  is strictly greater than the payoff associated to the meta strategy  $y$ , i.e.

$$u_i(x) > u_i(y), \forall i \in I_p.$$

*Remark 4.* The set of non dominated meta strategies with respect to the  $n_p$ -strict Pareto domination relation when  $n_p = n$  is a subset of the Pareto front.

#### 3.2 Nash - ascendancy

Similar to Pareto equilibrium a particular relation between strategy profiles can be used in order to describe Nash rationality. This relation is called Nash-ascendancy (NA).

A strategy is called Nash equilibrium [5] if each player has no incentive to unilaterally deviate i.e. it can not improve the payoff by modifying its strategy while the others do not modify theirs.

We denote by  $(s_i, s_{-i}^*)$  the strategy profile obtained from  $s^*$  by replacing the strategy of player  $i$  with  $s_i$  i.e.

$$(s_i, s_{-i}^*) = (s_1^*, s_2^*, \dots, s_{i-1}^*, s_i, s_{i+1}^*, \dots, s_n^*).$$

**Definition 5.** The strategy profile  $x$  Nash-ascends the strategy profile  $y$ , and we write  $x <_{NA} y$  if there are less players  $i$  that can increase their payoffs by switching their strategy from  $x_i$  to  $y_i$  then vice versa.

In [6] is introduced an operator

$$k : S \times S \rightarrow \mathbf{N},$$

$$k(y, x) = \text{card}\{i \in \{1, \dots, n\} | u_i(x_i, y_{-i}) \geq u_i(y), x_i \neq y_i\}.$$

$k(y, x)$  denotes the number of players which benefit by switching from  $y$  to  $x$ .

**Proposition 6.** *The strategy  $x$  Nash-ascends  $y$  ( $x$  is NA-preferred to  $y$ ), and we write  $x <_{NA} y$ , if the inequality*

$$k(x, y) < k(y, x),$$

holds.

According to [6] the set of all strategies from  $S$  non-dominated by respect of Nash ascendancy relation equals the set of Nash equilibria.

This result proves that the Nash ascendancy is the generative relation for the Nash equilibrium.

### 3.3 Differential generative relation of Nash equilibrium (DGN)

A new generative relation for Nash equilibrium is proposed. This relation relies on the payoff difference between perturbed and non perturbed strategies.

We introduce the measure

$$m(y, x) = \sum_{i \in \mathbf{N}} (u_i(x_i, y_{-i}) - u_i(y))$$

**Definition 7.** The strategy  $x$  dominates  $y$ , and we write  $x <_{DGN} y$ , if the inequality

$$m(x, y) < m(y, x),$$

holds.

### 3.4 Joint Nash–Pareto domination

Let us consider two meta-strategies  $x = (x_1|r_1, x_2|r_2, \dots, x_n|r_n)$  and  $y = (y_1|r_1, y_2|r_2, \dots, y_n|r_n)$ .

Let us denote by  $I_N$  the set of Nash biased players (N-players) and by  $I_P$  the set of Pareto biased players (P-players). Therefore we have  $I_N = \{i \in \{1, \dots, n\} | r_i = \text{Nash}\}$ .

We consider the operators  $k_P$  and  $k_N$  defined as:

$k_P(x, y) = \text{card}\{j \in I_P | u_j(x) > u_j(y), x \neq y\}$  and respectively  $k_N(x, y) = \text{card}\{i \in I_N | u_i(y_i, x_{-i}) \geq u_i(x), x_i \neq y_i\}$ .

*Remark 8.*  $k_P(x, y)$  measures the *relative efficiency* of the meta strategies  $x$  and  $y$  with respect to Pareto rationality and  $k_N(x, y)$  measures the *relative efficiency* of the meta strategies  $x$  and  $y$  with respect to Nash rationality.

**Definition 9.** The meta strategy  $x$  N–P dominates the meta strategy  $y$  if and only if the following statements hold

1.  $k_P(x, y) = n_P$
2.  $k_N(x, y) < k_N(y, x)$

In what follows we consider that efficiency relation induces a new type of equilibrium called *joined Nash-Pareto equilibrium*.

### 3.5 Joint Differential Nash Pareto domination

A new domination relation with respect to Nash-Pareto equilibrium is introduced by using differential generative relation of Nash equilibrium.

**Definition 10.** The meta strategy  $x$  DGN-P dominates the meta strategy  $y$  if and only if the following statements hold

1.  $k_P(x, y) = n_P$
2.  $m(x, y) < m(y, x)$

## 4 Detecting joint N-P equilibria in generalized games

Consider a three player non-cooperative game. Let  $r_i$  be the rationality type of player  $i$ . If  $r_1 = r_2 = r_3 = \text{Nash}$  then all players are Nash biased and the corresponding solution concept is the Nash equilibrium. If  $r_1 = r_2 = r_3 = \text{Pareto}$  then all players are Pareto biased and the corresponding equilibria are described by the set of strictly non dominated strategies (Pareto front).

We also intend to explore the joint cases where one of the players is Nash biased and others are Pareto and the one where one is Pareto and the others are Nash biased.

In order to detect the joined Nash-Pareto equilibria of the generalized game an evolutionary approach is used. For a certain equilibrium the corresponding generative relation allows the comparison of two meta-strategies. This comparison may guide the search towards the game equilibrium.

Let us consider an initial population of meta strategies for the generalized three player game. Each member of the population has the form

$$x = (s_1|r_1, s_2|r_2, s_3|r_3).$$

Non domination (with respect to a generative relation) is considered for fitness assignment purposes. Evolutionary Multiobjective Optimization Algorithms [3] are efficient tools for evolving strategies based on a non domination relation.

The state of the art NSGA2 [2] has been considered to illustrate how generative relations can be used for evolutionary detection of proposed equilibria.

A population of 100 strategies has been evolved using a rank based fitness assignment technique. In all experiments the process converges in less than 30 generations.

## 5 Numerical experiments

In order to illustrate the proposed concepts the oligopoly Cournot model is considered (see for instance [4]).

Let  $q_1$ ,  $q_2$  and  $q_3$  denote the quantities of an homogeneous product - produced by three companies respectively. The market clearing price is  $P(Q) = a - Q$ , where  $Q = q_1 + q_2 + q_3$ , is the aggregate quantity on the market. Hence we have

$$P(Q) = \begin{cases} a - Q, & \text{for } Q < a, \\ 0, & \text{for } Q \geq a. \end{cases}$$

Let us assume that the total cost for the company  $i$  of producing quantity  $q_i$  is  $C_i(q_i) = c_i q_i$ . Therefore, there are no fixed costs and the marginal cost  $c_i$  is constant,  $c_i < a$ . Suppose that

the companies choose their quantities simultaneously. The payoff for the company  $i$  is its profit, which can be expressed as:

$$\begin{aligned} \pi_i(q_i, q_j) &= q_i P(Q) - C_i(q_i) \\ &= q_i [a - (q_i + q_j) - c_i]. \end{aligned}$$

Several experiments have been performed for this game by using RED technique [4].

The symmetric Cournot model with parameters  $a = 24$  and  $c_1 = c_2 = c_3 = 9$  is considered.

According to the data from the Table 1 in less than 30 generations the algorithm converges to the Nash equilibrium point (14.00, 14.00, 14.00) for each relation. We observe that the differential Nash domination provides more accurate results than the Nash ascendancy. We must consider however the particular nature of this Cournot game. For other types of games a normalisation of the deviations must be done in order to sum them.

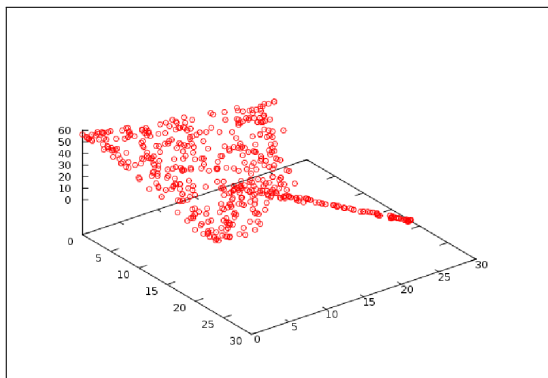


Figure 1: The payoffs for the Nash-Nash-Pareto front detected in less than 30 iterations for the symmetric Cournot game with the Nash-Pareto generative relation

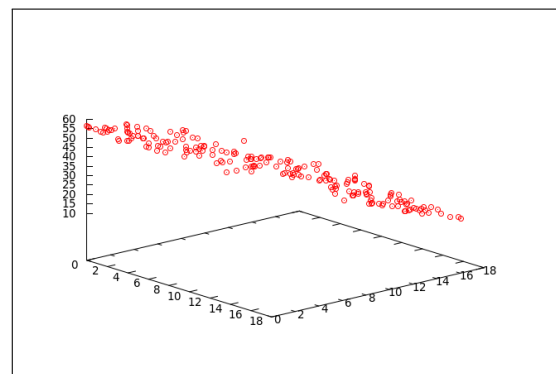


Figure 2: The payoffs for the Nash-Nash-Pareto front detected in less than 30 iterations for the symmetric Cournot game with the differential Nash-Pareto generative relation

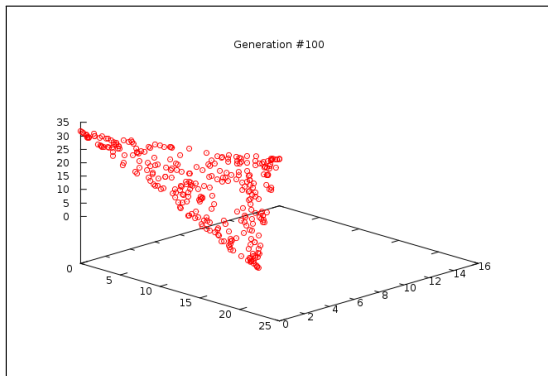


Figure 3: The payoffs for the Nash-Pareto-Pareto front detected in less than 30 iterations for the symmetric Cournot game with the Nash-Pareto generative relation.

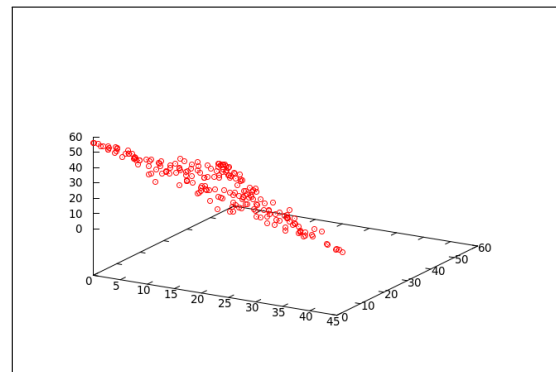


Figure 4: The payoffs for the Nash-Pareto-Pareto front detected in less than 30 iterations for the symmetric Cournot game with the differential Nash-Pareto generative relation.

The resulting front in the Nash-Nash-Pareto case spreads from the standard Nash equilibrium corresponding to the two player-Cournot game (25.00, 25.00) to the Nash equilibrium corresponding to the three player-Cournot game, and from there to the edges of Pareto front for

Table 1: Average payoff and standard deviation of the final populations in 30 runs with 100 meta-strategies after 30 generations for the symmetric Cournot model where all three players are Nash biased using Nash ascendancy and differential Nash generative relations.

| N-N-N    | Average payoff             |       |       | St. dev. |      |      | Maximum payoff |       |       | Minimum payoff |       |       |
|----------|----------------------------|-------|-------|----------|------|------|----------------|-------|-------|----------------|-------|-------|
| player   | p1                         | p2    | p3    | p1       | p2   | p3   | p1             | p2    | p3    | p1             | p2    | p3    |
|          | Nash ascendancy relation   |       |       |          |      |      |                |       |       |                |       |       |
| Average  | 14.05                      | 14.06 | 14.05 | 0.03     | 0.04 | 0.04 | 14.85          | 15.57 | 15.00 | 12.25          | 12.49 | 12.45 |
| St. Dev. | 0.02                       | 0.02  | 0.02  | 0.08     | 0.09 | 0.08 | 1.39           | 2.80  | 1.83  | 3.25           | 3.00  | 3.05  |
|          | Differential Nash relation |       |       |          |      |      |                |       |       |                |       |       |
| Average  | 14.06                      | 14.06 | 14.06 | 0.00     | 0.00 | 0.00 | 14.06          | 14.06 | 14.06 | 14.06          | 14.06 | 14.06 |
| St. Dev. | 0.00                       | 0.00  | 0.00  | 0.00     | 0.00 | 0.00 | 0.00           | 0.00  | 0.00  | 0.00           | 0.00  | 0.00  |

Table 2: Average payoff and standard deviation of the final populations in 30 runs with 100 meta-strategies after 30 generations for the symmetric Cournot model where two player are Nash biased and one is Pareto for both joint Nash–Pareto and joint Differential Nash–Pareto generative relations.

| N-N-P    | Average payoff                          |       |       | St. dev. |       |        | Maximum payoff |       |       | Minimum payoff |      |      |
|----------|---|-------|-------|----------|-------|--------|----------------|-------|-------|----------------|------|------|
| player   | p1                                      | p2    | p3    | p1       | p2    | p3     | p1             | p2    | p3    | p1             | p2   | p3   |
|          | Joint Nash–Pareto relation              |       |       |          |       |        |                |       |       |                |      |      |
| Average  | 10.99                                   | 11.01 | 29.80 | 52.81    | 53.02 | 182.28 | 25.92          | 25.71 | 56.24 | 0.00           | 0.00 | 0.49 |
| St. Dev. | 0.36                                    | 0.33  | 0.78  | 1.75     | 2.33  | 17.62  | 0.92           | 0.88  | 0.00  | 0.00           | 0.00 | 1.67 |
|          | Joint Differential Nash–Pareto relation |       |       |          |       |        |                |       |       |                |      |      |
| Average  | 8.50                                    | 8.48  | 35.53 | 22.76    | 22.67 | 165.84 | 18.47          | 18.92 | 56.24 | 0.00           | 0.00 | 7.60 |
| St. Dev. | 0.35                                    | 0.31  | 0.85  | 0.25     | 0.25  | 0.54   | 2.88           | 2.93  | 0.00  | 0.00           | 0.00 | 5.60 |

the Nash–Pareto equilibria (see Figure 1). For differential Nash–Pareto (see Figure 2) the front spreads from vicinity of the Nash equilibrium for Cournot game to the edge of the Pareto front corresponding to the Pareto player. The numerical results are presented in Table 2.

As we can see in the Figure 3 in the Nash–Pareto–Pareto case for Nash–Pareto generative relation the result is similar to the Pareto front. In the same case for differential Nash–Pareto generative relation (Figure 4) the Pareto front is deformed in the Nash player’s corresponding edge. The numerical results are presented in Table 3.

## 6 Conclusions and future work

A new generative relation for Nash equilibrium based on differences between perturbations is introduced. Generative relations between meta strategies induce corresponding solutions concepts named Joined Nash–Pareto equilibrium, respectively joint differential Nash–Pareto equilibrium.

An evolutionary technique for detecting approximations of the generalized equilibria is used. The ideas are exemplified for Cournot games with three players and two types of rationality. Results indicate the potential of the proposed technique.

Future work will address generalized games having other rationality types and other methods

Table 3: Average payoff and standard deviation of the final populations in 30 runs with 100 meta-strategies after 30 generations for the symmetric Cournot model where one player is Nash biased and the other two Pareto for both joint Nash–Pareto and joint Differential Nash–Pareto generative relations.

| N-P-P                                   | Average payoff |       |       | St. dev. |        |        | Maximum payoff |       |       | Minimum payoff |      |      |
|---|----------------|-------|-------|----------|--------|--------|----------------|-------|-------|----------------|------|------|
| player                                  | p1             | p2    | p3    | p1       | p2     | p3     | p1             | p2    | p3    | p1             | p2   | p3   |
| Joint Nash–Pareto relation              |                |       |       |          |        |        |                |       |       |                |      |      |
| Average                                 | 17.74          | 18.52 | 18.44 | 242.42   | 247.83 | 247.97 | 56.23          | 56.24 | 56.24 | 0.00           | 0.00 | 0.00 |
| St. Dev.                                | 0.40           | 0.36  | 0.42  | 7.36     | 6.98   | 6.82   | 0.04           | 0.00  | 0.00  | 0.00           | 0.00 | 0.00 |
| Joint Differential Nash–Pareto relation |                |       |       |          |        |        |                |       |       |                |      |      |
| Average                                 | 15.13          | 19.83 | 19.76 | 154.37   | 250.62 | 248.44 | 48.90          | 56.24 | 56.24 | 0.00           | 0.00 | 0.00 |
| St. Dev.                                | 0.86           | 0.77  | 0.56  | 0.78     | 0.29   | 0.32   | 3.02           | 0.00  | 0.01  | 0.00           | 0.00 | 0.00 |

of combining them.

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