THE EXTENDED BLACK-SCHOLES MODEL WITH–LAGS–AND "HEDGING ERRORS"

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1. Introduction

The Black-Scholes model is derived under the assumption that heding is done instantaneously. In practice, there is a "small" time that elapses between buying or selling the option and hedging using the underlying asset. Under the following assumptions used in the standard Black-Scholes analysis, the value of the option will depend only on the price of the underlying asset *S*, time *t* and on other variables assumed constants. These assumptions or "ideal conditions" as expressed by Black-Scholes are the following.

- The option is European,
- The short term interest rate is known,
- The underlying asset follows a random walk with a variance rate proportional to the stock price. It pays no dividends or other distributions.
- There is no transaction costs and short selling is allowed, i.e. an investor can sell a security that he does not own.
- Trading takes place continuously and the standard form of the capital market model holds at each instant.

The last assumption can be modified because in practice, trading does not take place in- stantaneouly and simultaneously in the option and the underlying asset when implementing the hedging strategy. We will modify this assumption to account for the "lag" .The lag corresponds to the elapsed time between buying or selling the option and buying or selling - delta units of the underlying assets. The main attractions of the -Black-Scholes model are that their formula is a function of "observable" variables and that the model can be extended to the pricing of any type of option. All the assumptions are conserved except the last one.

2. The Modified Black-Scholes Model and the Hedging Errors

We assume that the option price is a function of the single source of uncertainty, namely stock price and time to maturity, $c(S, t)$ and that over "short" time intervals, Δt, a hedged portfolio consisting of the option, the underlying asset and a riskless security can be formed, where portfolio weights are chosen to eliminate" market risk" , Black-Scholes expressed the expected return on the option in terms of the option price function and its partial derivatives. In fact, following Black-Scholes, it is possible to create a hedged position consisting of a sale of $\frac{1}{\left[\frac{\partial c(S,t-\tau)}{\partial S}\right]}$ options against one share of stock long.

We denote by T the delay time which corresponds to the elapsed time that induces the hedging error when the investor can not implement simultaneously and instantaneously the hedge using the option and the underlying asset. If the stock price changes by a small amount ΔS, the option changes by an amount $\left[\frac{\partial c(S,t-\tau)}{\partial S}\right]\Delta S$. Hence, the change in value in the long position (the stock) is approximately offset by the change in $\frac{1}{\left[\frac{\partial c(S,t-\tau)}{\partial S}\right]}$ options. This hedge can be maintained continuously so that the return on the hedged position becomes completely independent of the change in the underlying asset value, i.e. the return on the hedged position becomes certain.

The value of equity in a hedged position, containing a stock purchase and a sale of $\frac{1}{\left[\frac{\partial c(S,t-\tau)}{\partial S}\right]}$ options is $S - \frac{C(S,t)}{\left[\frac{\partial c(S,t-\tau)}{\partial S}\right]}$

Over a short interval Δt, the change in this position is

$$\Delta S - \frac{\Delta c(S,t)}{\left[\frac{\partial c(S,t-\tau)}{\partial S}\right]} \tag{1}$$

Where $\Delta c(S,t)$ is given by $c(S + \Delta S, t + \Delta t) - c(S,t)$.

Using stochastic calculus for $\Delta c(S,t)$ gives

$$\Delta c(S,t) = \frac{\partial c(S,t)}{\partial S}\Delta S + \frac{1}{2}\frac{\partial^2 c(S,t)}{\partial S^2}\sigma^2 S^2 \Delta t + \frac{\partial c(S,t)}{\partial t}\Delta t \tag{2}$$

2.1. The First Method and the Lags

The change in the value of equity in the hedged position is found by substituting $\Delta c(S,t)$ from (2) into equation (1) or:

$$\Delta S \left[\frac{\frac{\partial c(S,t-\tau)}{\partial S} - \frac{\partial c(S,t)}{\partial S}}{\frac{\partial c(S,t-\tau)}{\partial S}} \right] - \frac{\left(\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 c(S,t)}{\partial S^2} + \frac{\partial c(S,t)}{\partial t} \right) \Delta t}{\frac{\partial c(S,t-\tau)}{\partial S}}$$

By the Taylor development, we have:

$$\frac{\partial c(S,t-\tau)}{\partial S} - \frac{\partial c(S,t)}{\partial S} = -\tau \frac{\partial^2 c}{\partial t \partial S}(S,t-\tau)$$

Here, we can define the delay time as follows:
$\tau = a.\Delta t$

and we have:

$\Delta S.\ \Delta t = a\ (\mu S \Delta t + \sigma S \Delta w)\ \Delta t = a\ .\ (\mu S\ (\Delta t)^2 + \sigma S \Delta w\ \Delta t) = 0$

So, that first term of the previous equation is zero.

Since the return to the equity in the hedged position is certain, it must be equal to $r\Delta t$ where r stands for the short term interest rate. Hence, the change in the equity must be equal to the value of the equity times $r\Delta t$, or

$$\frac{\left(\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 c(S,t)}{\partial S^2} + \frac{\partial c(S,t)}{\partial t} \right)}{\frac{\partial c(S,t-\tau)}{\partial S}} \Delta t = \left[S - \frac{c(S,t)}{\frac{\partial c(S,t-\tau)}{\partial S}} \right] r\Delta t$$

Dropping the time and rearranging gives the modified Black-Scholes partial differential equation

$$\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 c(S,t)}{\partial S^2} - rc\ (S,t) + \frac{\partial c(S,t)}{\partial S} + rS \frac{\partial c(S,t-\tau)}{\partial S} = 0 \qquad (3)$$

or

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c(S,t)}{\partial S^2} + rS\frac{\partial c(S,t)}{\partial S} - rc(S,t) + \frac{\partial c(S,t-\tau)}{\partial S} = 0$$

This equation shows that the hedging ratio or the option delta is affected because of the hedging error. In fact, the term including the hedge ratio $rS\frac{\partial c(S,t-\tau)}{\partial S}$ which is different from the standard delta is the Black-Scholes derivation. We can simplify this equation when we use the Taylor development:

$$\frac{\partial c(S,t-\tau)}{\partial S} = \frac{\partial c(S,t)}{\partial S} - \tau\frac{\partial^2 c(S,t)}{\partial S\partial t}$$

In the presence of hedging errors, the modified Black and Scholes equation becomes

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c(S,t)}{\partial S^2} + rS\frac{\partial c(S,t)}{\partial S} - rS,\tau\frac{\partial^2 c(S,t)}{\partial St} - rc(S,t) + \frac{\partial c(S,t)}{\partial t} = 0 \quad (4)$$

The term $-rS\tau\frac{\partial^2 c(S,t)}{\partial S\partial t}$ describes the lag effect "retard effect". The partial differential equation (3) must be solved under the boundary condition expressing the call's value at maturity date $c(S, t^*.) = \max[0, S_{t^*} - K]$ where K is the option's strike price.

For the European put, the equation must be solved using the condition $P(S, t^*.) = \max[0, K - S_{t^*}]$.

We denote g as the "lag" or "retard" function. We assume that:

$$c(S,t-\tau) = g(\tau)\,.\,c(S,t)$$

with $g(0) = 1$. Let $r' = r.\, g(\tau)$.

Using the following substitution, we obtain:

$$c(S,t)=e^{r(t-t^*)}y\left[\frac{2}{\sigma^2}\; r`\frac{\sigma^2}{2}\left(\ln\frac{S}{K}-(r`\frac{1}{2}\sigma^2)(t^*-t),\right)-\frac{2(t^*-t)}{\sigma^2}(r`\frac{1}{2}\sigma^2)^2\right] \quad (5)$$

Using this substitution, the differential equation becomes $\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial S^2}$ This differential equation is the heat transfer equation of physics. The boundary condition is rewritten as $y(u, 0) = 0$

If $u < 0$ otherwise $y(u, 0) = K \left[e^{\left(\frac{\frac{1}{2}u\sigma^2}{r` \frac{1}{2}} \right)} - 1 \right]$

The solution to this problem is the solution to the heat transfer equation.

$$y(u, s) = \frac{1}{\sqrt{2\Pi}} \int_{\frac{-u}{\sqrt{2s}}}^{\infty} K \left[e^{\frac{1/2(u + q\sqrt{2}s)\ \sigma^2}{r` - 1/2\ \sigma^2}} - 1 \right] e^{(-q^2/2)}dq$$

Substituting from this last equation into (5) gives the following solution for the European call price with $T = t^* - t$.

$$c(S,T) = e^{T(r`-r)}\, SN(d_1) - Ke^{-rT}\, N(d_2)$$

which can be expressed as

$$c(S,T) = e^{-Tr(1-g(\tau))}\, SN(d_1) - Ke^{-Tr\, g(\tau)}N(d_2)$$

with $g(\tau) = e^{-}\beta^{T^*}$ is the delay coefficient for the case without lags, we have $g(0) = 1$ and for very large delay $(T \longrightarrow \infty)$

$\lim_{T\to\infty} g(\tau) = 0$ with means that the interest rate will converge to zero.

$$d`_1 = \frac{1n(S/K) + (r` + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$d'_2 = d'_1 - \sigma\sqrt{T}$ where *N(.)* is the cumulative normal density function.

It is important to note that the option value is independent of its underlying asset expected return. This may sound rather strange. One intuitive way to account for this is to say the expected return on the stock is already embedded into the stock price itself. The value of the put option can be obtained from that of the call option using the put-call parity relationship.

2.2. *The Second Method and The Hedging Errors*

Following the previous analysis, we can consider the delay time as $\tau = a\,\Delta t$. We have $\Delta S.\,\Delta t = a\,(\mu S\Delta t + \sigma S\Delta w)\,\Delta t = a\,.\,(\mu S\,(\Delta t)^2 + \sigma S\Delta w\,\Delta t) = 0$. Since the return to the equity in the hedged position is certain, it must be equal to $r_1\Delta t$ where r_1 stands for the short term interest rate. Hence, the change in the equity must be equal to the value of the equity times $r_1\Delta t$, or

$$-\frac{\left(\frac{1}{2}\sigma^2 S^2\,\frac{\partial^2 c(S,t)}{\partial S^2} + \frac{\partial^2 c(S,t)}{\partial t}\right)\Delta t}{\frac{\partial c(S,t-\tau)}{\partial S}} = \left[S - \frac{c(S,t)}{\frac{\partial c(S,t-\tau)}{\partial S}}\right] r_1\,\Delta t \qquad (6)$$

with $r_1 = r + \lambda W(l)$ where *l* is "the temporal advantage in recuperating information", "l'avantage temporelle de la récupération de l'information" and *W(l)* is the function that - determines the recuperation of information".

We propose $W(l) = 1 - e^{\cdot}\sigma^{l}$ where the advantage coefficient a is defined a

$$W(0) = 0$$
$$\lim_{T\to\infty} W(l) = 1$$

Dropping the time and rearranging gives the partial differential equation

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 c(S,t)}{\partial S^2} - r_1 c(S,t)\,\frac{\partial c(S,t)}{\partial S} + r_1 S\,\frac{\partial c(S,t-\tau)}{\partial S} = 0 \qquad (7)$$

or

$$\frac{1}{2} s^2 S^2 \frac{\partial^2 c(S,t)}{\partial s^2} + r_1 S \frac{\partial c(S,t-\tau)}{\partial s} - r_1 c(s,t) + \frac{\partial c(S,t)}{\partial t} = 0 \qquad (7)$$

We can simplify this equation when we use the Taylor development

$$\frac{\partial c(S,t-\tau)}{\partial s} = \frac{\partial c(S,t)}{\partial s} - T \frac{\partial^2 c(S,t)}{\partial s \partial t}$$

so that the modified Black and Scholes equation becomes

$$\frac{1}{2} \sigma^2 S^2 \frac{\partial^2 c(S,t)}{\partial s^2} + r_1 s \frac{\partial c(s,t)}{\partial s} - r_1 st + \frac{\partial^2 c(s,t)}{\partial s \partial t} - r_1 c(s,t) + \frac{\partial c(s,t)}{\partial t} = 0$$

The term $- r_1 s\tau \frac{\partial^2 c(S,t)}{\partial s \partial t}$ describe the lag "retard" effect. The partial differential equation (7) can be solved under the call's boundary condition $c(S,t^*) = \max [0, S_{t^*} - K]$ where *K* is the option's strike price.

For the European put, the equation must be solved using :

$P(S,t^*) = \max [0, K – S_{t^*}]$

We assume that : $c(S,t - \tau) = g(\tau) . c(S,t)$

with $g(0) = 1$ and $g(\tau) = e^{-b^T}$ where b is the delay coefficient.

We denote by g the lag or "retard" function.

Let $r' = r_1 . g(\tau)$ .The equation of the modified Black and Scholes can be fund with the following substitution :

$$c(S,t) = e^{r'_1 (t-t^*)} y \left[\frac{2}{\sigma^2} (r' - \frac{\sigma^2}{2}) \left(\ln \frac{S}{K} - (r' - \frac{1}{2} \sigma^2)(t^* - t), \right) - \frac{2(t^* - t)}{\sigma^2} (r' - \frac{1}{2} \sigma^2) \right]^2 \qquad (8)$$

with $r' = r^1 g(\tau) = [r + \lambda W(1)] g(\tau)$

Using this substitution, the differential equation becomes $\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial s^2}$

The boundary condition is rewritten as y (u, 0) = 0 if u < 0

$$\text{otherwise } y(u, 0) = K\left[e^{\left(\frac{1/2u\sigma^2}{r` - 1/2\sigma^2}\right)} - 1 \right]$$

The solution is that of the heat transfer equation

$$y(u, s) = \frac{1}{\sqrt{2\Pi}} \int_{\frac{-u}{\sqrt{2s}}}^{\infty} K \left[e^{\frac{1/2\,(u + \sqrt{q}\ 2s)\,\sigma^2}{r` - 1/2\ \sigma^2}} - 1 \right] e^{(-q^2/2)} dq$$

Substituting from this last equation into (8) gives the European call price with: $T = t'' - t$

$$c(S,T) = e^{T(r_1 - r')} N(d_1) - Ke^{-r'T} N(d_2)$$

or :

$$c(S,T) = e^{-Tr1(1-g(\tau))}\ S\ N(d_1) - Ke^{-r1g(\tau)} N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r` + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

where *N(.)* is the cumulative normal density function.

We obtain the compensation between the delay and the temporal advantage.

When $r = r'$: $r = [r + \lambda W(l)]\, g(\tau)$

with:

$$W(l) = 1 - e^{-al}$$

$$g(\tau) = \frac{r}{r + \lambda W(l)} \qquad \text{or}$$

$$g(\tau) = \frac{r}{r + \lambda\,(1 - e^{-\alpha l})}$$

Summary

This paper provides a modified Black-Scholes (1973) formula in the presence of hedging errors. The formula accounts for a "lag" resulting from the non-simultaneous trading in the option and its underlying asset when implementing the hedging portfolio. Two methods are proposed and two formulas are provided to account for "lags" in the hedging process. The lags may reflect problems of liquidity in the market place.

Reference

Black, F., & Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, *81*, 637-659.