AN INTERPRETATION OF THE AHP GLOBAL PRIORITY AS THE EIGENVECTOR SOLUTION OF AN ANP SUPERMATRIX

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ABSTRACT

Not only the local priority vectors but also the global synthesized priority vector for the alternatives can be obtained from the eigenproblem solution for the ANP supermatrix. This global priority vector retains the main property of any AHP vector – to present the mean dominance of each of the items over the others with respect to the goal of the hierarchy.

Keywords: AHP local and global vectors, eigenproblem of supermatrix, ANP supermatrix.
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"*When making a decision of minor importance, I have always found it advantageous to consider all the pros and cons. In vital matters, however, such as the choice of a mate or a profession, the decision should come from the unconscious, from somewhere within ourselves. In the important decisions of personal life, we should be governed, I think, by the deep inner needs of our nature*." Sigmund Freud.

1. Introduction

In the article, *An Interpretation of the AHP Eigenvector Solution for the Lay Person,* in the previous issue of this journal (Lipovetsky, 2010), we showed how the priority vector for an Analytic Hierarchy Process (AHP) pairwise comparison judgment matrix can be derived through an iterative process and that the vector obtained in this way is an eigenvector of the original judgment matrix. Similarly an eigenproblem solution can be obtained for an AHP problem, formulated in the supermatrix of the Analytic Network Process (ANP) rather than as a hierarchy; it gives the overall relative priorities of the alternatives with respect to the goal, and also the synthesized relative priorities of the alternatives with respect to all the other nodes in the model as well.

2. The Hierarchical Synthesis Process

Synthesizing local priority vectors into global priority vectors for an AHP model can be easily performed by multiplying the matrix Z comprised of the local eigenvectors of the alternatives with respect to the criteria, times a column vector X of criteria priorities with respect to the goal. This is equivalent to performing additive hierarchical composition.

Consider the example shown in Table 1 of "Choosing the best house" (Saaty, 1996, pp. 26-31). This classical AHP example has also been considered in (Saaty & Kearns, 1985, ch.3; Saaty and Vargas, 1994, ch.1). This data was also used for testing some new techniques in (Lipovetsky, 1996, 2005; Lipovetsky & Tishler, 1999; Lipovetsky & Conklin, 2002). In this example three houses are evaluated with respect to 8 criteria. The top row in Table 1 contains the weights of the criteria obtained by pairwise comparing the criteria with respect to the goal. This row vector may be labeled X' (where the prime denotes transposition) so that X is a column vector. The matrix Z is the 3×8 matrix containing the local priority vectors of the alternatives with respect to the criteria in Table 1. The last column contains the product of ZX, the global priority vector. As Z is 3×8 and the X is 8×1, ZX is a column vector of order 3×1.

Table 1
Local and global priority vectors for "Choosing the best house" (Saaty, 1996)

Criteria weights	size	yard	trans-port	facil-ities	neigh-borhd	condi-tion	age	finan-ce	Global vector
	.1730	.0540	.1881	.0175	.0310	.0363	.1669	.3332	
house A	.7536	.6738	.2331	.7466	.7536	.2000	.3333	.0719	.3338
house B	.1811	.1007	.0545	.0601	.0653	.4000	.3333	.6491	.3365
house C	.0653	.2255	.7124	.1933	.1811	.4000	.3333	.2790	.3296

The ANP supermatrix for an AHP model that includes inner-dependent criteria has been considered by Saaty (1994, pp. 245-6; 1996, pp. 132-133; 2010, p. 190) and is shown below (with a first row of zeros for the goal):

$$W = \begin{pmatrix} 0 & 0 & 0 \\ X & Y & 0 \\ 0 & Z & I \end{pmatrix}, \qquad (1)$$

Our house model can be arranged into a supermatrix W with components X, Y and Z in the form shown in equation (1) from (Saaty, 1996, p. 97). The supermatrix W of our house hierarchy consists of a goal with n criteria and m alternatives. The top row of zeros contains the priorities of the goal node with respect to all the other nodes in the model. The goal is a source node that connects only *to* other nodes and not *from* them, so it has only zero priorities with respect to the other nodes. The X component is an $n \times 1$ column vector of weights derived by pairwise comparing the criteria with respect to the goal, Y is an $n \times n$ order matrix of priority vectors derived for inner-dependent criteria, Z is an $m \times n$ matrix of priority vectors of the alternatives with respect to the criteria, and for a hierarchy, it is necessary to include an identity matrix I of order $m \times m$ in the right hand bottom corner of W, thus the matrix W is of the order $m + n + 1$.

This formulation leads to a supermatrix W of the 12th order, shown in Table 2. If there were subcriteria, the supermatrix would need to be expanded to accommodate them and all the rows would no longer be stochastic. Any column can be made stochastic by normalizing it. The vector X is the 8×1 vector of criteria priorities with respect to the Goal (blue); Y is an 8×8 matrix of priority vectors, one for each of the criteria in the case where the criteria are inner dependent (shown in yellow), but *Y* contains only zeros as for a hierarchy where the criteria are not inner dependent; *Z* is the 3×8 matrix of priority vectors of alternatives with respect to the criteria (in green); and *I* is the 3×3 identity matrix (pink).

Following Saaty's approach of raising this matrix to powers until it reaches a stable solution (at the second power in this example), we obtain the global priority vector, with respect to the goal, in the first column in Table 3. It is the same as the results in Table 1 obtained using hierarchic composition. The results for hierarchic composition coincide with the vector product *ZX*.

Table 2
The supermatrix *W* containing the local priority vectors for the house hierarchy

		GOAL	CRITERIA								ALTERNATIVES		
		Goal	Size	Yard	Transprt	Facilities	Nghbrhd	Condition	Age	Finance	House A	House B	House C
GOAL	Goal	0	0	0	0	0	0	0	0	0	0	0	0
CRITERIA	Size	0.173	0	0	0	0	0	0	0	0	0	0	0
	Yard	0.054	0	0	0	0	0	0	0	0	0	0	0
	Transprt	0.1881	0	0	0	0	0	0	0	0	0	0	0
	Facilities	0.0175	0	0	0	0	0	0	0	0	0	0	0
	Nghbrhd	0.031	0	0	0	0	0	0	0	0	0	0	0
	Condition	0.0363	0	0	0	0	0	0	0	0	0	0	0
	Age	0.1669	0	0	0	0	0	0	0	0	0	0	0
	Finance	0.3332	0	0	0	0	0	0	0	0	0	0	0
ALTERNATIVES	House A	0	0.7536	0.6738	0.2331	0.7466	0.7536	0.2000	0.3333	0.0719	1	0	1
	House B	0	0.1811	0.1007	0.0545	0.0601	0.0653	0.4000	0.3333	0.6491	0	1	0
	House C	0	0.0653	0.2255	0.7124	0.1933	0.1811	0.4000	0.3333	0.2790	0	0	0

Table 3
The limit supermatrix *W* for the house example; the global priority vector for the alternatives is shown in blue

		GOAL	CRITERIA								ALTERNATIVES		
		Goal	Size	Yard	Transprt	Facilities	Nghbrhd	Condition	Age	Finance	House A	House B	House C
GOAL	Goal	0	0	0	0	0	0	0	0	0	0	0	0
	Size	0	0	0	0	0	0	0	0	0	0	0	0
	Yard	0	0	0	0	0	0	0	0	0	0	0	0
	Transprt	0	0	0	0	0	0	0	0	0	0	0	0
CRITERIA	Facilities	0	0	0	0	0	0	0	0	0	0	0	0
	Nghbrhd	0	0	0	0	0	0	0	0	0	0	0	0
	Condition	0	0	0	0	0	0	0	0	0	0	0	0
	Age	0	0	0	0	0	0	0	0	0	0	0	0
	Finance	0	0	0	0	0	0	0	0	0	0	0	0
	House A	0.3339	0.7536	0.6738	0.2331	0.7466	0.7536	0.2000	0.3333	0.0719	1	0	0
ALTERNATIVES	House B	0.3365	0.1811	0.1007	0.0545	0.0601	0.0653	0.4000	0.3333	0.6491	0	1	0
	House C	0.3296	0.0653	0.2255	0.7124	0.1933	0.1811	0.4000	0.3333	0.2790	0	0	1

We shall now show a second method of obtaining the synthesized global priority vector. Instead of raising the supermatrix W of equation (1) to powers, we solve the transpose of this matrix in a straightforward manner for its left eigenvectors. The left eigenvectors of (2) coincide with the right eigenvectors of W but are obtained from the transposed matrix:

$$W'\gamma = \lambda\gamma \tag{2}$$

For a hierarchy of only three levels, such as the house model, the columns of W are stochastic, that is, they sum to one. The left eigenvectors are needed for the transposed matrix because they correspond correctly to the calculations by its blocks for the needed product *ZX* that gives the synthesized global vector. The left eigenvectors and their eigenvalues of W', the transpose of the matrix in Table 2, are given in Table 4. Three of the columns have eigenvalues equal to 1, and these columns contain the synthesized global (row) vector for the alternative houses in the Goal node row: House C = 0.3296, House B = 0.3365, and House A = 0.3339. In this solution the order of the columns is the reverse of the order of the rows. The synthesized result of the alternatives with respect to each criterion can also be read from the rows of Table 4.

It is useful to note the following to clarify the problem. As is well known, Perron proved that every positive matrix has a largest real eigenvalue that is greater than the absolute value of any of the others (including the complex ones). Frobenius carried the work on to matrices with zeros in them, like our supermatrix in Table 2. He proved there could be *more* than one largest eigenvalue. And the supermatrix W' is a transposed stochastic one whose eigenvalues are all one. So when we transpose it in (2) it becomes a stochastic matrix with the rows equaling one, as in our example which has multiple eigenvalues of 1.

Table 4
All twelve eigenvectors (columns) and associated eigenvalues (bottom row) for the supermatrix with the synthesized global priorities for the houses in row 1.

	House C	House B	House A	Fin-ance	Ag-e	Con-dition	Neig-hbo-ur	Fac-Ilities	Tran-sport	Ya-rd	Si-ze	Go-al
Goal	0.3296	0.3365	0.3339	-1	-1	-1	-1	-1	-1	-1	-1	1
Size	0.0653	0.1811	0.7536	0	0	0	0	0	0	0	0	0
Yard	0.2255	0.1007	0.6738	0	0	0	0	0	0	0	0	0
Transport	0.7124	0.0545	0.2331	0	0	0	0	0	0	0	0	0
Facilities	0.1933	0.0601	0.7466	0	0	0	0	0	0	0	0	0
Neighbour	0.1811	0.0653	0.7536	0	0	0	0	0	0	0	0	0
Condition	0.4	0.4	0.2	0	0	0	0	0	0	0	0	0
Age	0.3333	0.3333	0.3333	0	0	0	0	0	0	0	0	0
Finance	0.279	0.6491	0.0719	0	0	0	0	0	0	0	0	0
HouseA	0	0	1	0	0	0	0	0	0	0	0	0

HouseB	0	1	0	0	0	0	0	0	0	0	0	0
HouseC	1	0	0	0	0	0	0	0	0	0	0	0
Eigen-Values	1	1	1	0	0	0	0	0	0	0	0	0

Inverting the arrangement of the three main eigenvectors to put the houses back into the proper order with which we started, and extracting the synthesis information we are interested in, we obtain Table 5. It shows the global synthesized priorities with respect to the Goal in the first row along with the global synthesized priorities of all the other nodes (in rows). The global priority vectors are the same as the original local priority vectors for a simple three level hierarchy. However, for a hierarchy of more than 3 levels, the synthesis is more interesting as the global priorities are not usually the same as the local priorities for the other nodes in the model.

Table 5
Local and global priorities obtained from the main eigenvectors from the supermatrix for the "Choosing the best house" example

Names of nodes in hierarchy	Eigenvectors of the Supermatrix		
	House A	House B	House C
Goal	.3338	.3365	.3296
Size	.7536	.1811	.0653
Yard	.6738	.1007	.2255
Transport	.2331	.0545	.7124
Facilities	.7466	.0601	.1933
Neighbor	.7536	.0653	.1811
Condition	.2000	.4000	.4000
Age	.3333	.3333	.3333
Finance	.0719	.6491	.2790
House A	1	0	0
House B	0	1	0
House C	0	0	1
eigenvalues λ	1	1	1

3. Conclusion

The eigenvectors in Tables 4 and 5 give the same results in the first row for the global synthesis vectors as that obtained using the regular AHP synthesis shown in Table 1. Thus, the global AHP priority can be found by solving the supermatrix' eigenproblem for its left eigenvectors, which in a way similar to the local eigenvectors represents the mean dominance of each item over the others.

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