THE EIGENVECTOR IN LAY LANGUAGE

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ABSTRACT

Decision making depends on identifying a structure of criteria and alternatives of a decision. It also depends on experience and judgments to select the best alternative. In the Analytic Hierarchy Process (AHP) for decision making the criteria and alternatives are prioritized by forming matrices of judgments and from these judgments priorities are derived for each matrix in the form of the principal eigenvector. An *eigenvector* is a technical mathematical idea that would benefit from a simplifying explanation. That is what this note does - in two ways.

Keywords: AHP, decision making, principal eigenvector

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1. Motivation

An eigenvector is the solution of a problem (a system of equations) that depends on several factors. In the case of decision problems, one is concerned with the identification of the best alternative among several alternatives and the solution represents the best decision. In geometry we know that to determine the position of a point in space three Cartesian coordinates are required. Therefore, the factors on which the solution depends are only three. Generally, however, in decision problems the parameters to be taken into account to reach a decision may be many. For example, when we choose the city where to live we need to examine and identify the criteria that qualify the best city for us to live in. Therefore, **criteria** such as employment opportunities, entertainment, good schools for our children, not too high a standard of living are our system of equations. A**lternatives** such as Orlando, Pittsburgh, Washington are our system of unknowns to find the best among them. The d**ecision** about **which city is best** (the goal) is our solution.

At this point, once we understand what an eigenvector is, it is essential to understand how the assessment process occurs to identify the best solution. The process is based on the comparison judgments that we express in terms of preference for a solution rather than for another. Here we have introduced the concepts of many criteria and many alternatives in a decision and the need for an eigenvector in each set of comparisons. Thus we have many systems of equations each having its own eigenvector. One then also needs to talk about synthesis. This synthesis can be obtained from the supermatrix, a stochastic matrix with largest eigenvalue equal to one and whose corresponding principal eigenvector is the synthesis of all other eigenvectors.

2. What we learn when we have measurement

The following explanation of what an eigenvector is begins with a consideration of comparing items (apples) in pairs whose weight is known. We then show that the same process can be used to derive relative weights for apples where their weights are not known, but we can guess as we compare them in pairs. The latter way of estimating the pairwise comparisons can be used to compare items that are not measurable, for example on critical issues where judgments must be made as to which issue is more important with a certain goal in mind. In this case, too, the measurements are unknown in advance and must be derived from the quantitative judgments. Notice that all the solutions are presented as relative numbers.

Suppose we have three apples, A, B, and C, whose weights in ounces are known to be 6, 3, and 1; to derive their relative weights we compare them in pairs and enter the comparison as shown in Figure 1 below. To form a comparison judgment the size of the apple at the left of the row is written in the numerator and the size of the apple at the top of the column is written in the denominator as shown in Figure 1.

Size Comparison	Apple A	Apple B	Apple C
Apple A	6/6	6/3	6/1
Apple B	3/6	3/3	3/1
Apple C	1/6	1/3	1/1

Figure 1 Pairwise comparison matrix with known weights

The weights of these apples are given in ounces but if the weights are transformed into pounds or kilograms the absolute values would be different but the proportions in the comparisons and hence the fractions would be the same. It does not matter what scale is used to make the measurements if proportions are used rather than absolute measurements. To get the proportionate measures, we add the numbers obtained from using any scale of measurement and divide each value by the total. For example the respective relative weights of the apples A, B, C using their measurements in ounces are:

$$\frac{6}{6+3+1}=.6, \frac{3}{6+3+1}=.3, \frac{1}{6+3+1}=.1.$$

These fractions give us the part of the whole for each apple, regardless of the scale used to make the measurements. The decimals .6, .3, and .1 are referred to as the priorities for apples A, B, and C according to weight.

The table of comparisons can be rewritten using these priorities instead of the original weights as shown in Figure 2.

Size Comparison	Apple A	Apple B	Apple C
Apple A	.6/.6	.6/.3	.6/.1
Apple B	.3/.6	.3/.3	.3/.1
Apple C	.1/.6	.1/.3	.1/.1

Figure 2 Matrix of pairwise comparisons using priorities

2. What to do if we have no measurements

If the weights of the apples are unknown, we estimate their relative sizes by using judgments; Apple A is 2 times the size of Apple B; Apple B is 1/2 the size of Apple A and so on. Each apple is compared in its row with all the apples using the smaller apple as the unit and estimating the larger as a multiple of that unit. The priority vector on the right in Figure 3 is any of the column entries of the comparisons matrix normalized to one or the sum of each of the the three rows normalized using their total sum of all rows.

Size Comparison	Apple A	Apple B	Apple C	Resulting Priority Eigenvector
Apple A	1	2	6	6/10
Apple B	1/2	1	3	3/10
Apple C	1/6	1/3	1	1/10

Figure 3 Comparison according to volume

How do we then obtain the relative weights of the apples from our judgments? In that case we follow what we did when we have measurements above, but because we don't have priorities for the apples we assume they are unknowns and we need to find the true values. Thus we multiply by three unknowns in place of the three priority measurements .6, .3, .1 . We note that the one in the first row and first column in this table represents the comparison of the large apple with itself, but the one in the middle position represent comparison of the medium apple with itself and the one in the third row and third column represents comparison of the small apple with itself. These *ones* do not convey the information about the size of the apple being compared. We must introduce the weights of the apples to make the different ones mean what they say. But as we just said, the weights are unkown and must be indicated for now by unknown variables x, y, z. Thus we multiply the numbers in the first column by x, those in the second second column by y and those in the third column by z. Now we can add the weighted numbers in each row. Following our example above we require that the solution should be a constant multiple *a* (whose value in the example above was equal to 3). For this example, we have for the sum of the weighted rows respectively:

$$x+2y+6z=ax$$

$$\frac{x}{2}+y+3z=ay$$

$$\frac{x}{6}+\frac{y}{3}+z=az$$

and for the sum of the weighted columns we have

$$\frac{10}{6}x+\frac{10}{3}y+10z=a$$

We must solve a system of four equations in four unknowns x, y, z, a.

Here is how we get that solution. If we multiply the second equation by two and subtract the resulting equation from the first equation we get $y=\frac{x}{2}$. If we multiply the third eqaution by six and subtract the resulting equation from the first equation, we get $z=\frac{x}{6}$.

If we now normalize by dividing each of x, y, z by the sum of their values we get:

$$\frac{x}{x+y+z}=\frac{x}{x+\frac{x}{2}+\frac{x}{6}}=\frac{6}{10}$$

$$y=\frac{x}{2}=\frac{3}{10}$$

$$z=\frac{x}{6}=\frac{1}{10}$$

and from the fourth eqaution we have on substituting these values for x,y and z that $a=3$.

The three variables together (.6, .3, .1) define what is known in mathematics as an eigenvector of the matrix of the numerical coefficients found in the first three equations, and the constant *a* is called an eigenvalue of that matrix.

Note: (Saaty, 1996) It is known in mathematics that an *n* x *n* matrix has *n* eigenvalues and *n* corresponding eigenvectors. The eigenvalues may not be distinct and in general are often complex conjugates. When a matrix has positive entries, Perron's theory assures us that the matrix has a real positive eigenvalue that dominates all other eigenvalues in modulus and a corresponding eigenvector that has positive entries. These are respectively called the principal eigenvalue and the principal eigenvector of that matrix. Perron-Frobenius theory extends the idea to non-negative matrices with slight modification on the dominance of the principal eigenvalue over the other eigenvalues and the entries of the eigenvector, some of which may now be zero. Frobenius further showed that when a matrix is "irreducible" and thus has a block of zeros of the form:

$$\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$$

the principal eigenvalue need not strictly dominate the moduli of the other eigenvalues but may be equal to them.

3. A second interpretation

The priority of the importance of each apple according to its relative is the sum of all the numbers that represent the judgments in its row of the comparisons. These numbers indicate how much it dominates every other element. But all the elements are not equally important. If we know how important they are we would use that priority of importance to weight each judgment by it and then add the weighted numbers in each row and get these priorities back. Not knowing the priorities, we assume that all the elements are equally important and use the same constant number to weight the judgments in each row and add over that row. Doing that, we get a first estimate of the priorities. This estimate is the exact priorities when we have measurements. We use this first estimate of the priorities to weight the judgments in each row and add the weighted numbers in each row to get a new estimate of the priorities of the elements. We stop if the first set of priorities is identical to the second set. Otherwise, again we use this second set of priorities to weight the judgments in each row and add the weighted numbers to get a third estimate. We continue the process until the last estimate of the priorities is close enough for our need of accuracy to the one before it. Now we have the priorities we are looking for. Computing the eigenvector does exactly what we just described above.

4. Ratio Scales and Interval Scales as Utilities

Some researchers of Multiple-Criteria Decision-Making prefer to obtain the priorities using the utility values instead of the eigenvector. Some of them are critics of the eigenvector, because of a misunderstanding according to Harker & Vargas (1987) of the theoretical foundation of the AHP or reluctance to move away from traditional methods used in multi-attribute utility theory which uses interval instead of the customary ratio scales used for measurement in science. To convert measurements from a ratio scale to an interval scale, take any column of priorities and identify the smallest among the three values, subtract this value from each of the three values, then divide by the maximum value among the resulting three values as illustrated in the three columns of Figure 4. Several ratio scale values can lead to the same interval scale value resulting in loss of information.

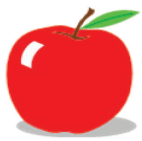
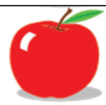
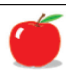
	Weight [in ounces, a ratio scale measurement]	Normalized Eigenvector	Utility
 Apple A	6	0.6	(0.6 - 0.1)/0.5 = 1
 Apple B	3	0.3	(0.3 - 0.1)/0.5 = 0.4
 Apple C	1	0.1	(0.1 - 0.1)/0.5 = 0

Figure 4 Converting ratio scale measurements to interval scale measurements

Let us suppose that Apple A is rotten and someone replaces it with Apple D, weighing 4 oz; the new values for the eigenvector and utility priorities are shown in Figure 5.

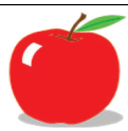
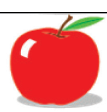
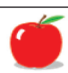
	Weight [ounces]	Normalized Eigenvector	Utility
 Apple D	4	0.500	1
 Apple B	3	0.375	0.667
 Apple C	1	0.125	0

Figure 5 Replacing Apple A with Apple D

As we can see, only the utility of the weight of Apple B will be changed from 0.4 to 0.667, so the heaviest and the lightest apples will have their utility unchanged, even though the heaviest now weighs 50% less. The eigenvector is more sensitive to capture and portray this kind of change that happens often in real world decisions where factors are measured in days, dollars, people and so on.

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