PRIORITY VECTOR ESTIMATION: CONSISTENCY, COMPATIBILITY, PRECISION

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ABSTRACT

Various methods of priority vector estimation are known in the Analytic Hierarchy Process (AHP). They include the classical eigenproblem method given by Thomas Saaty, developments in least squares and the multiplicative approach, robust estimation based on transformation of the pairwise ratios to the shares of preferences, and other approaches. In this paper, the priority vectors are completed with validation of data consistency, comparisons of vectors' compatibility, and estimation of precision for matrix approximation by vectors. The numerical results for different data sizes and consistency show that the considered methods reveal useful features, are simple and convenient, and capable of facilitating practical applications of the AHP in solving various multiple-criteria decision making problems.

Keywords: AHP priority vector estimations; consistency measures; *S*-index and *G*-index of compatibility; precision of fitting

1. Introduction

The Analytic Hierarchy Process (AHP) is a widely used methodology and a set of methods for solving various problems of prioritization. Developed by Thomas Saaty and expanded in numerous works by many authors, it is currently one of the main approaches for managers and practitioners who need to apply multi-criteria decision making to reach their goals. In this work, the term AHP is used not in its whole rich entirety but in a narrower sense as a method of finding local priority vectors by a pairwise comparison matrix. Estimations of priority vectors in the AHP include the classical eigenproblem method (EM) proposed by Saaty (1977, 1980, 1994, 1996, 2005), the least squares (LS) solution and the multiplicative or logarithmic (LN) least squares described in Saaty and Vargas (1984, 1994) and Lootsma (1993, 1999), and numerous other modifications (for instance, Lipovetsky, 1996, 2009, 2013; Lipovetsky and Tishler, 1999). Particularly, priority vector robust estimation (RE) which is not prone to possible inconsistencies in pairwise judgements can be based on the ratio transformation to the shares of preferences and obtained by Markov chain modeling for steady-state probabilities (Lipovetsky and Conklin, 2002, 2015).

The current work presents the results of comparisons between the EM, LN, LS, and RE using several characteristics of closeness for the obtained solutions, including pair

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correlations, the so-called Saaty compatibility index (*S*-compatibility) (Saaty, 2005; Saaty and Peniwati, 2007), and the Garuti compatibility index or *G*-compatibility (Garuti, 2007; Garuti and Salomon, 2011). For different sizes and consistency of the matrices of judgement used in the classical AHP literature, the priority vectors are calculated, their compatibility indices estimated, and characteristics of the matrix fit by the vectors are described. In general, the explored methods are simple and convenient and can significantly facilitate practical applications of the AHP for optimum solutions in various problems.

The paper is organized as follows: Section 2 describes the methods of priority estimation, Section 3 defines the measures of compatibility and quality of fit, Section 4 discusses several numerical examples, and Section 5 concludes on the obtained results.

2. Priority vector estimations

Let us briefly describe several main methods of priority vector estimations. The general form of the AHP pairwise priority ratios matrix can be written as follows:

$$A = \begin{pmatrix} 1 & a_{12} & \dots & a_{1n} \\ a_{21} & 1 & \dots & a_{2n} \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ a_{n1} & a_{n2} & \dots & 1 \end{pmatrix}. \quad (1)$$

This is a Saaty matrix of pairwise judgements among n items, elicited from an expert. Each element a_{ij} shows a quotient of preference of the i-th item over the j-th item in their comparison, so we have the transposed-reciprocal elements $a_{ij} = a_{ji}^{-1}$. A theoretical Saaty matrix of pair comparisons defines each ij-th element as a ratio of the unknown priorities w_i and w_j:

$$W = \begin{pmatrix} w_1/w_1 & w_1/w_2 \;\dots & w_1/w_n \\ ------ & -------- & ---- \\ w_n/w_1 & w_n/w_2 \;\dots & w_n/w_n \end{pmatrix} = w * (\frac{1}{w})'. \quad (2)$$

The vector-column w consists of the elements $w_1, w_2, \dots, w_n$, the vector-row $(1/w)'$ contains the reciprocal values $1/w_1, 1/w_2, \dots, 1/w_n$, and the right-hand side of the relationship (Equation 2) shows the outer product of these two vectors (where the prime denotes transposition). From Equation (2), it is easy to find the identical relation $Ww = nw$ for the theoretical matrix and vector. For the obtained matrix (Equation 1), a similar relationship can be presented as the eigenproblem:

$$A\alpha = \lambda\alpha\,, \quad (3)$$

where the first eigenvector alpha for the maximum eigenvalue λ defines the vector of priorities. This is the eigenvector method EM of the classical AHP.

Another known way is the least squares estimation for the priority vector which can be expressed via the following eigenproblem:

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$$(AA')\alpha = \lambda^2 \alpha \,, \tag{4}$$

The main vector alpha yields the priority vector in the LS approach.

The third popular approach to priority estimation is called the multiplicative or logarithmic technique. It can be reduced to calculating the elements of the priority vector as the geometric means of the elements in each row of the matrix (Equation 1):

$$\alpha_i = \sqrt[n]{\prod_{j=1}^{n} a_{ij}} \ . \tag{5}$$

The obtained AHP priority vectors are also standardized by the total of the elements, so a solution is divided by the total of all of the elements and the sum of the normalized components equals one:

$$\alpha_{i_{normalized}} = \alpha_i / sum(\alpha_i) \ . \tag{6}$$

This is the priority vector estimation in the LN approach.

The solution with robust estimation (RE) is less prone to possible inconsistencies in the pairwise judgements; let us introduce a theoretical matrix of shares as follows:

$$U = \begin{pmatrix} w_1/(w_1+w_1) & w_1/(w_1+w_2) & \dots & w_1/(w_1+w_n) \\ ----- & ------- & & ---- \\ w_n/(w_n+w_1) & w_n/(w_n+w_2) & \dots & w_n/(w_n+w_n) \end{pmatrix}, \tag{7}$$

Each element u_{ij} in Equation (7) is defined as i-th priority in the sum of i-th and j-th priorities:

$$u_{ij} = \frac{w_i}{w_i+w_j} = \frac{w_i/w_j}{1+w_i/w_j} \ . \tag{8}$$

To estimate the priority vector using the matrix (Equation 7), we write identical equalities:

$$\begin{cases} \frac{w_1}{w_1+w_1}(w_1+w_1) + \frac{w_1}{w_1+w_2}(w_1+w_2) + \dots + \frac{w_1}{w_1+w_n}(w_1+w_n) = nw_1 \\ - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \\ \frac{w_n}{w_n+w_1}(w_n+w_1) + \frac{w_n}{w_n+w_2}(w_n+w_2) + \dots + \frac{w_n}{w_n+w_n}(w_n+w_n) = nw_n \end{cases} . \tag{9}$$

Then, with notation (Equation 8) we present the system (Equation 9) as:

$$\begin{cases} (u_{11} + \sum_{j=1}^{n} u_{1j})w_1 + u_{12}w_2 + \dots + u_{1n}w_n = nw_1 \\ - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \\ u_{n1}w_1 + u_{n2}w_2 + \dots + (u_{nn} + \sum_{j=1}^{n} u_{nj})w_n = nw_n \end{cases} . \tag{10}$$

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In the matrix form the system (Equation 10) can be written as:

$$(U + diag(Ue))w = nw, \tag{11}$$

where U is the matrix (Equation 7), e denotes a uniform vector of n-th order, and $diag(Ue)$ is a diagonal matrix of totals in each row of matrix U.

In the classical AHP, the pair ratios w_i/w_j (Equation 2) are estimated by the elicited values a_{ij} (Equation 1). Using a_{ij} in Equation (8), we obtain the empirical estimates b_{ij} of the pairs' shares:

$$b_{ij} = \frac{a_{ij}}{1+a_{ij}} . \tag{12}$$

This transformation of the elements of a matrix A (Equation 1) yields a pairwise share matrix B with the elements (Equation 12). These elements (Equation 12) are positive, less than one, and have a property $b_{ij} + b_{ji} = 1$. This means that the transposed elements b_{ij} and b_{ji} are skew-symmetrical off the diagonal b_{ii}=0.5, so $b_{ij} - b_{ii} = -(b_{ji} - b_{ii})$.

For the empirical Saaty matrix A (Equation 1), we have the eigenproblem (Equation 3) in place of the theoretical relationship (Equation 2). Similarly, using the empirical skew-symmetric matrix B (Equation 12) in place of theoretical matrix U, we represent the system (Equation 11) as the eigenproblem:

$$(B + diag(Be))\alpha = \lambda\alpha, \tag{13}$$

where α as the main eigenvector. This is the RE vector of priority, and its properties have been studied in the works of Lipovetsky and Conklin (2002, 2015).

3. Measures of consistency, compatibility, and precision

Due to the general methodology of the AHP, the so-called consistency index (CI) equals

$$CI = \frac{\lambda - n}{n-1} \tag{14}$$

where λ is the maximum eigenvalue of the matrix in the problem (Equation 3), and n is the matrix order. The so-called random consistency index (RI) is a constant tabulated in the AHP for various n, and the consistency ratio (CR) equals the following value:

$$CR = \frac{CI}{RI} . \tag{15}$$

A value of CR up to 10% is considered to indicate a small inconsistency in the matrix of the pairwise comparisons (Equation 1), and therefore an acceptable matrix; however, if the CR > 10%, a review of the elicited judgements could be required.

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For comparisons between the obtained solutions, several characteristics can be applied. Among those are the pairwise correlation between the elements of two vectors, which can be reduced to the expression:

$$r(x,y)=\frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i-\bar{x})^2\sum_{i=1}^{n}(y_i-\bar{y})^2}}=\frac{\sum_{i=1}^{n}x_i y_i-\frac{1}{n}}{\sqrt{\sum_{i=1}^{n}x_i{}^2-\frac{1}{n}}\sqrt{\sum_{i=1}^{n}y_i{}^2-\frac{1}{n}}}\,, \tag{16}$$

where a bar above the variables denotes the mean values and those equal $1/n$ for the vectors normalized by Equation (6). Without the items $1/n$ for centering (and this value is small for a bigger n), the measure (Equation 16) coincides with the cosine as a normalized projection of one vector onto another one. The closer a correlation or cosine is to 1, the higher the similarity of the two solutions. The cosine values repeat the correlations but are slightly bigger, so for a more conservative measure the correlation is preferred.

Another good measure of closeness between two vectors is the so-called Saaty compatibility index (S-compatibility) (Saaty, 2005; Saaty and Peniwati, 2007; Garuti and Salomon, 2011). This index can be built as follows. For two vectors x and y of an n-th order, build a matrix X with its elements defined as quotients $X_{ij}=x_i/x_j$ of the components of the vector x, and a matrix Y with its elements defined as quotients $Y_{ij}=y_i/y_j$ of the components of the vector y. Take the transposed matrix Y' with the elements $Y'_{ij}=y_j/y_i$ and find the Hadamard element-wise product of these two matrices $X*Y'$, then the S-index is defined as the normalized total of the elements of this matrix:

$$S=\frac{1}{n^2}\sum_{i,j=1}^{n}X_{ij}Y'_{ij}=\frac{1}{n^2}\sum_{i,j=1}^{n}\frac{x_i}{x_j}\frac{y_j}{y_i}\,. \tag{17}$$

If two vectors coincide, this index equals 1. Within 10% of discrepancy, when $S\leq 1.1$, the vectors are considered compatible; otherwise, when $S>1.1$, they are incompatible (Saaty and Peniwati, 2007).

A further development of a compatibility measure in the so-called compatibility index G was proposed in Garuti (2007) and Garuti and Salomon (2011) where it was defined as:

$$G=\sum_{i=1}^{n}\frac{min(x_i,y_i)}{max(x_i,y_i)}\frac{x_i+y_i}{2}\,. \tag{18}$$

Due to recommendation in Garuti (2007), the values $G<0.9$ correspond to incompatible vectors, otherwise the vectors are compatible.

To check a precision of fit for the pairwise judgements by the priority vector estimate, we can use a definition of the elements a_{jk} as quotients of preference between each pair of j-th and k-th items. With a vector-column of priority estimate alpha, we find its element-reciprocal vector-row (1/alpha)*'* and build their outer product by the same pattern as used in Equation (2). With this outer product we find the quality of its fit for the matrix A Equation (1). The standard error (STE) is a measure of the mean distance between the observed and estimated pairwise ratios:

$$STE=\sqrt{\frac{1}{n^2}\sum_{j,k=1}^{n}\left(a_{jk}-\frac{\alpha_j}{\alpha_k}\right)^2}\,\,. \tag{19}$$

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Another convenient measure of the precision for a matrix approximation by the vectors outer product is the mean absolute error (MAE):

$$MAE = \frac{1}{n^2}\sum_{j,k=1}^{n}\left|a_{jk} - \frac{\alpha_j}{\alpha_k}\right| \ . \tag{20}$$

The smaller the values of fit (Equations 19-20), the better the quality of the vector estimate. The measures of STE and MAE can be obtained by using Equations (19-20) only for the off-diagonal pairwise ratios equal or above 1 because they correspond to the elicited quotients of preference, and the reciprocal values below 1 are simply added at completion of the matrix (Equation 1) of pairwise judgements.

Besides the characteristics of the residual mean values assessed via standard deviation (Equation 19) or absolute deviation (Equation 20), the quality of approximation of the pairwise judgements by the obtained priority vectors can be checked by a measure reminding the coefficient of multiple determination R^2 that is widely used in regression analysis. As shown in Lipovetsky (2009), this coefficient can be defined via the observed and estimated paired ratios of the priorities:

$$R^2 = 1 - \frac{RSS}{ESS} = 1 - \frac{\sum_{j,k=1}^{n}\left(a_{jk} - \frac{\alpha_j}{\alpha_k}\right)^2}{\sum_{j,k=1}^{n}\left(a_{jk} - 1\right)^2} \ . \tag{21}$$

In the numerator (Equation 21) the residual sum of squares (*RSS*) of the estimated priority deviations from the elicited values is used, and the denominator is presented by the equivalent sum of squares (*ESS*) which assumes all the same preferences $\alpha_j / \alpha_k \equiv 1$. The coefficient (Equation 21) shows how much the found priorities outperform the case of absence of preferences among the alternatives. The better the approximation of the paired judgements by the estimated priorities is the closer the *RSS* is to zero, so the coefficient of determination R^2 is bigger and closer to one. In the absence of preferences $\alpha_j/\alpha_k = 1$, the numerator equals the denominator, and $R^2 = 0$. For the exact fit $a_{jk} = \alpha_j/\alpha_k$ for all judgements, $RSS = 0$, and $R^2 = 1$.

The value R^2 commonly belongs to the interval from 0 to 1, which makes it a very convenient measure for comparison of the priority vectors obtained by different techniques. Only really poor estimates can produce a residual total *RSS* above the value of the equivalent residuals *ESS*, and it would be indicated by a negative R^2. The characteristic (Equation 21) corresponds to the *STE* measure (Equation 19) of squared deviations, but it is possible to build the other estimates, for example, using the MAE residuals (Equation 20) as well.

4. Numerical comparisons for priority estimations

Let us consider numerical examples of the priority estimations for three classical AHP problems.

Example 1: the problem of "Choosing the best home" as described in Saaty and Kearns (1985), Saaty and Vargas (1994) and Saaty (1996). This matrix is also used for checking

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some new approaches (Lipovetsky, 1996; Lipovetsky and Tishler, 1999; Lipovetsky and Conklin, 2002, 2015). The criteria of comparison are: 1 – size of house, 2 – location to bus, 3 – neighborhood, 4 – age of house, 5 - yard space, 6 – modern facilities, 7 – general condition, 8 – financing. The matrix of pairwise comparisons *A* (Equation 1) for this problem is presented in Table 1a.

In this example with *n*=8, the maximum eigenvalue (Equation 3) of the matrix in Table 1a equals $\lambda = 9.669$. With the random consistency for this case *RI*=1.41, the consistency index and consistency ratio (Equations 14-15) are:

$$CI = \frac{9.669-8}{7} = 0.238, \quad CR = \frac{0.238}{1.41} = 0.169.$$

A value of CR up to 10% is considered to indicate some inconsistency, so the obtained result of 17% is acceptable with a reservation, when the data could require a review of the elicited judgements, and in Lipovetsky and Conklin (2002) it was shown how to identify and to adjust the data in this case.

Table 1a
Example 1: Choosing the best home problem. Pairwise comparison matrix.

item	1	2	3	4	5	6	7	8
1	1	5	3	7	6	6	1/3	1/4
2	1/5	1	1/3	5	3	3	1/5	1/7
3	1/3	3	1	6	3	4	6	1/5
4	1/7	1/5	1/6	1	1/3	1/4	1/7	1/8
5	1/6	1/3	1/3	3	1	1/2	1/5	1/6
6	1/6	1/3	1/4	4	2	1	1/5	1/6
7	3	5	1/6	7	5	5	1	1/2
8	4	7	5	8	6	6	2	1

Several methods of priority estimation for this data are presented in Table 1b. In its upper part, there are four estimates of the priority vector as follows: the classical EM solution (Equation 3), the LS estimation (Equation 4), the LN technique (Equation 5), and the robust estimation RE (Equation 13). All the vectors are normalized by the total of their elements equaling one (Equation 6). Judging by eye, all of the solutions are very similar by weight and the smallest and the biggest by importance are the age and financing of the house, items 4 and 8, respectively.

For comparison between the obtained four priority vectors we applied the measures (Equations 16-18) of correlations, S-compatibility, and G- compatibility, presented after the vectors in the three matrices in Table 1b. Judging by the correlations, all of the vectors are close enough by their structure and the LS is a bit further from the other three. The measure of *S*-compatibility proves that the EM, LN, and RE vectors are similar, within less than the required threshold of 10% of the *S*-index deviation from one. The more sensitive *G*-compatibility demonstrates that the pair of EM and LN vectors are close with *G*=0.927, and the two vectors LN and RE are close with *G*=0.912, which are

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values above the threshold 0.9 needed for viewing the corresponding vectors as compatible.

Table 1b
Example 1: Choosing the best home problem. Priority vector estimations

Item	EM	LS	LN	RE
1. size of house	0.173	0.199	0.175	0.150
2. location to bus	0.054	0.100	0.063	0.054
3. neighborhood	0.188	0.148	0.149	0.141
4. age of house	0.018	0.017	0.019	0.022
5. yard space	0.031	0.045	0.036	0.037
6. modern facilities	0.036	0.065	0.042	0.041
7. general condition	0.167	0.184	0.167	0.163
8. financing	0.333	0.242	0.350	0.392
Correlations				
EM	1	0.935	0.988	0.972
LS	0.935	1	0.933	0.881
LN	0.988	0.933	1	0.991
RE	0.972	0.881	0.991	1
S-compatibility				
EM	1	1.113	1.015	1.028
LS	1.113	1	1.071	1.122
LN	1.015	1.071	1	1.010
RE	1.028	1.122	1.010	1
G-compatibility				
EM	1	0.774	0.927	0.865
LS	0.774	1	0.809	0.742
LN	0.927	0.809	1	0.912
RE	0.865	0.742	0.912	1
Precision				
STE	2.071	1.849	1.813	1.831
MAE	1.079	1.083	0.958	0.934
R^2	0.423	0.540	0.558	0.549

The last segment at the bottom of Table 1b displays the precision by Equations (19-21) for each vector solution. By the minimum standard error *STE*, the best model is the LN, and by the mean absolute error MAE, the best model is the RE. The values of the MAE also suggest that an average deviation from the observed pair judgements evaluated by the obtained quotients from a priority vector is not more than one unit. The coefficient of multiple determination R^2 in the last row of Table 1b shows by its maximum values that the LN and RE models outperform the other two models, though all R^2 values are not high, which indicates a difficulty in approximation of inconsistent judgements by a priority vector in any estimation.

Example 2: the problem of "Distance from Philadelphia" is one of the first AHP problems described by Saaty (1977). The distance of six cities from Philadelphia was estimated by the criterion: for each pair of cities, how many times farther is the more distant city located from Philadelphia than the nearer one? The elicited data is presented in Table 2a.

Table 2a
Example 2: Distance from Philadelphia problem. Pairwise comparison matrix

Airport	CAI	TYO	ORD	SFO	LGW	YMX
Cairo.CAI	1	0.333	8	3	3	7
Tokyo.TYO	3	1	9	3	3	9
Chicago.ORD	0.125	0.111	1	0.167	0.2	2
SanFrancisco.SFO	0.333	0.333	6	1	0.333	6
London.LGW	0.333	0.333	5	3	1	6
Montreal.YMX	0.143	0.111	0.5	0.167	0.167	1

The maximum eigenvalue (Equation 3) in this example equals $\lambda = 6.454$. The random consistency for n=6 is RI=1.24, then the consistency index and consistency ratio Equations (14-15) are:

$$CI = \frac{6.454-6}{5} = 0.091, \quad CR = \frac{0.091}{1.24} = 0.073\ .$$

The value of CR=7.3% is less than the 10% permitted, which allows one to conclude that the data on pair judgements is sufficiently consistent.

Table 2b presents the results of priority estimations for this example and is organized as the previous Table 1b, but with one additional column of the actual shares of distances known in this case.

Table 2b
Example 2: Distance from Philadelphia problem. Priority vector estimations

City	EM	LS	LN	RE	actual
1. Cairo	0.262	0.254	0.260	0.239	0.278
2. Tokyo	0.397	0.305	0.399	0.447	0.361
3. Chicago	0.033	0.047	0.035	0.034	0.032
4. San Francisco	0.116	0.186	0.116	0.104	0.132
5. London	0.164	0.184	0.163	0.147	0.177
6. Montreal	0.027	0.024	0.027	0.029	0.019
Correlations					
EM	1	0.943	1.000	0.990	0.991
LS	0.943	1	0.941	0.898	0.973
LN	1.000	0.941	1	0.991	0.990
RE	0.990	0.898	0.991	1	0.962
Actual	0.991	0.973	0.990	0.962	1
S-compatibility					
EM	1	1.064	1.000	1.009	1.024
LS	1.064	1	1.064	1.106	1.045
LN	1.000	1.064	1	1.008	1.027
RE	1.009	1.106	1.008	1	1.060
Actual	1.024	1.045	1.027	1.060	1
G-compatibility					
EM	1	0.821	0.993	0.900	0.914
LS	0.821	1	0.820	0.753	0.854
LN	0.993	0.820	1	0.905	0.908
RE	0.900	0.753	0.905	1	0.823
Actual	0.914	0.854	0.908	0.823	1
Precision					
STE	1.390	1.340	1.333	1.523	2.295
MAE	0.696	0.862	0.686	0.790	1.012
R^2	0.794	0.809	0.810	0.753	0.438

We see that in general the vectors are similar and each one makes sense as proportionally scaled distances from Philadelphia to other cities in the USA, as well as to other countries and continents. The pair correlations also show that the vectors are closely related to the actual distances, and the same is supported by the *S*-compatibility index. *G*-compatibility indicates that the EM and LN vectors are compatible with the actual shares of distances. The precision of the reproduction of the judgement matrix is high, especially with the LS and LN methods. The precision measured by *STE*, *MAE*, and R^2 of the actual distances is the worst one within the other values in the last rows of Table 2b. This means that the pair judgements on distances correspond to the priority vectors rather than to the actual distance shares. Therefore, in this data we do not need to use the actual data to consider compatibility among the vectors.

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Example 3. The data for this problem is given in Whitaker (2007) where the area of five geometric figures was compared – see the matrix of pair judgements in Table 3a.

Table 3a
Example 3: Geometric figures' area problem. Pairwise comparison matrix

Figure	Circle	Triangle	Square	Diamond	Rectangle
Circle	1	9	2.5	3	6
Triangle	0.111	1	0.2	0.286	0.667
Square	0.4	5	1	1.7	3
Diamond	0.333	3.5	0.588	1	1.5
Rectangle	0.167	1.5	0.333	0.667	1

The maximum eigenvalue of this matrix is $\lambda = 5.026$. The random consistency for *n*=5 is *RI*=1.12, so the consistency index and consistency ratio (Equations 14-15) equal the following values:

$$CI = \frac{5.026-5}{4} = 0.006, \quad CR = \frac{0.006}{1.12} = 0.006\ .$$

CR=0.6% which proves a very high level of consistency of this data. This can be explained by the pairwise ratios that were used where not only the integer numbers but also the rational numbers (like 2.5 or 3.5) were permitted in the preference evaluation.

Table 3b presents the priority estimation results for this example, and is organized as Table 2b, with the additional column of the actual shares of the areas measured for these figures.

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Table 3b
Example 3: Geometric figures' area problem. Priority vector estimations

	EM	LS	LN	RE	actual
1. Circle	0.488	0.464	0.487	0.496	0.470
2. Triangle	0.049	0.050	0.049	0.049	0.050
3. Square	0.233	0.248	0.233	0.225	0.240
4. Diamond	0.148	0.159	0.148	0.147	0.140
5. Rectangle	0.082	0.078	0.082	0.083	0.090
Correlations					
EM	1	0.998	0.999	0.999	0.999
LS	0.998	1	0.998	0.995	0.998
LN	0.999	0.998	1	0.999	0.999
RE	0.999	0.995	0.999	1	0.998
actual	0.999	0.998	0.999	0.998	1
S-compatibility					
EM	1	1.003	1.000	1.000	1.003
LS	1.003	1	1.003	1.004	1.007
LN	1.000	1.003	1	1.000	1.003
RE	1.000	1.004	1.000	1	1.003
actual	1.003	1.007	1.003	1.003	1
G-compatibility					
EM	1	0.948	0.999	0.982	0.955
LS	0.948	1	0.948	0.931	0.953
LN	0.999	0.948	1	0.982	0.956
RE	0.982	0.931	0.982	1	0.940
actual	0.955	0.953	0.956	0.940	1
Precision					
STE	0.253	0.195	0.253	0.289	0.279
MAE	0.145	0.110	0.145	0.162	0.165
R^2	0.987	0.992	0.987	0.983	0.985

All the vector estimates in this data practically coincide; the pair correlations are very high, and both the *S*- and *G*- indices prove compatibility among the estimates and with the actual observations. The precision measured by the *STE*, *MAE*, and R^2 characteristics demonstrates a high quality of fit of the data by any of the estimated vectors of priority and by the actual values as well.

5. Conclusions

The paper considered several methods of priority vector evaluation in the AHP. They include the classical eigenproblem method, least squares, multiplicative or logarithmic approach, and a robust estimation based on transformation of the pairwise ratios to the shares of preferences. Together with estimation of the vectors, validation of data consistency and comparison of vectors by correlations, the *S*- and *G*- compatibility indices were also completed. The numerical results for different data sizes and consistency indices demonstrate that all of the methods produce compatible results for the consistent data, otherwise a discrepancy between the different methods of the priority estimation would be observed. Therefore, the data consistency should always be proved before the vector evaluation.

Another important conclusion concerns the precision assessment for the data matrix approximation by the obtained priority vectors. Any regular statistical modeling requires a verification of the produced results by some quality characteristics. For example, in regression analysis, measures like the residual standard error *STE*, mean absolute error *MAE*, and coefficient of multiple determination R^2 are commonly employed. Applying them in the AHP environment can enrich the evaluation and interpretation of the results on priority modeling and is demonstrated on the numerical estimations performed in the paper. For instance, in the data for Example 1 with a low consistency, the R^2 values are also not high which indicates a difficulty of approximation of inconsistent judgements by a priority vector in any estimation, and by *MAE* values a mean deviation of the quotients of a priority vector's elements from the observed pair judgements could be as big as one unit. In Example 2 with a good consistency, the precision of the reproduction of the judgement matrix by the found priority vectors is high enough, although at the same time the actual distances occurred to yield the worst vector for approximation of the elicited pairwise priority matrix Therefore, in this data we should not use the actual data on distances to check the compatibility with the obtained estimates of the vectors. Example 3 with a perfect consistency yields all vectors of high compatibility and of a great quality of the elicited judgements reconstruction by each vector's quotients of preference.

The considered methods of priority vector estimation and characteristics of their quality are convenient and helpful in practical applications of the AHP for solving various multi-criteria decision making problems.

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