SAATY 1977: THE BUILDING BLOCKS

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ABSTRACT

Saaty's 1977 article is his first comprehensive publication of the ideas behind AHP. He reveals his creativity in a new method for ratio measurement that includes pairwise ratio matrices, derived ratio scales from those matrices, and checks on the consistency of data. His ingenuity in using ratio measures is revealed by the use of hierarchical structures to display priorities and then a rescaling of them in a manner that allows synthesis for a composite ratio result. Face validity is provided by many supporting examples and mathematical validity is provided by the solution to many theorems.

Keywords: AHP origins; ratio measurement; creativity, ingenuity

1. Introduction

In the early 1970's, Thomas Saaty was developing and testing new concepts for decision-making. He already had an exemplary reputation in queuing theory, operations research and mathematics, and was turning his attention to a comprehensive methodology that could solve many types of problems, including ones central to him such as peace and conflict resolution. What emerged was the Analytic Hierarchy Process (AHP) that gained extreme popularity, widespread use, and strong scrutiny. Saaty's subsequent work centered around this framework but always in an evolving manner. He was a creative individual who sought improvements, new insights and novel applications for his methodology. What started as AHP for complex hierarchies grew into the Analytic Network Process (ANP) for even more complex structures that had interdependencies.

In 1977, Saaty published '*A Scaling Method for Priorities in Hierarchical Structures'* (Saaty, 1977). I consider that article to be the genesis of AHP and the kernel for ANP. In that article, Saaty presented all the theory and validations for the work he had been doing in the previous years. The words "process" and "hierarchy" are used extensively throughout the article, yet nowhere is there reference to the Analytic Hierarchy Process. That title came a few years later.

Noteworthy about the 1977 article is Saaty's choice for the first comprehensive release of his ideas, *The Journal of Mathematical Psychology*. It is the prime outlet frequented by measurement theorists. This signifies that the foundations of AHP/ANP are in measurement theory. Yet Saaty states in the article that his core ideas were improvised, grew completely out of applications, and then had to be integrated into the main stream of literature. This means that his methodology was not an advancement of a particular thread of research; rather, it was an integration of many concepts more akin to the software of Microsoft or microcomputers of Apple that were disruptive technologies of that era. AHP was disruptive as well. Like a Bill Gates or Steve Jobs, Saaty took pre-existing concepts along with creativity, ingenuity and determination to develop a dominant product.

The depth and breadth of Saaty's knowledge and his ability to synthesize concepts is displayed in the article. It contains all the building blocks of AHP and the foundations for the evolution to ANP. There are sections on:

1. Ratio scales from reciprocal matrices, supported by nine rigorous theorems proving facts such as:
   - λ_{max}.=n for a consistent matrix
   - proof of existence of a ratio solution for $Aw=\lambda_{max}w$
   - uniqueness and existence of the eigenvector solution
   - proof that the limit of A is the normalized eigenvector times a constant
   - proof that an order type w_i/w_j involves all the information in A and its powers

2. Justification for the scale via validation with real life examples (distance from Philadelphia, inverse law of light, wealth of nations, weight estimation)

3. Hierarchies that can be used to represent both the structural and functional relations of a system. Examples are given for Sudan planning and backward and forward planning hierarchies

4. Formal properties of hierarchies for getting composite answers from local priorities. Support comes from rigorous definitions, 3 additional theorems and five examples (school selection, psychotherapy, choosing a job, vacation selection and conflict resolution)

5. Methods for decomposition, aggregation and clustering in a hierarchy, including the number of comparisons required to create vectors of the hierarchy

6. Relations to other research and work

It is not the purpose here to go into the details of the article. Saaty elaborates on those concepts in subsequent publications and books. What is covered here is the historical context of Saaty's invention, advice from Saaty that bears re-emphasis, and some explanations regarding the uniqueness of Saaty's contribution. Regarding the contribution in a historical context, it should be noted that ratio measurement, pairwise comparison matrices, eigenvectors and hierarchies were well established in 1977. Saaty was novel in how he used and combined the concepts.

The following comments are divided into three sections: personal reaction to the article, creativity illustrated in the micro aspects and ingenuity in the macro aspects.

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2. Personal reaction to the article

To study his work, to meet him, and to communicate and discuss with Thomas Saaty is a very personal experience. Those experiences rate "nine" on my AHP preference scale and explain why I am writing in the first person.

I first came across Saaty's 1977 article in the spring of 1981 in Melbourne, Australia while visiting Monash University during sabbatical leave. Being aware of both the benefits and deficiencies of quantitative techniques, I was investigating how to add qualitative factors to quantitative models. A colleague from Canada, Ernie Love, passed through Melbourne and paid a visit. Knowing my interests, he gave me a short article by Charles Whaley, entitled simply, '*Fuzzy Decision Making'* (Whaley, 1979).

That article was not about Fuzzy Set Theory that was popular at the time, but it did have two components that were to completely guide my future avenue of research. The first was a very simple computer program written in BASIC, called Fuzzy. The second was the reference to Saaty (1977).

The software program was not pure AHP, because it did not use pairwise comparison matrices to get priorities for alternatives. Nonetheless, it did use Saaty's eigenvector routine to develop priorities for criteria. I programmed Whaley's software on Monash's mainframe and then invited many volunteers to test the procedure. I connected them to the mainframe via the communication mode of that time, an acoustic coupler attached to a telephone, facsimile machines and paper printouts. Today, the computer is on the telephone, accessing the cloud, and instantaneously available. Since a computer is essential for AHP calculations, widespread microcomputer advances and early AHP programs such as Expert Choice were essential for the spread of AHP. Later, Super Decisions played the same function for the dissemination of ANP.

My 1981 tests with many people were successful. Uncannily, the paired comparison procedure captured people's preferences. Everyone seemed to agree that the eigenvector routine returned priorities that represented their values. One humorous test by my 13-year old daughter both shocked and sold me on the technique. As with all of my volunteers, I explained to my daughter that she should use the technique for a multi-criteria problem – one that had a number of alternatives that were evaluated according to different criteria. She immediately accepted the assignment, saying, "I will use it to choose a boyfriend!" She dutifully entered the names of Dean and Steven who frequently came around for poolside swims. Then she entered John, Emelio, David and …. Knowing that the number of comparisons can increase with a larger n, I politely suggested she limit the alternative list after she reached seven. Next, she proceeded to enter criteria: "personality", "looks", and behold: "kissability!" Shocked by the "kissability" criterion from my 13-year old, I probably should have terminated the session right there (users must be experienced and able to recall the phenomena). Nonetheless, we continued. That was a good choice, because I now realize that "preference for kissability" is not the same as actual "kissability" – just as the preference for the weight of a suitcase (lighter is better) is not the same as the actual weight of the suitcase. In my daughter's test, Dean (the athlete) was ranked first, and Steven (the brain) was close behind. Seven years later, my daughter married Steven.

Stories like that and other experiences by users made me become one of the populist supporters of this new technique. Of course, my curiosity led me to the Saaty (1977) reference at the end of the Whaley article. I was able to find and photocopy it at Monash's library. Over the years, I have read the article many times, each time learning something new.

So why did Saaty's AHP become such a phenomenal success? Some explanation is provided by the creativity and ingenuity displayed in the article. Herein, I use the word "creativity" to imply a novel invention and "ingenuity" to imply cleverness in combining concepts.

3. The micro aspects – creativity

To understand the creativity of Thomas Saaty, it is important to consider the micro aspects of the article. This includes how he used the pairwise comparison matrix, how he derived a ratio scale from it, and how he used matrix information to measure consistency.

3.1 The pairwise ratio matrix

Saaty characterizes his pairwise comparison matrix as a dominance matrix as opposed to proximity, profile or conjoint matrices that were then in common use. Thurstone (1928) and subsequent researchers used comparison matrices with ordinal or probabilistic statements about one stimulus being more important than another. Saaty's questioning procedure started out in a similar manner, but went one step farther. First, he began with an ordinal question that identified the dominant of two objects, and then a ratio question regarding how many times more dominant. This second question establishing the intensity of dominance means that each comparison within the matrix creates an estimated ratio scale between the two objects: $[a_{ij}, a_{jj}=1]^T$. With many of these comparisons in the matrix and with unity along the diagonal, each column vector is a different estimate of the ratio relations. Those multiple vectors as estimates indicate that redundancy is built into the comparisons of the matrix.

Ratio estimates is what distinguishes Saaty's comparison matrix from other matrix types. To denote this difference, it should be called the Pairwise Ratio Matrix (PRM). There are many techniques for deriving a summary ratio scale from a PRM. Backed by mathematical justification, Saaty selected and staunchly defended the principal right eigenvector as the best technique.

3.2 Magnitude estimation vs eigenvector estimation

By choosing ratios for psychometric measurement, Saaty entered an arena that had a long history of controversy. In 1932, the British Association for the Advancement of Science appointed a committee from two divisions (mathematical/physical sciences and psychology) to debate whether or not qualitative factors could be measured (Stevens, 1946). Eight years later, after numerous meetings, they were still at a stalemate. Stevens, from the United States, was the main proponent for psychophysical measurement. By 1977, he had developed several methods involving ratios, the most popular of which was Magnitude Estimation (Stevens, 1971). From the alternative objects, this technique required the evaluator to specify a standard stimulus (a modulus) with a specified magnitude value such as 10. The evaluator then considers other objects and assigns

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numerals to them that are relative in intensity to the standard stimulus. In ratio form, such numbers can be renormalized to any desired unit.

As Narens (1996) observed, any object can be the standard stimulus for Magnitude Estimation and that object can be given the number 1 as the subjective norm. Other objects take their intensities from that unit. Using this unity modulus approach, Magnitude Estimation could be used to establish each column of the PRM. In general, however, Magnitude Estimation is used to establish a single ratio vector under assumed error-free conditions.

Saaty's technique was different and went much farther. Saaty knew and accepted the fact that comparison estimates are likely to have imprecision. He also recognized that the comparison ratio to the dominated object implies the reverse ratio of the dominated to the dominant. By adopting the reciprocal property for the PRM (a_{ji} =1/a_{ij},), redundancy in estimation is achieved with half the number of possible comparisons and no necessity of generating every column according to Magnitude Estimation procedures.

Since each cell in Saaty's PRM is an estimated conversion factor between two objects, transitive impacts can be calculated across multiple rows and columns (e.g. $a_{ij}*a_{jk}=a_{ik}$). Such indirect estimation of cell values is instrumental in both the eigenvector calculation and the summary statistic of consistency. It is from the redundant comparisons that Saaty's procedure provides more information – not only an elegant ratio average, but also a check on the consistency of the underlying data. Magnitude Estimation and other techniques did not do that.

3.3 Accuracy vs. consistency

In developing his theory, Saaty started with the assumption that a true ratio scale actually exists for the phenomena being measured. We do not know that true scale – our problem is to derive it for n objects based upon the estimates in the PRM. As Saaty points out, if all estimates are perfectly accurate, the PRM will be perfectly consistent and the true scale can be derived from any row, column or spanning tree of the matrix. PRMs, however, are estimates from humans. They are seldom perfectly consistent. This means that the derived priorities will not be a perfect replication of the underlying true ratio scale.

In his article Saaty (1977) was very cognizant of this distinction. For the first three pages of the section on ratio scales, he used Aw=nw to refer to the true matrix of ratios and $A'w'=\lambda_{max}w'$ for the estimated matrix. Thereafter, he dropped the primes and used $Aw=\lambda_{max}w$. This may have been unfortunate, because some users think consistency reduction and λ_{max} closer to n is the ultimate goal. Saaty, however, was quite clear. He noted that, "improving consistency does not mean getting an answer closer to the "real" life solution. It only means that the ratio estimates in the matrix, as a sample collection, are closer to being logically related than to being randomly chosen …." (Saaty, 1977)

Probably guided by this statement, Saaty later proposed the Consistency Ratio (CR) as the ratio of a PRM's Consistency Index (CI) to the average Consistency Index of randomly generated PRMs (i.e., CR=CI/RCI). He suggested a rule of thumb that being 90% away from random (i.e. CR<.10) was acceptable and CR<.20 as tolerable.

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Interestingly, he never pursued a statistical test for the Consistency Index that he mentions in the paper.

Saaty invited and looked upon a certain amount of inconsistency as being desirable. He felt that it was the inconsistent part of worldly phenomena that sparked and motivated new discoveries and human progress. If there was perfect consistency and homeostasis there would be no dynamic to promote change. In my personal contact with Saaty, I found him to be about 80-90% consistent, 10-20% inconsistent, and usually right. Our discussions were about the 10-20% part.

3.4 Variability of intangibles

Based upon pairwise comparison estimates, Saaty's procedure produces a derived scale from the data. Such a procedure could be used to create a stable scale for some tangible phenomenon that never changes. Saaty's procedure, however, had much more flexibility.

As noted, Saaty's scale is a derived scale. It determines the measures rather than using some predefined scale to measure. In the case of intangible and qualitative factors, preferences can change. For example, the preference for "kissability" can change with the first kiss. Saaty's procedure has the flexibility to capture such changes. Substitute updated comparisons, calculate a new derived scale, and then ask the decision maker whether those priorities represent the new situation. Most prior techniques failed to recognize the transient nature of preferences.

4. The macro aspect – ingenuity

Saaty's PRM procedure can be used to derive ratio scales for many different types of objects: alternatives, criteria, strategies, scenarios, etc. What he needed was some way to integrate the derived scales from those different aspects into a unifying structure. From there, he could aggregate to get a composite answer.

4.1 Hierarchies

To achieve structure for problems, he chose the concept of hierarchies. In the 1977 article, Saaty comments favorably on how Herbert Simon enriched the concept of a hierarchy (Simon, 1962; Simon and Ando, 1961). Referring to many different types of systems in nature, economics and society, Simon had developed an important theory of how complex systems could be structured and analyzed as a hierarchy. Along with that, Simon provided the mathematical analog for aggregating variables within the hierarchy. He observed that subsystem variables at a lower level might be an aggregate of variables at a still lower level.

Saaty used this hierarchical framework to organize and display the elements of a decision. Hierarchies not only provide a cognitive map of relationships but they also act as placeholders for the values of each element. Saaty realized that the eigenvector values from his PRM procedure could become the values for each cluster. From the outset, he adopted the convention of calling them priorities and normalizing them to sum to unity. Since each cluster summed to unity, the implication is that all cluster priorities represent within-cluster effects. Saaty needed some method to determine hierarchical effects.

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4.2 Rescaling

In ratios, Saaty had the answer. The only permissible transformation of a ratio scale without destroying ratio relationships is multiplication by a positive constant. Since such multiplication maintains ratios but not values, the new values signify that the ratio measurement is now in a different unit of measure. Starting at the top of the hierarchy with a value of unity for the whole hierarchy, Saaty used the scaling concept to make lower levels aggregate to the value in a higher level. By multiplying lower level priorities by the value of the higher connecting node, each cluster which formerly summed to unity is rescaled to a new sum which is the value of its parent node. Thus, Saaty used Simon's concept that the value at one level can be the aggregate of the values immediately below.

Note that Saaty's rescaling changes priority values within the hierarchy, but not the underlying relative ratios. Ultimately, each element of the hierarchy receives its portion of the unit value at the top of the hierarchy. Put differently, rescaling puts all values into a single unit representing elements and influences of the entire hierarchy. Rescaled to a common hierarchical unit, it is possible to sum values of the same object that are in different parts of the hierarchy. In this manner, Saaty aggregates to get a comprehensive solution. Unlike other methods, that composite answer possesses ratio properties.

It should be noted that if an alternative is removed from lower levels and remaining alternatives keep their same values, then hierarchical composition produces the same ratio results. This is because there has been no further rescaling (change of unit) in the hierarchy. However, if the remaining alternatives are renormalized once again to unity, then:

(1) the unit of measurement for those remaining cluster alternatives has changed (values are different)
(2) hierarchic rescaling still maintains the original ratio relationships between those remaining alternatives (values within a cluster change but ratios do not).
(3) hierarchical synthesis to get a composite result can lead to changes in the ratios of composite results (different values are composed)
(4) If the change in composite ratios is sufficient enough, the remaining alternatives can display a reversal in rank

In the early days of AHP, too much time and effort were expended on the rank reversal debate. It seems incomprehensible that such a debate would be carried out on ordinal terms when AHP is a ratio method. Resorting to a lower order of measurement also lowered the possibilities for analysis (Stevens, 1946). I believe that the debate would have been resolved had it been conducted on a ratio basis. Perhaps the debate was deflected by how easy it is to convert ratios to ordinals or intervals. Many do! That, however, is not the intent of AHP. To get its full worth, one should use its measurement in ratios.

4.3 Networks

Saaty makes reference to concepts that would eventually become ANP. He notes that, "any system can be represented by a large interaction matrix whose rows and columns are components of the system." He comments that, "… when component i and component j interact strongly, the i, jth entry is near ±1" and "… when they do not interact, the entry is near zero." He also commented that, "… through the concept of a reachability matrix

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and its powers, a distinct hierarchic structure is often discerned." With entries positive or zero, those concepts are the attributes of an ANP matrix and the ANP process (Saaty, 1977).

5. Summary

A Web of Science search reveals that Saaty's 1977 article has been cited over 2400 times. That is remarkable for a technical publication. AHP and ANP have been very successful. But why? Was it ratio paired comparisons, derived scales, consistency measurement, hierarchies, rescaling, composition or structure? Of course, AHP is an amalgam of all of those. I claim that the introduction of ratio comparisons to pairwise comparison matrices, the derivation of a ratio vector from those matrices, and the provision of a consistency check on those matrices was the creative and novel aspect of Thomas Saaty's invention. The ability to foresee that those derived ratios could be analyzed in hierarchical structures illustrates his ingenuity. He was both a master of minutiae (the micro) and innovator of integration (the macro). What he wrote in the 1977 article provided the building blocks for AHP and ANP.

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