ACHIEVING THE DESIRED LEVEL OF DEPENDENCY IN ANP DECISION MODELS

Orrin Cooper
University of Memphis, Fogelman College of Business
Memphis, TN
olcooper@memphis.edu

Guoqing Liu
University of Pittsburgh
Pittsburgh, PA
liugq1990@gmail.com[1]

ABSTRACT

When designing an ANP model it is important to acknowledge and properly address whether the elements in the model are dependent on or independent of each other. The decision maker must perform criteria cluster weighting comparisons individually for the criteria clusters in each column of the Supermatrix to correctly model when the criteria and alternatives are dependent on one another to accurately capture the dependence. Failing to recognize that the criteria in a criteria cluster in one column of the Supermatrix is not necessarily equal in weight to the criteria in that same criteria cluster but in another column can lead to misrepresented rankings in the final priorities. In the extreme case, it can remove all dependence from an ANP model. Two models are used to demonstrate this unintended effect on the final priorities, and also demonstrate a crucial contribution that this effect is independent of the tangibility of the criteria considered. In the third model, the solution is discussed and implemented. A proof is provided in the appendix. This criteria cluster weighting approach further extends the applicability of the ANP to additional decisions when a decision maker wishes to represent a fully-dependent ANP decision. *https://doi.org/10.13033/ijahp.v9i1.450*

Keywords: Analytic Network Process (ANP); dependence; intangible elements; criteria weights

1. Introduction

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The Analytic Hierarchy Process (AHP), and its generalization the Analytic Network Process (ANP), take advantage of our natural ability to structure decisions as hierarchies or networks and make relative comparisons or judgments (Saaty, 1996). When one decides to create an AHP or ANP model certain assumptions must be made about how the decision at hand will be framed. Some of the questions one must consider are:

1) What level of dependence (or independence) is there among the criteria and alternatives?
2) What is the meaning of the final priority vector that will be obtained?
3) Are there both tangible and intangible criteria in the model?

The first two questions about the meaning of the final priority vectors and level of dependence are addressed in further detail below. Neither question has a universally correct answer, but different answers to the questions will lead to different model designs with different results. It will also be shown herein that the answers to the first two questions are independent of the third question relating to the tangibility of the elements; and contrary to prior claims that it is the tangibility/intangibility of the elements that is the determinant of how to weight the criteria clusters it is the answer to the first two questions that will determine how the criteria clusters should be weighted. When a decision maker wishes to model a decision where the alternatives are fully-dependent on the defined system or decision and to obtain a final priority vector that is of the form $a_i / \sum_{i=1}^{n} a_i$ in the Limit matrix, which provides the relative priority or contribution of an alternative a_i with respect to the system of n alternatives being considered, careful attention must be paid to the weighting method that will be used to weight the criteria clusters to convert an unweighted Supermatrix which generally contains multiple priority vectors in each column that sum to one and must be weighted to obtain the "weighted" Supermatrix that is column stochastic. In this paper, a model that provides final priority vectors of the form $a_i / \sum_{i=1}^{n} a_i$ will be termed Fully-Dependent ANP models. The pairwise comparisons performed to obtain the criteria cluster weights must be performed individually for each column in the Supermatrix to obtain a fully-dependent ANP model. Whereas the current approach is to apply the same criteria cluster weight equally across all columns in the Supermatrix to weight the unweighted Supermatrix (Saaty, 2005, 2008a, 2008b, 2011). The use of a single cluster weight across all columns of the Supermatrix within a given cluster fails to recognize that a "one" here, in a given criteria cluster in a certain column, does not necessarily equal a "one" there representing another priority vector in a different column of the Supermatrix even though that priority vector is in the same criteria cluster (see Figure 1). Understanding and applying that statement is at the core of this paper. If the decision maker does not properly address the desired level of dependency sought in the model they will be left with a model that does not necessarily reflect a fully-dependent ANP model and can lead to unintended and misleading results in the final, or global, priority vectors in the Limit Matrix. Both the alternative and criteria final priority vectors will be impacted for length and the focus will be on the impact to the final priority vector for alternatives.

It has been shown previously that unique criterion weighting for each alternative must be used when tangible elements are considered and/or the model must be validated against

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actual results (Harker & Vargas, 1990, Saaty, 2011). However, as will be demonstrated in the first example in the Model section, this process of weighting the priority vectors in the Supermatrix is clearly independent of the tangibility of the elements being considered. This distinction is clearly shown here in this paper for the first time and hence requires decision makers to conscientiously determine the desired level of dependency in their model independent of the tangibility of the elements and then take the necessary steps to achieve it.

This paper is not a proposition for a universal level of dependence for all decision models rather that the determination of the level of dependence must be explicitly determined by the decision maker at the outset of the problem and then the proper steps be followed to achieve that desired level of dependence in the model. Herein, dependency will be categorized into two categories with meaningful interpretations as: independent (AHP) and Fully-Dependent (ANP). The solution provided in this paper should be used if the decision maker decides they want a model where the alternatives are dependent on the criteria and vice versa. It is also worth noting that a formal acknowledgement of the categories of dependency can also not only separate and distinguish but even provide solutions for many of the criticisms of the AHP/ANP.

The need for and the potential benefits of the suggested method will be heightened with a review of the relevant literature. Subsequently, in the Models section, the unintended consequences of not fully capturing the dependence by not providing unique cluster weights in each column of the Supermatrix are identified and shown to be independent of the tangibility of the elements. The solution is proposed, demonstrated in an example, and generalized in the proof. Obtaining meaningful final priority vectors should be at the core of a decision maker's objectives. In a Fully-Dependent ANP model performing unique criteria cluster weighting comparisons for each column of the unweighted Supermatrix will lead to a more meaningful final priority vector in the Limit matrix.

2. Literature review

2.1 The AHP and independent criteria weights

The first publication using the AHP ranked transportation projects in Sudan (Saaty, 1977b). The AHP/ANP is now the most widely published multi criteria decision making method (Wallenius et al., 2008). A more comprehensive review of the theory is found in the following references (Saaty, 1977a; Saaty, 1986; Saaty, 1996, 2005; Saaty & Cillo, 2008; Lipovetsky, 2011, 2013; Lipovetsky & Conklin, 2015). The process of creating an AHP model is summarized below in nine steps. The criteria weights calculated in an AHP model are independent of the alternatives (Dyer, 1990; Saaty, 1986; Saaty, 1996; Schoner, Wedley, & Choo, 1993). In an AHP model because there is no inner and outer dependence step 5 and step 6 are one in the same and there is no effect on the independence of the criteria from the alternatives. In the AHP literature, there are arguments for and against independent criteria weights (Belton & Gear, 1983; Dyer, 1985; Harker, 1987; Schoner et al., 1993; Wedley & Choo, 2011).

Rather than concluding that independent criteria weights are right or wrong; in general, it is more meaningful to frame the question in the context of what is the desired outcome; that is, if we are dealing with AHP models or fully-dependent ANP models. There are many examples where independent criteria weighting can be and should be used. A common use of criteria weighting that is independent of the alternatives is for candidate selection like college program admission formulas. The weighting for each category like grade point average and standardized test scores are set independently from the alternatives or applicants that are evaluated during each cycle. Another common application is the use of independent criteria weights in budget/resource allocation (Arbel, 1983; Zahedi, 1986). The decision maker should ask the question at the outset, "Am I intending to evaluate my alternatives against a "standardized" or independent set of criteria?". If the answer is yes, then the current method for weighting criteria clusters can be used. In other instances, decision makers may not want to use criteria weights that are independent of the alternatives and might say, "I am looking to find the relative value of an alternative with respect to the values that all the alternatives provide to the entire system." The model that allows decision makers to measure the relative contribution of each alternative with respect to the entire set of alternatives is the ANP.

Step 1: Define the problem
Step 2: Organize the criteria and alternatives into a hierarchy/network
Step 3: Perform pairwise comparisons
Step 4: Consistency ratio estimation
Step 5: Formation of the initial unweighted Supermatrix
Step 6: Weighting the unit sum clusters in each of the columns unweighted Supermatrix
Step 7: Formation of the weighted Supermatrix
Step 8: Raise the Supermatrix to powers
Step 9: Calculation of global priority vectors

2.2 The Analytic Network Process

The ANP is the generalized form of the AHP, when designing an ANP model a decision maker follows the steps as provided and the decision maker is not restricted to a hierarchal organization but may organize the criteria and alternatives into a network. A network design is a richer model that allows for dependence among the criteria and alternatives (Saaty, 2005; Saaty & Vargas, 2006). In the Supermatrix there may be multiple priority vectors in each column that sum to 1. It is then necessary to perform pairwise comparisons among the criteria clusters to aggregate them with respect to their relative contributions and to ensure that the weighted Supermatrix is column stochastic (Saaty, 2005). The weighted Supermatrix is then raised to powers to calculate the global priorities and weights. The process of weighting the criteria clusters will be reviewed in greater detail.

2.3 Fully-dependent ANP weighting

While much has been written to dispute the validity of AHP/ANP the purpose of this publication is not to join these arguments for or against the AHP or the ANP but rather to focus on the problem definition at the outset of setting up a model (Belton & Gear, 1983; Dyer, 1990; J. W. R. E. Dyer, 1985; Harker & Vargas, 1990; Saaty, 1986; Saaty, 1990).

A specific form of the desired final priority vector in the Limit matrix is $p = a_i / \sum_{i=1}^{n} a_i$, which provides the relative priority or contribution of an alternative a_i with respect to the system of n alternatives being considered. In other words, a Fully-Dependent model is one where the final priority vector depends on the contribution of an alternative with

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respect the contributions of everything else in the system. In order to obtain a final priority vector of the form $a_i / \sum_{i=1}^{n} a_i$, the pairwise comparisons to weight the criteria clusters performed as part of step 6 weighting the unit sum clusters in each of the columns of the unweighted Supermatrix must be done **individually** for each criteria cluster in each column regardless of whether the criteria are tangible or intangible. Harker and Vargas (1990) explained that "if the [criterion importance] vary with the chosen alternative (the reference point) then the supermatrix technique described in Harker and Vargas (1987) and Saaty (1990) must be employed." Their example uses an ANP model and obtains a vector of the form $a_i / \sum_{i=1}^{n} a_i$; however, their proof only includes a single criterion cluster. In this paper, the Supermatrix technique, demonstrated by Harker and Vargas (1990) for the case of a single criteria cluster, is extended to a fully-dependent ANP model; that is, a model where not only the individual criterion weights for each alternative vary but also the criteria cluster weights vary with each alternative.

There are three additional properties of ANP models that will further clarify this point. First, further investigation into the overall unit of measurement in an ANP model will allow criteria clusters to be combined into a general ANP model and facilitate comparisons across criteria clusters. (Choo, Schoner, & Wedley, 1999; Wedley & Choo, 2011). Second, according to the current literature this claim would depend on the tangibility of criteria being considered. Finally, an important property of the priority vectors that are combined in a Supermatrix further underscores the need for the weight for each criteria cluster to be calculated individually for every cluster in each individual column. These properties and their application to a Fully-Dependent ANP model are explained in greater detail below.

2.4 Criteria weights and the unit of measurement

Criteria weights in general are misunderstood and misused and there is no consensus on the meaning or manner of deriving criteria weights (Choo, Schoner, & Wedley, 1999). Furthermore, one may not want the criteria weights to be calculated in a way that is independent of how they are used in a decision model. Normalization, in and of itself, does not remove the units from the criteria being considered. According to Saaty (2004), relative scales do not need a unit of measurement. However, any multiplication by a constant *b*, b> 0, changes the unit of measure for a ratio vector. The fact that the value for a particular object has changed is evidence of a new unit being used. Wedley & Choo (2011) explain that ratio scales in the AHP have a unit of measure and the unit of measure is important and useful. The unit of measurement is derived from the topmost node in the total network. The scale that one can obtain from such a unit is transient, depending on the alternatives being considered, but so is the ratio scale itself. Focusing on the ratios rather than the rank will improve the efficacy of the AHP. Wedley & Choo (2011) conclude, "therein lie both the advantage and dilemma of AHP. We do not need explicit knowledge of the underlying unit of measure to derive a ratio scale, yet the derived scale has a unit." This understanding that the unit of measurement is derived from the topmost node in the hierarchy provides a unit to use as the basis for comparing criteria across clusters. Applying this unit concept to ANP facilitates comparing clusters together at the individual level of each cluster with respect to their contribution to the

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other criteria in that column. Each column must be treated separately because the weights within one column are only dependent on the topmost node of that column, the column title, in other words everything in that column is dependent on the perspective of the node at the top.

The design of the network emphasizes that the common unit of measurement is to be interpreted in the limit matrix. While in a network there is no topmost node of a network the following steps explain how a common unit is obtained:

(1) In the unweighted Supermatrix there are several units in each column – one unit sum scale for each priority vector of a set of comparisons.
(2) In the weighted Supermatrix, each of unit sum vector of a column is linked to others so that they are commensurate within the column.
(3) When the Supermatrix is then made stochastic, each column is in the unit of the totality of influences of each column.
(4) In the limit matrix where all columns are the same, each column is in the unit of all influences of the entire Supermatrix.

After reviewing these important points about criteria weights, the tangibility of the criteria also becomes relevant.

2.5 Tangibility

The concept of individual criterion weights for each alternative has been used to capture dependency and validate models with "tangible" criteria (Harker & Vargas, 1990; Schoner et al., 1993). While it has been shown that "column normalization" must be used in the case of tangibles, the opposite has been argued for intangibles (Harker & Vargas, 1990; Saaty, 2011; Saaty, 2016). The need to individually weight each criteria cluster in each column regardless of the tangibility of the elements in dependent models is demonstrated in Model 1 and Model 2. In both models, the unintended consequences of not fully representing the dependence are highlighted as the final results will seem counterintuitive because the criteria weights are obtained and applied universally across all columns in the Supermatrix and unintended consequences arise. In a decision model the value the elements from one alternative provide towards the alternative's overall value in most cases will not be equal to what those elements in another alternative provide towards the value of the other alternative. This again is the idea that a one here does not necessarily equal a one there. However, if the criteria clusters are given a single weight for all the columns the individual clusters now provide equal contributions regardless of the differences between the clusters in each column which leads to the unintended effect. This effect is clearly demonstrated in the first examples in the Model section where further clarification is provided.

It is important to emphasize that this extension is not about using particular ratio scales in the priority vectors whether dealing with tangibles or intangibles (Saaty, 2011). Rather, this example underscores the importance of using priorities and not just ratio scales in decisions that involve either tangibles or intangibles, or the combination of both. The following example demonstrates the potential difference between ratio scales and priorities. If one had no money but wanted to go to a movie that cost \$5.00 and was asked, "which amount of money would you prefer receiving: \$5.25 or \$4.90?" One should expect the individual to extremely prefer the \$5.25 to the \$4.90 which would

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correspond to a rating of 9 in Saaty's 1-9 scale. An additional validation of ratings in the movie example is the use of the consistency index. Assume a third value was introduced like $5.50. If the inconsistency of the pairwise comparisons were under the desired value of .1, then the decision maker's consistency could demonstrate that the choice of such extreme ratings was not just random but reflect the individual's preferences. "Tangibly" speaking $5.50 is not even 2 times better than $4.90; however, in terms of priorities and the goal to see a $5.00 movie there is an extreme difference. This example underscores the focus on using priorities even while using a "tangible" example because the individual's preferences do not map to a monetary ratio scale. This is the first important clarification about tangibility of the elements. Additionally, the models in the next section go one step further and use intangible criteria that do not have a direct conversion or ratio scale to show that the need to recognize and address that a one here is not necessarily equal to a one there is independent of the "tangibility" of the elements.

Without tangible data or a scale to interpret the ratings and results one might ask, "then how can the results be validated?" The results from the examples are validated by using a similar approach to Wedley and Choo "the ability of each method to replicate the true composite ratios as a measure of effectiveness. Ratio preservation rather than rank preservation is chosen as the measure of effectiveness because AHP produces ratio answers. With perfect accuracy consistency cannot be the reason why ratios are deflected from their true values" (Wedley & Choo, 2001). This methodology will be followed herein; however, with intangibles it could be more difficult to compare ratios and differences in ratios so an example where the ratios change to such an extent that rank also changes will be used. Hence, one is not initially concerned with "rank preservation/reversal" but that the relative ratios have changed to such an extent that it is obvious that ratio preservation has not been maintained due to the effect that can occur with independent criteria cluster comparisons.

2.6 Property of Supermatrix vectors

Significant insight is provided by determining the unit of measurement in an ANP model and that all the elements are related. There is also a need for clarification about the unit of measure of each of the eigenvectors from the paired comparison across the criteria and alternatives that are entered into the Supermatrix. Zahir (2007) and Wedley (2013) explain that the "unity of normalization does not bear the same level of meaning – neither across the criteria nor across the decision space." Zahir (2007) originally coined the term "a one here does not necessarily equal to a one there." The application was in the case of an AHP model and did not address the concept of a criteria cluster in one column versus another criteria cluster in another column. This concept can be more easily understood by referring to Figure 1. As a Supermatrix is an aggregation of multiple individual eigenvectors each summing to one, when they are weighted there is no reason to assume that the contributions of each eigenvector whether in the same column or rows provides an equal contribution to the overall system. This is demonstrated in the illustration in Figure 1 that is similar to one displayed by Wedley (2013).

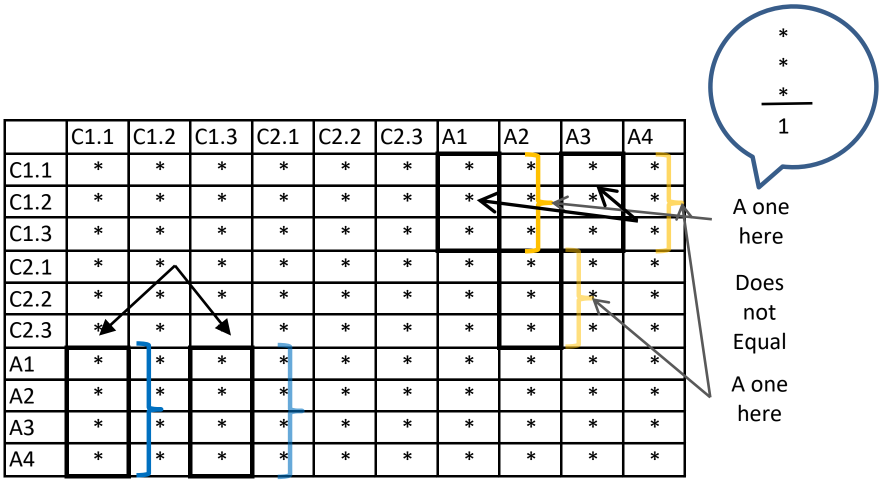

Figure 1. Sample unweighted Supermatrix with unequal eigenvectors

Figure 1 represents an unweighted Supermatrix with four alternatives evaluated with respect to six criteria. The criteria have been organized as three subcriteria within two main criteria clusters. Five eigenvectors representing sets of comparisons that sum to unity have been outlined. The three in the upper right hand corner consist of two from the same criteria cluster but from different alternatives and the third from a separate alternative in a separate criteria cluster. With the exception that A1 and A3 were identical alternatives while each eigenvector sums to unity there is no reason that a one here is equal to a one there or in other words that the clusters of dependent criteria provide an equal contribution or the same weight to each alternative. This example further exemplifies the effect of this unintended result because assigning the same criteria weights assumes the units of measurement are the same and can lead to unintended results. In the next section, a numerical example is used to demonstrate the outcomes of this effect.

3. Models

3.1 Problem definition

Three ANP models are provided below to demonstrate the problem and solution. In each example one must choose which of three individuals to speak at a graduation ceremony. It is important to mention upfront that these examples provide a meaningful context wherein to evaluate the results of the current and proposed methods of weighting criteria clusters and are not meant to be comprehensive or universal models for choosing a graduation speaker. While the weightings are not in any shape or form the “optimal” weights for this type of model, nor are they unique to invoking the problem that is presented, they are reasonable and highlight the proposed concerns. In every example three unique criteria clusters are used to evaluate the potential speakers: Prestige, Public Speaking Skills, and Availability. It is also worth noting that it is necessary to use more than one criteria cluster because in a model with a single cluster of criteria the criteria cluster is already column stochastic. The plural number of clusters used provides additional insight to the discussion in the literature review in sections 2.3-2.6.

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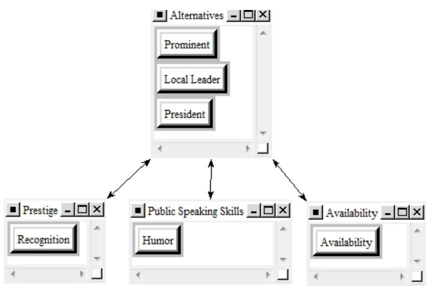
The three alternatives in the examples are: 1) the President of a nation, 2) a governor or more local but very Prominent leader, and finally 3) a Local community leader. In the first example, each criteria cluster will have a single criterion within each cluster. In the subsequent examples, multiple criteria will be included within the criteria clusters. The counterintuitive results in the first two examples of this sample decision model using intangible criteria are used to demonstrate the potential unintended consequences of not capturing the dependence by performing individual cluster weighting comparisons in each column, and also to show that this unintended effect is independent of the tangibility of the elements.

3.2 Model 1



	President	Prominent	Local		Criteria Weights
Prestige - Recognition	High	Medium-High	Low		0.5
Public Speaking - Humor	Medium	Medium	Medium		0.2
Availability	Low	Medium-Low	High		0.3

Figure 2. ANP model and ratings for Model 1 – Unintended Independence

Figure 2 displays the ANP model and initial ratings of each of the alternatives with respect to the criteria that will be used in Model 1. The criteria weights for the 3 separate criteria clusters here are chosen to represent an emphasis on Prestige. From the priority vector for the criteria weights (Figures 2 and 3) one could conclude that the value placed on Prestige is as important as everything else put together. This vector is obtained by asking the three questions in SuperDecisions as shown in Figure 3 with respect to the entire network.

Figure 3. Model 1 Criteria cluster weights

While one may assume that the president will be the preferred option because of the emphasis on Prestige, the results in Table 1 demonstrate that with the use of a single criterion in each of the 3 separate criteria clusters in this ANP model and that by performing criteria comparisons as is currently prescribed and demonstrated in Figure 3 that this ANP model is actually nothing more than an AHP model. The dependency of the alternatives on the criteria is lost. Just as with an AHP model both the criteria cluster weights and the individual elements in this ANP model are independent of the alternatives and this occurred even though the elements are intangibles.

The first matrix in Table 1 is the unweighted Supermatrix. The entries in the top right hand side are all equal to one because there is only a single criterion in each criteria cluster. The three clusters in the unweighted Supermatrix are then weighted to become the weighted ANP Supermatrix. Because according to the current approach a criteria cluster is given the same weight across all the columns in the Supermatrix this set of three identical vectors could be replaced by this same vector in an AHP model with a goal node and achieve the same results. For example, the Prestige of the Local leader is given the same weight as the Prestige of the President. Likewise, the Availability of the President is also given the same weight as the Availability of the Local leader and hence the criteria cluster weights are independent of the alternatives.

In the 2nd example, because there is more than a single criterion in each cluster there will be some level of dependence but ultimately that dependence is compromised just as in this first example. Although it is not as clear as when a single criterion is used in each cluster, the effect is similar and hence not limited to the special case of a single criterion. The final results for Model 1 in the Limit matrix (Table 1) show that every alternative is basically equal. Not addressing the dependency resulted in the alternatives appearing almost identical. The important findings in the first example are a simple demonstration of the effect of not performing criteria cluster weighting comparisons individually for each alternative and that the undesirable effect applies to both tangible and intangible criteria and therefore is independent of the tangibility of the elements.

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Table 1
Model 1 Unweighted and Weighted Supermatrix and Limit matrix

Unweighted Supermatrix

		Criteria			Alternatives		
		Recognition	Humor	Availability	President	Prominent	Local
Criteria	Prestige - Recognition	0	0	0	**1**	**1**	**1**
	Public Speaking - Humor	0	0	0	**1**	**1**	**1**
	Availability	0	0	0	**1**	**1**	**1**
Alternatives	President	0.4901	0.3333	0.0738	0	0	0
	Prominent	0.4507	0.3333	0.1218	0	0	0
	Local	0.0592	0.3333	0.8044	0	0	0

Weighted Supermatrix

		Criteria			Alternatives		
		Recognition	Humor	Availability	President	Prominent	Local
Criteria	Prestige - Recognition	0	0	0	**0.5000**	**0.5000**	**0.5000**
	Public Speaking - Humor	0	0	0	**0.2000**	**0.2000**	**0.2000**
	Availability	0	0	0	**0.3000**	**0.3000**	**0.3000**
Alternatives	President	0.4901	0.3333	0.0738	0	0	0
	Prominent	0.4507	0.3333	0.1218	0	0	0
	Local	0.0592	0.3333	0.8044	0	0	0

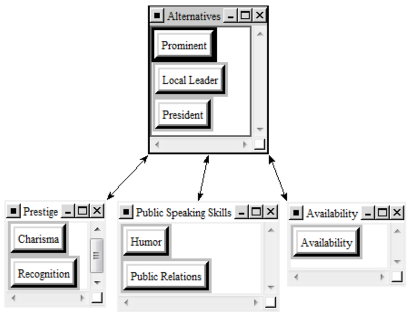
Limit Matrix

		Criteria			Alternatives		
		Recognition	Humor	Availability	President	Prominent	Local
Criteria	Prestige - Recognition	0	0	0	0.5000	0.5000	0.5000
	Public Speaking - Humor	0	0	0	0.2000	0.2000	0.2000
	Availability	0	0	0	0.3000	0.3000	0.3000
Alternatives	President	**0.3338**	**0.3338**	**0.3338**	0	0	0
	Prominent	**0.3286**	**0.3286**	**0.3286**	0	0	0
	Local	**0.3376**	**0.3376**	**0.3376**	0	0	0

This example with a single criterion in each criteria cluster clarifies that the need to perform criteria cluster comparisons for each alternative individually is not affected by the "tangibility" of the elements. Regardless of whether the elements are tangible or intangible the criteria weights in this example are independent of the alternatives. There is no longer any dependency in this ANP model with intangible criteria even though the needed connections were made in this ANP model to achieve dependency. By not making the criteria cluster weighting comparisons separately for each alternative the desired dependency is lost. In the next example, the specific case of a single criterion is extended to a more general case where there are *n+1* criteria in *m* clusters.

3.3 Model 2

The second example is similar to the first in that one is choosing from the same three alternatives and using the same three criteria clusters. Now with multiple criteria in some clusters, the impact of this effect can be seen in the current application of the ANP. Once more, the final results will seem counterintuitive because the criteria cluster weightings are not performed individually for each alternative in each column. An additional criterion is added in two of the three criteria clusters. The Public Speaking Skills criteria cluster will now be measured with two individual elements: Humor and Public Relations skills. The Prestige criteria cluster will also now be measured with two individual elements: Prestige and Charisma. The Availability cluster will continue with a single element.



	President	Prominent	Local
Recognition	High	Medium-High	Low
Charisma	Medium	Medium	Medium
Humor	Medium	Medium	Medium
Public Relations	Medium	Medium	Medium
Availability	Low	Medium-Low	High

Figure 4. ANP model and ratings for Models 2-3

For simplification purposes, it is assumed that the candidates have equally perceived values in the amounts of Charisma, Humor and Public Relations. While this assumption is not necessary it is only used to provide clarity when the results and the potential unintended consequences of not modeling the dependence are discussed. With that assumption in place, the real differences between candidates is between their Availability and Prestige.

Depending on the user defined criteria weightings if the Availability criterion is the most important, a less prestigious candidate will be chosen, whereas if the Prestige criteria are determined to be more important a more prestigious leader will be chosen. Figure 4 lists each alternative (candidate), the criteria, and a verbal rating for each alternative. Similar criteria cluster weights are used in this model as those in the previous example; however, in an effort to counter the argument that the solution is to simply put more weight on the Prestige cluster that argument is addressed in this example. Figure 5 displays the new criteria cluster weights. These weights are used to weight the criteria clusters in the unweighted Supermatrix to obtain the weighted Supermatrix and the Limit matrix in Table 2. Once again, because of the unintended consequences of failing to fully capture the dependency in the model, the president is not the preferred alternative.

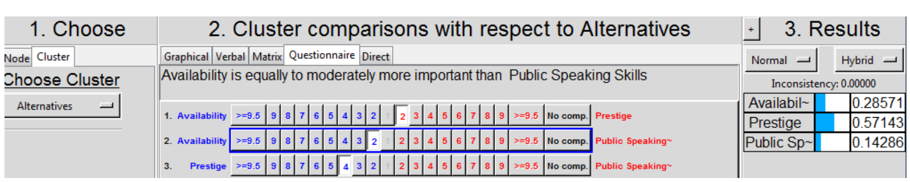


Figure 5. Model 2 Criteria cluster weights

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Table 2
Model 2 Unweighted and weighted Supermatrix and Limit matrix

Semi-Dependent - Unweighted Supermatrix

		Criteria					Alternatives		
		Recognition	Charisma	Humor	Public Relations	Availability	President	Prominent	Local
Criteria	Recognition	0	0	0	0	0	**0.9**	**0.8**	**0.1**
	Charisma	0	0	0	0	0	**0.1**	**0.2**	**0.9**
	Humor	0	0	0	0	0	**0.5**	**0.5**	**0.5**
	Public Relations	0	0	0	0	0	**0.5**	**0.5**	**0.5**
	Availability	0	0	0	0	0	**1**	**1**	**1**
Alternatives	President	0.4901	0.3333	0.3333	0.3333	0.0738	0	0	0
	Prominent	0.4507	0.3333	0.3333	0.3333	0.1218	0	0	0
	Local	0.0592	0.3333	0.3333	0.3333	0.8044	0	0	0

Semi-Dependent - Weighted Supermatrix

		Criteria					Alternatives		
		Recognition	Charisma	Humor	Public Relations	Availability	President	Prominent	Local
Criteria	Recognition	0	0	0	0	0	**0.5143**	**0.4571**	**0.0571**
	Charisma	0	0	0	0	0	**0.0571**	**0.1143**	**0.5143**
	Humor	0	0	0	0	0	**0.0714**	**0.0714**	**0.0714**
	Public Relations	0	0	0	0	0	**0.0714**	**0.0714**	**0.0714**
	Availability	0	0	0	0	0	**0.2857**	**0.2857**	**0.2857**
Alternatives	President	0.4901	0.3333	0.3333	0.3333	0.0738	0	0	0
	Prominent	0.4507	0.3333	0.3333	0.3333	0.1218	0	0	0
	Local	0.0592	0.3333	0.3333	0.3333	0.8044	0	0	0

Semi-Dependent Limit Matrix

		Criteria					Alternatives		
		Recognition	Charisma	Humor	Public Relations	Availability	President	Prominent	Local
Criteria	Recognition	0	0	0	0	0	**0.3231**	**0.3231**	**0.3231**
	Charisma	0	0	0	0	0	**0.2483**	**0.2483**	**0.2483**
	Humor	0	0	0	0	0	**0.0714**	**0.0714**	**0.0714**
	Public Relations	0	0	0	0	0	**0.0714**	**0.0714**	**0.0714**
	Availability	0	0	0	0	0	**0.2857**	**0.2857**	**0.2857**
Alternatives	President	**0.3098**	**0.3098**	**0.3098**	**0.3098**	**0.3098**	0	0	0
	Prominent	**0.3108**	**0.3108**	**0.3108**	**0.3108**	**0.3108**	0	0	0
	Local	**0.3794**	**0.3794**	**0.3794**	**0.3794**	**0.3794**	0	0	0

The results in the Limit matrix in Table 2 identify the Local leader as the preferred alternative with a priority that is 1.22 times greater than the priority for the President. These results may seem somewhat counterintuitive given the criteria weights used in this example. Prestige was considered the most important criteria (.5714); the candidates have identical amounts of Public Relations (.1429), and Availability was half as important as the Prestige (.2857). Why then is the local leader the preferred alternative? It is in this model, where there are multiple criteria clusters with multiple elements, that a definition for a fully-dependent ANP model becomes essential. A fully-dependent ANP model is one where not only the weights of the individual criterion elements for each alternative vary from one alternative to the next to reflect the specific contribution each makes to the respective alternative, but also the weighting of the criteria clusters must also reflect the specific contribution made individually to each alternative.

Because the Prestige cluster is weighted the same for each alternative, the criteria cluster weightings are ultimately independent of the alternatives. It is true that the weight given to the Prestige cluster is distributed differently among the two criteria elements within that cluster in each alternative. However, because the criteria cluster receives the same weighting for each alternative the contribution of the Prestige of the Local leader is over

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weighted and the remainder of the Prestige weight for the Local leader is then allocated to its Charisma (refer to the highlighted cells in the middle of Table 2).

The Prestige and Charisma of each alternative are victims of not addressing the model as fully-dependent and the alternatives appear much more similar than they really are. In regards to the Public Relations cluster, even though the Public Relations is the same for each alternative, the cluster should probably not provide the same relative contribution to the overall value for each alternative because the relative contribution with respect to each individual alternative will most likely be different. In the Availability cluster (also highlighted in the middle of Table 2) each alternative has an equal contribution even though the comparisons represented in the left quadrant of the Supermatrix seem to imply the Availability of each alternative differs significantly. A solution to avoiding these confusing results is to create a Fully-Dependent ANP model by performing individual criteria cluster weighting comparisons for each column in the Supermatrix. The proof is provided in the appendix and supported with an example incorporating its results in the next section.

The desired eigenvector to be obtained from the synthesis of the Fully-Dependent Supermatrix is defined in the form $a_i / \sum_{i=1}^{n} a_i$, which provides the relative priority or contribution of an alternative ai with respect to the system of n alternatives being considered. In other words, it is equal to the relative contribution of all the attributes which an individual alternative possesses with respect to the contribution of all the alternatives considered in a decision.

The first step in the proof is to show that the $\lambda_{\max}$ =1 in a weighted Supermatrix which is a column stochastic matrix by the Gershgorin Circle theorem; the importance of this finding has also been emphasized by Lipovetsky who demonstrates that when solving $Ax = \lambda_{\max} x$ that $\lambda_{\max}$ =1 with multiple lambda equal to 1 (Lipovetsky, 2011). Next, by invoking the Perron–Frobenius theorem one can obtain the limiting priority vector. According to the Perron–Frobenius theorem there is a unique solution to the equation $Ax = 1x$ and by restricting x to be of a particular form $a_i / \sum_{i=1}^{n} a_i$ therefore the Supermatrix *A* must also be of a particular form wherein the criteria weightings are performed for each alternative individually. For the complete proof refer to the appendix.

4. Achieving full-dependency

The third example begins with the same initial ratings and unweighted Supermatrix which was used in Model 2. Criteria cluster weighting comparisons will again be used to weight the criteria clusters, but now according to the method shown in the proof. In summary, the criteria cluster weighting comparisons will be performed individually in each column. Obtaining unique criteria cluster weights for each column will provide the necessary weights to obtain a column stochastic Supermatrix. The cluster weights are obtained for each column using the same pairwise comparison process. The only difference is the questions are asked with respect to an individual alternative and then

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repeated for each alternative; the criteria cluster weights and inconsistency indices are presented in Table 3.

4.1 Model 3

It is worth noting the differences in the criteria cluster weights in Table 3. The President obtains most of her value from the Prestige cluster and little from the Availability cluster, while the value of the Local leader is more that he is Available than that he is Prestigious. The criteria cluster weights are unique to each alternative and the contributions of the alternatives are weighted according to the unique contribution of said criteria to the alternative. This is the advantage of recognizing that a “one here does not necessarily equal a one there” and designing a Fully-Dependent ANP model by obtaining criteria cluster weights for each individual cluster in each column. In Table 4 the weights from Table 3 are used to obtain the Weighted Supermatrix. This weighted Supermatrix is raised to powers and the Limit matrix is obtained.

Table 3
Fully-Dependent cluster weights and inconsistency index

Criteria Cluster Comparisons by Column

Cluster Comparisons - President

	Prestige	Public Speaking	Availability	Eigenvector
Prestige	1	8	9	0.8142
Public Speaking	0.125	1	2	0.1140
Availability	0.1111	0.5	1	0.0718
Inconsistency				0.0516

Cluster Comparisons - Prominent

	Prestige	Public Speaking	Availability	Eigenvector
Prestige	1	6	8	0.7514
Public Speaking	0.1667	1	2	0.1782
Availability	0.125	0.5	1	0.0704
Inconsistency				0.0280

Cluster Comparisons - Local

	Prestige	Public Speaking	Availability	Eigenvector
Prestige	1	0.5	0.25	0.1494
Public Speaking	2	1	0.5	0.3764
Availability	4	1	1	0.4742
Inconsistency				0.05156

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Table 4
Model 3 Unweighted and Weighted Supermatrix and Limit matrix - Fully-Dependent model

Fully-Dependent - Unweighted Supermatrix

		Criteria					Alternatives		
		Recognition	Charisma	Humor	Public Relations	Availability	President	Prominent	Local
Criteria	Recognition	0	0	0	0	0	**0.9**	**0.8**	**0.1**
	Charisma	0	0	0	0	0	**0.1**	**0.2**	**0.9**
	Humor	0	0	0	0	0	**0.5**	**0.5**	**0.5**
	Public Relations	0	0	0	0	0	**0.5**	**0.5**	**0.5**
	Availability	0	0	0	0	0	**1**	**1**	**1**
Alternatives	President	0.4901	0.3333	0.3333	0.3333	0.0738	0	0	0
	Prominent	0.4507	0.3333	0.3333	0.3333	0.1218	0	0	0
	Local	0.0592	0.3333	0.3333	0.3333	0.8044	0	0	0

Fully-Dependent - Weighted Supermatrix

		Criteria					Alternatives		
		Recognition	Charisma	Humor	Public Relations	Availability	President	Prominent	Local
Criteria	Recognition	0	0	0	0	0	**0.7328**	**0.6011**	**0.0149**
	Charisma	0	0	0	0	0	**0.0814**	**0.1503**	**0.1344**
	Humor	0	0	0	0	0	**0.0570**	**0.0891**	**0.1882**
	Public Relations	0	0	0	0	0	**0.0570**	**0.0891**	**0.1882**
	Availability	0	0	0	0	0	**0.0718**	**0.0704**	**0.4742**
Alternatives	President	0.4901	0.3333	0.3333	0.3333	0.0738	0	0	0
	Prominent	0.4507	0.3333	0.3333	0.3333	0.1218	0	0	0
	Local	0.0592	0.3333	0.3333	0.3333	0.8044	0	0	0

Fully-Dependent Limit Matrix

		Criteria					Alternatives		
		Recognition	Charisma	Humor	Public Relations	Availability	President	Prominent	Local
Criteria	Recognition	0	0	0	0	0	**0.4778**	**0.4778**	**0.4778**
	Charisma	0	0	0	0	0	**0.1209**	**0.1209**	**0.1209**
	Humor	0	0	0	0	0	**0.1064**	**0.1064**	**0.1064**
	Public Relations	0	0	0	0	0	**0.1064**	**0.1064**	**0.1064**
	Availability	0	0	0	0	0	**0.1885**	**0.1885**	**0.1885**
Alternatives	President	**0.3593**	**0.3593**	**0.3593**	**0.3593**	**0.3593**	0	0	0
	Prominent	**0.3495**	**0.3495**	**0.3495**	**0.3495**	**0.3495**	0	0	0
	Local	**0.2912**	**0.2912**	**0.2912**	**0.2912**	**0.2912**	0	0	0

The results from the Limit matrix for Model 3 show the preferred speaker is the President. The Prominent leader is the next most preferred and the Local leader is now the least preferred. Ratio preservation has been achieved by performing the criteria cluster weighting comparisons for each column (alternative) individually. These differences can be seen in the highlighted cells in the Weighted Supermatrix in the middle of Table 4 and compared with the same entries in the Weighted Supermatrix in Model 2 in Table 2. This contrast in the Weighted Supermatrices in Tables 2 and 4 underscores the benefits of calculating the criteria cluster weights separately for each column in an ANP model. This final priority vector is fully-dependent on the alternatives and criteria; the final priority vector now reflects the relative contribution of each alternative with respect to the entire network.

4.2 Independence of irrelevant alternatives

While we agree with Wedley and Choo (2011) that the argument should be concerned with what is the unit of measurement and rank preservation, we will demonstrate that by implementing Step 6, as proposed herein, a Fully-Dependent ANP model is not subject to rank reversal when adding both irrelevant and identical alternatives. Rank reversal is still a phenomenon worth resolving (Hefnaway & Mohammed, 2014). The independence of irrelevant alternatives (IIA) hangs on the findings of the Gershgorin Circle theorem from which we can conclude there is a unique solution to the equation

$$Ax = \lambda_{\max} x \quad (10)$$

In this paper, three alternatives were used and their priorities are equal to the following priority vectors in the limit matrix where $\mathbf{A_i}$ represents the total influence of each alternative i:

$$a_i / \sum_{i=1}^{n} a_i = \left(\frac{A_1}{A_1+A_2+A_3}, \quad \frac{A_2}{A_1+A_2+A_3}, \quad \frac{A_3}{A_1+A_2+A_3}\right) \quad (11)$$

In a fully-dependent ANP model because the columns were normalized individually the portion of the Supermatrix representing the distribution of the criteria among the alternatives does not need to be changed or modified. Rather a new column representing the distribution of the criteria among the new alternative is added. The distributions of the alternatives among each criterion are updated only by comparing the new alternative against the original alternatives. Adding copy of $\mathbf{A_1}$ will result in the following vector:

$$\left(\frac{A_1}{A_1+A_2+A_3+A_1}, \quad \frac{A_2}{A_1+A_2+A_3+A_1}, \quad \frac{A_3}{A_1+A_2+A_3+A_1}, \quad \frac{A_1}{A_1+A_2+A_3+A_1}\right)$$

While the individual priorities of each alternative will be smaller, the relative weight of each alternative with respect to the other alternatives remains unchanged. The IIA property is another advantage of Fully-Dependent ANP models.

While the advantages of performing cluster comparisons individually for each column has been demonstrated with numerical examples the generalization in general mathematical terms in the proof in the appendix provides an even stronger argument for the advantages of performing the additional cluster comparisons to achieve Fully-Dependent ANP models.

5. Conclusion

The AHP and ANP can be categorized as disruptive technology. They are amazing frameworks that are used to model independent criteria and fully-dependent criteria as demonstrated above. Because a Fully-Dependent ANP model requires additional pairwise comparisons it is crucial that decision makers determine upfront what they want the final priority vector to represent; that is, whether they want a final priority vector that represents independence or dependence among the criteria and alternatives. The need to address this question upfront in the decision-making process is further underscored by the research discussed in the literature review regarding AHP and ANP. Fully-Dependent ANP models as put forth in this paper are not intended to be a universal approach to all ANP models; as was stated earlier, there are very good reasons to model the criteria independent of the alternatives. The same can be said for modeling the criteria as fully-dependent on the alternatives. Future research is needed to determine the meaning and interpretation of final priority vectors if one wishes to use a form of the semi-dependence

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among the criteria and alternatives where the criteria clusters are weighted equally across columns in an ANP model. Additionally, it will be useful to quantify the impact of criteria cluster weighting techniques on the coherency of the ANP Supermatrix.

What is most important in this paper is that if one wishes to model a fully-dependent system which will provide a final priority vector of the form in the Limit matrix, which is interpreted as the relative priority or contribution of an alternative a_i with respect to, or dependent on, the system of n alternatives being considered then the criteria cluster weighting comparisons must be performed individually in each column of the Supermatrix. It is also important to recognize that this undesirable effect exists independent of the tangibility of the elements. This point was demonstrated through the discussion in the literature review regarding tangibility and clarified through models contained herein. One potential reason this issue has not been addressed in the literature previously is that prior proofs contained a single criteria cluster and did not contain multiple criteria clusters. The proof contained herein contains multiple criteria clusters and provides the generalization of the proposed method. Fully-Dependent ANP models allow the decision maker to capture a greater level of dependence in the ANP than is currently available and will further aid decision makers to make better decisions.

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Here we consider a strongly connected system of n alternatives $A_1, A_2, \cdots, A_n$ with m clusters with p_k criteria in cluster k. Let $a_{ij}^{(k)}$ represents the perceived or elicited values of the priorities for each criteria i in cluster k possessed by alternative j. The initial data matrix has the following form:

$$A = \begin{pmatrix} a_{11}^{(1)} & a_{21}^{(1)} & \cdots & a_{p_1 1}^{(1)} & a_{11}^{(2)} & a_{21}^{(2)} & \cdots & a_{p_2 1}^{(2)} & \cdots & a_{11}^{(n)} & a_{21}^{(n)} & \cdots & a_{p_n 1}^{(n)} \\ a_{12}^{(1)} & a_{22}^{(1)} & \cdots & a_{p_1 2}^{(1)} & a_{12}^{(2)} & a_{22}^{(2)} & \cdots & a_{p_2 2}^{(2)} & \cdots & a_{12}^{(n)} & a_{22}^{(n)} & \cdots & a_{p_n 2}^{(n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{1m}^{(1)} & a_{2m}^{(1)} & \cdots & a_{p_1 m}^{(1)} & a_{1m}^{(2)} & a_{2m}^{(2)} & \cdots & a_{p_2 m}^{(2)} & \cdots & a_{1m}^{(n)} & a_{2m}^{(n)} & \cdots & a_{p_n m}^{(n)} \end{pmatrix} \qquad (1)$$

Furthermore, assume one wishes to determine which alternative A_j is most "preferred" or "valued" such that $\frac{\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{ij}^{(k)}}{\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{ij}^{(k)}}$ represents the relative value of an alternative j (the sum of the values of the criteria which it possesses) with respect to all the alternatives being considered. In this way, we may define the following vector,

$$x = \left(\frac{\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{i1}^{(k)}}{\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{ij}^{(k)}}, \frac{\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{i2}^{(k)}}{\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{ij}^{(k)}}, \cdots, \frac{\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{in}^{(k)}}{\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{ij}^{(k)}}\right)^T \qquad (2)$$

which represents the relative weight or relative value of each alternative.

Without loss of generality, we will set $m = 3$, and then the Supermatrix is of the form:

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$$\begin{pmatrix}
0 & 0 & \cdots & 0 & x_{11} & x_{21} & \cdots & x_{p_1 1} & y_{11} & y_{21} & \cdots & y_{p_2 1} & z_{11} & z_{21} & \cdots & z_{p_3 1} \\
0 & 0 & \cdots & 0 & x_{12} & x_{22} & \cdots & x_{p_1 2} & y_{12} & y_{22} & \cdots & y_{p_2 2} & z_{12} & z_{22} & \cdots & z_{p_3 2} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & x_{1n} & x_{2n} & \cdots & x_{p_1 n} & y_{1n} & y_{2n} & \cdots & y_{p_2 n} & z_{1n} & z_{2n} & \cdots & z_{p_3 n} \\
u_{11} & u_{12} & \cdots & u_{1n} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
u_{21} & u_{22} & \cdots & u_{2n} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
u_{p_1 1} & u_{p_1 2} & \cdots & u_{p_1 n} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
v_{11} & v_{12} & \cdots & v_{1n} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
v_{21} & v_{22} & \cdots & v_{2n} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
v_{p_2 1} & v_{p_2 2} & \cdots & v_{p_2 n} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
w_{11} & w_{12} & \cdots & w_{1n} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
w_{21} & w_{22} & \cdots & w_{2n} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
w_{p_3 1} & w_{p_3 2} & \cdots & w_{p_3 n} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0
\end{pmatrix} \tag{3}$$

Where $u_{ij} = \frac{a_{ij}^{(1)}}{\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{ij}^{(k)}}$, $v_{ij} = \frac{a_{ij}^{(2)}}{\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{ij}^{(k)}}$, $w_{ij} = \frac{a_{ij}^{(3)}}{\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{ij}^{(k)}}$, $x_{ij} = \frac{a_{ij}^{(1)}}{\sum_{j=1}^{n} a_{ij}^{(1)}}$, $y_{ij} = \frac{a_{ij}^{(2)}}{\sum_{j=1}^{n} a_{ij}^{(2)}}$, $z_{ij} = \frac{a_{ij}^{(3)}}{\sum_{j=1}^{n} a_{ij}^{(3)}}, \forall i, j.$

For simplicity, denote the matrix S as,

$$S = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \tag{4}$$

In order to verify the convergence of the matrix power operation it is sufficient to consider the even power as the power index approaches ∞. we have,

$$S^{2n} = \begin{pmatrix} (BC)^n & 0 \\ 0 & (CB)^n \end{pmatrix} \tag{5}$$

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Since $(CB)^n = C(BC)^{n-1}B$, let us focus on the matrix BC and denote the matrix BC as $(\Delta_{ij})_{n\times n}$.

Theorem 1. The matrix BC is a positive stochastic matrix. Furthermore, the maximal eigenvalue of the matrix BC is 1.

Proof. Based on the assumption that S is strongly connected, every entry in BC cannot be equal to 0 which is positive. Because the summation of each column in B or C is equal to 1. By calculating the produce, the summation of each column of the matrix BC is 1. That is, BC is a positive stochastic matrix. Now, by the Gershgorin circle theorem,

$$|\lambda - \Delta_{ii}| \leq \sum_{j\neq i} \Delta_{ji} \tag{6}$$

Hence,

$$|\lambda| \leq \sum_{j} \Delta_{ji} = 1$$

Furthermore, 1 is the eigenvalue of the matrix BC by computing $\det(BC - Id) = 0$.
Therefore, the maximal eigenvalue of matrix BC is 1.

Moreover, direct calculation implies the following properties:

Lemma 2. Consider the previously defined vector

$$x = \left(\frac{\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{i1}^{(k)}}{\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{ij}^{(k)}}, \frac{\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{i2}^{(k)}}{\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{ij}^{(k)}}, \cdots, \frac{\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{in}^{(k)}}{\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{ij}^{(k)}}\right)^T \tag{7}$$

and define

$$y = (1, 1, \cdots, 1)^T$$

Then $(BC)x = x$, $(BC)^T y = y$ and $x^T y = 1$.

So x is the eigenvector corresponding to eigenvalue 1. As a result, we can use the Perron's Theorem in Linear Algebra, to get the following main convergence theorem:

Theorem 3. $(BC)^n \to L$ as $n \to \infty$ where $L = xy^T$.

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Thus, we get the convergent limit of the power operation of matrix BC, which is

$$\begin{pmatrix} \frac{\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{i1}^{(k)}}{\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{ij}^{(k)}} & \frac{\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{i1}^{(k)}}{\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{ij}^{(k)}} & \cdots & \frac{\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{i1}^{(k)}}{\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{ij}^{(k)}} \\ \frac{\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{i2}^{(k)}}{\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{ij}^{(k)}} & \frac{\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{i2}^{(k)}}{\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{ij}^{(k)}} & \cdots & \frac{\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{i2}^{(k)}}{\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{ij}^{(k)}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{in}^{(k)}}{\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{ij}^{(k)}} & \frac{\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{in}^{(k)}}{\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{ij}^{(k)}} & \cdots & \frac{\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{in}^{(k)}}{\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{ij}^{(k)}} \end{pmatrix} \tag{8}$$

Here each column is equal to the "preferred" or "valued" vector x.

Next by using $(CB)^n = C(BC)^{n-1}B$, the convergence limit of power operation of matrix CB is CLB which is a matrix with identical columns of the form:

$$\begin{pmatrix} \frac{\sum_{j=1}^{n} a_{1j}^{(1)}}{\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{ij}^{(k)}} \\ \frac{\sum_{j=1}^{n} a_{2j}^{(1)}}{\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{ij}^{(k)}} \\ \vdots \\ \frac{\sum_{j=1}^{n} a_{p_1 j}^{(1)}}{\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{ij}^{(k)}} \\ \frac{\sum_{j=1}^{n} a_{1j}^{(2)}}{\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{ij}^{(k)}} \\ \frac{\sum_{j=1}^{n} a_{2j}^{(2)}}{\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{ij}^{(k)}} \\ \vdots \\ \frac{\sum_{j=1}^{n} a_{p_2 j}^{(2)}}{\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{ij}^{(k)}} \\ \vdots \\ \frac{\sum_{j=1}^{n} a_{1j}^{(m)}}{\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{ij}^{(k)}} \\ \frac{\sum_{j=1}^{n} a_{2j}^{(m)}}{\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{ij}^{(k)}} \\ \vdots \\ \frac{\sum_{j=1}^{n} a_{p_m j}^{(m)}}{\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{i=1}^{p_k} a_{ij}^{(k)}} \end{pmatrix} \tag{9}$$

Each entry in this column represents the contribution of criteria i with respect to all the criteria in the system.

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