NEW ADVANCES OF THE COMPATIBILITY INDEX"*G*"IN WEIGHTED ENVIRONMENTS

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ABSTRACT

This article addresses the problem of measuring closeness in weighted environments (decision-making environments). This article is relevant because of the importance of having a dependable cardinal measure of distance in weighted environments. A weighted environment is a non-isotropic structure where the different directions (axes) may have different importance (weight) hence, privileged directions exist. In this kind of a structure, it would be very important to have a cardinal reliable index that is able to show how close or compatible the set of measures of one individual is with respect to the group or to any other, or how close one pattern of behavior is to another. A few common examples of the application of this are the interaction between actors in a decision making process (system values interaction), matching profiles, pattern recognition, and any situation where a process of measurement with qualitative variables is involved.

Keywords: Weighted environments; measurement; compatibility index *G*; order topology

1. Introduction

This article addresses the problem of measuring closeness in weighted environments (decision-making environments) using the concept of compatibility of priority vectors and value systems. When using the concept of closeness the question immediately comes to mind, what does it mean to be close (when does close really mean close)? Thus, when measuring closeness or proximity we should have a point of comparison (a threshold) that makes it possible to compare or make a decision if our positions, system values or priorities are really close.

For our purposes, compatibility is defined as the proximity or closeness between vectors within a weighted space (Garuti, 2014). We will propose a compatibility index able to measure closeness in a weighted environment. Thus, we will be able to assess pattern recognition, medical diagnosis support measuring the degree of closeness between disease-diagnosis profiles, buyer-seller matching profiles, measuring the degree of closeness between homebuyer and seller projects, or degree of matching for employment, measuring the degree of closeness between a person's profile with the desired position profile, conflict resolution in curricula network design, measuring closeness of two different value systems (the ways of thinking) by identifying and measuring the discrepancies, and in general measuring the degree of compatibility between any priority vectors in cardinal measure bases (order topology) (Garuti, 2016, 2014, 2012).

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First, the paper presents some theory behind distance (measurement) and closeness concepts in different cases and discusses a nice point of view from statistical and set theory for measuring distance and similarity. Then, the concept of scales is presented, including compatibility, compatibility index *G* and some analogies between *G* and distance concept. Then, a comparison with other compatibility indices present in the literature is shown, highlighting the advantages of *G*in relation to the others (especially within weighted environments). Next, a necessary threshold that allows "when close really means close" in weighted environments is presented. Finally, three relatively simple examples are developed, each one presenting a different application for the compatibility index *G*. One example asks if the order of choice is necessary to say if two rankings are compatible or not, the second is for quality testing (testing Saaty's consistency index through compatibility index *G*), and the third is for measuring comparability between two different rules of measurement (two different points of view).

2. Literature review

In metric topology the particular function of distance D(a,b) is used to assess the closeness of two points a, b as a real positive function that keeps 3 basic properties (Garuti, 2012):

1.- $D(a,b) > 0$ and $D(a,b) = 0$ if $a=b$ (definition of zero distance)
2.- $D(a,b) = D(b,a)$ (symmetry)
3.- $D(a,b) + D(b,c) \geq D(a,c)$ (triangular inequality)

The general function of distance used to calculate the separation between two points is:

$D(a,b)= \lim_{n \to k} (\Sigma_i(a_i-b_i)^n)^{1/n}$ (i=1,...,n; n= dimensionofthespace).

When applying different values of k, different Norms of distance appear:

For k = 1, then: $D(a,b)= \Sigma_i \text{Abs}(a_i-b_i)$. Norm1, absolute Norm or path Norm; this Norm measures the distance from *a* to *b* within a 1D line, "walking" over the path, in one line-dimension.

For k = 2, then: $D(a,b)= [(\Sigma_i(a_i-b_i)^2)]^{1/2}$. Norm2 or Euclidean Norm, this Norm measures the distance from *a* to *b*, within a 2D plane (X-Y plane) getting the shortest path (the straight line).

For k = +∞, then: $D(a,b)= \text{Max}_i (\text{abs}(a_i-b_i))$. Norm∞ or Norm Max; this Norm measures the distance from *a* to *b* within a ∞D hyperplane, getting the shortest path (the maximum coordinate) from all the possible paths.

In the field of statistics, an interesting case of distance calculation is known as distance of Mahalanobis which meets the metric properties shown above (Mahalanobis, 1936). This distance takes into consideration the parameters of statistics like deviation and covariance which can be assimilated to concepts of weight and dependence in the AHP/ANP world.

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Its formal presentation is:

$d_m(x,y) = \sqrt{(X - Y) \sum^{-1} (X - Y)}$ With Σ^{-1} the matrix of covariance between X and Y.

But, for a more simple case (without dependence), this formula can be written as:

$$d_2(x_1,x_2) = \sqrt{\left(\frac{(x11 - x12)}{\sigma 1}\right)^2 + \left(\frac{(x21 - x22)}{\sigma 2}\right)^2} \text{ or } d_e(X_1, X_2) = \sqrt{(X_1 - X_2)^T S^{-1} (X_1 - X_2)},$$

with S^{-1} the diagonal matrix with the standard deviation of variables X, Y.

It is interesting to see that the importance of the variable (to calculate distance) is dependent on the deviation value (the bigger the deviation the smaller the importance).This shows that the importance of the variable is not dependent on the variable itself but just on the level of certainty of the variable. However, is this statement always true?

In the field of botany, there exists another beautiful formula to measure the concept of similarity among species, this time coming from the Set Theory domain. This is the “Jaccard index of similarity” developed by Paul Jaccard (1868-1944)and states in a very simple way that the similarity of two sets of objects is given by its ratio of intersection and union, that is: J = A∩B /(A∪B), which can be written as: (ΣMin(A,B)) /(ΣMax(A,B)) considering that the minimum quantity of elements present simultaneously in two sets is given by its intersection and the maximum by its union (Jaccard, 1901).An approximate vector expression of the Jaccard index (using the dot product expression) can be written as: J = (A•B) / (A•A + B•B – A•B) considering that the dot product represents the intersection of two sets (vectors) A and B. If A and B are parallel vectors, then there is a total intersection, and when they are perpendicular vectors there is null intersection. The subtraction in the denominator is to avoid the double counting of elements. Thus, the intersection is a way to measure the degree of projection that two vectors may have. However, what happens when the elements in the set have different importance or weight? This could be a relevant issue as will be shown (working within weighted environments).

We discuss these approaches since factors such as weight and dependence are at the base of the structure of the AHP and ANP (Saaty, 2001, 2010). However, instead of having to understand and deal with probabilities and statistics (which by the way are not easy to build and later interpret),the idea here is to apply the natural way of thinking which is based more on priorities than in probabilities. Indeed, we can manage the same information in a more comprehensive, complete and easy to explain form by combining AHP/ANP with the compatibility index *G* and working with priorities. This avoids the need to create a big database or have to understand and interpret complex statistical functions. It is important to remember that priorities can include probabilities, but not vice-versa.

Therefore, the MCDM approach using the AHP/ANP method provides a very nice tool for our investigation and treatment of the knowledge and experience that experts possess in their different fields, and at the same time stays within the decision making domain (order topology domain) which avoids the need for a huge and costly database where knowledge about the individual behavior can be lost or misinterpreted.

3. Hypotheses/objectives

In order topology, measurement deals with dominance between preferences (intensity of preference). For instance: D(*a*,*b*)=3, means that dominance or intensity of preference of "*a*" over "*b*" is equal to 3, or that, *a* is 3 times more preferred than *b*. When talking about preferences a relative absolute ratio scale is applied. The term relative is used because priority is a number created as a proportion of a total (percent or relative to the total) and has no need for an origin or predefined zero in the scale. The term absolute is used because it has no dimension since it is a relationship between two numbers of the same scale leaving the final number with no unit. The term ratio is used because it is built in a proportional type of scale (6kg/3kg=2) (Garuti, 2012). So, making a general analogy between the two topologies, one might say that: "Metric topology is to distance as Order topology is to intensity" (Garuti, 2012; 2014)

An equivalent concept of distance is presented in order to make a parallel between the three properties of distance of metric topology (Garuti, 2014; 2012). This is applied in the order topology domain, considering a compatibility function (Equation 1) similar to distance function, but over vectors instead of real numbers.

Consideration: A, B, C are priority vectors of positive coordinates and $\Sigma_i a_i = \Sigma_i b_i = \Sigma_i c_i = 1$.

G(A,B) is the compatibility function expressed as:

$$G(A,B) = ½\sum\left((ai+bi)\frac{Min(ai,bi)}{Max(ai,bi)}\right) \qquad (1a)$$

When working in distributive or relative mode, or the priority vectors A, B comes from an unknown model.

When the model is known (its means the vector W with the criteria weights is known) and we are working in the ideal or absolute mode of measurement (the rating mode), using rating scales instead of comparing the alternatives, then *G* takes the form:

$$G(A,B) = \sum\left((wi)\frac{Min(ai,bi)}{Max(ai,bi)}\right) \qquad (1b)$$

With w_i the global weight of criterion i. And (a_i,b_i) the local values of the profiles A and B.

G function presents the following properties: (Garuti, 2014, 2012)

1. $0 \leq G(A,B) \leq 1$ (Non negative real number)

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The compatibility function G, returns a non-negative real number that falls in the 0 - 1 range. With G(A,B)=0, if A and B are perpendicular vectors (A⊥B), and represent the definition of total incompatibility between priority vectors A, B. (A∘B=0).
Also, G(A,B)=1, if A and B are parallel vectors, (A=B for normalized vectors), and represent the definition of total compatibility between priority vectors A and B. (A∘B=1)
2. G(A,B) = G(B,A)(Symmetry)
Symmetry condition, the compatibility measured from A to B is equal to the compatibility measured from B to A. This is easy to prove, just by interchanging A for B and B for A in the compatibility function G.
3. G(A,B) + G(B,C) ≥ G(A,C) (Triangular inequality)
4. If A***C***B and B***C***C =><=A***C***C (Non transitivity of compatibility)
If A is compatible with B and B compatible with C, this does not imply that A is necessarily compatible with C.

For property 3, it is easy to prove that if A, B and C are compatible priority vectors (i.e. $0.9 \le G_i \le 1.0$ for A, B, C), then property 3 is always satisfied. But, this property is also satisfied for the more relaxed (and interesting) condition where only two of the three vectors are compatible. For instance, if A is compatible with B (G(A,B)≥0.9) and A is compatible with C (G(A,C)≥0.9), or some other combination of A, B and C, then condition 3 is also satisfied. This more relaxed condition allows compatible and non-compatible vectors to be combined while property 3 is still satisfied.

This situation can be geometrically viewed in Figure 1.

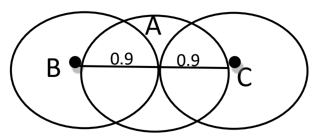

Figure1. Maximum circle of compatibility for position A, related to B and C (Garuti, 2012)

Figure 1 shows the compatibility neighborhood for A, in relation with B and C, with its minimum compatibility value of 0.9 represented by the radius of the circle (in the center the compatibility reaches its maximum value of 1.0). Thus, G(A,B)=G(A,C)=0.9 represents the minimum compatibility point, or the maximum distance for positions B and C to still be compatible with position A. Of course, G(B,C)<0.9 represents a non-compatible position for points B and C. Notice that with property 3,G(A,B) + G(B,C) ≥ G(A,C) is still valid, indeed any combination that one can make will keep the inequality satisfied since if C gets closer to A (increasing the right side of the equation), then G(B,C) will also grow. The extreme case is when C is over A, (G(A,C)=1.0) then G(B,A)+G(B,C)=0.9+0.9=1.8>1.0, keeps the inequality satisfied (Garuti, 2012). We may also define the incompatibility function as the arithmetic complement of the compatibility where Incompatibility=1–Compatibility. Thus, incompatibility is equivalent to (1–G). The incompatibility concept is closer to the idea of distance since the greater the distance the greater the incompatibility (Saaty, 2010; Garuti, 2014, 2012).

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4. Generating a threshold for compatibility index *G*

To answer the question, "When does close really mean close?", it is necessary to first have a reliable index of compatibility. However, that is not sufficient as it is also necessary to have a second condition which is a limit or threshold for the index. It is necessary to have a limiting lower value (minimum threshold) to indicate when two priority vectors are compatible or close to being compatible in order to precisely define when close really means close.

We have four different ways to define a minimum threshold for compatibility (Garuti, 2012):

1) Considering that compatibility ranges between 0-100% (0<□Cosα<1) with 100% being a case of total compatibility (represented by parallel vectors), it is reasonable to define a value of 10% of tolerance (1/10th of 100%) as a maximum threshold of incompatibility to consider two vectors as compatible vectors (which means a minimum of 90% of compatibility to consider two vectors compatible).This explanation is based on the idea of one order of magnitude for an admissible perturbation for measurement. This lower bound is also based on the accepted 10% used in AHP for the consistency index. In the comparison matrix of the AHP, the 10% limit of tolerance inconsistency comes from the consistency ratio (CR) obtained from the consistency index compared to a random index (CR=CI/RI), that in general it has to be less than10%. This says that the farther the CI is from the RI (random index response) the better the CR is. It's interesting to recall that CR is built as a comparison from the statistical analysis of RI (this idea will be reviewed in the last case analysis).
2) The compatibility index is related to a topological analysis since compatibility is related to the measure of closeness in weighted environments (weighted spaces). In Figure 2 below a sequence of 2, 3, 4 and 5 dimension vectors is presented. The first or initial vector is obtained as an isotropic space situation, that is, with equal values (1/*n*) in each coordinate (no privileged direction in the space); the second one is a vector obtained by perturbing (adding or subtracting) 10% on each coordinate, creating "small crisps" or little privileged directions, then the incompatibility index is calculated with the 5 different formulas(all formulae can be used because it is a near flat space, no singularities, where every formula works relatively well).

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	Case Sensitization for 2-3-4 and 5D Homogeneous Vectors (perturbing flat space to near flat space)									
DIM	Coordinates Perturbing 10% the initial vector of coordinates and normalizing					(Saaty's Index) (%)	*G* *(%)*	Hilber (%)	IVP (%)	Norm 1 (%)
2D	0,500	0,500	*Initial*			1,010	9,523	8,715	1,01	5,00
	0,450	0,550	*Pertur*							
3D	0,333	0,333	0,333	*Initial*		0,898	8,192	8,715	0,89	4,30
	0,354	0,290	0,354	*Pertur*						
4D	0,250	0,250	0,250	0,250	*Initial*	0,910	9,523	8,715	1,01	5,00
	0,275	0,225	0,275	0,225	*Pertur*					
5D	0,200	0,200	0,200	0,200	0,200	0,969	8,964	8,715	0,97	4,70
	0,215	0,176	0,215	0,176	0,215					

Figure 2. Defining a possible threshold of 10% for *G* function

When looking at the outputs for incompatibilities, it is possible to observe a good response for everyone (equal or less than 10%), with *G* and Norm1 circa 10% and 5% as the upper bound in every case.

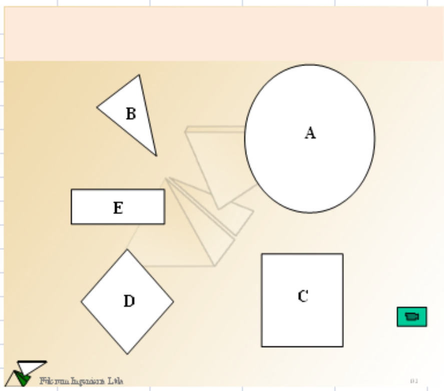
3) In the Figure 3 below, a simple test was run over an Excel spreadsheet using the common area example of AHP where the result (the importance of the area of the figures) can be calculated precisely with the typical geometric formulas and then its values normalized to obtain the exact priorities as a function of the size of their areas. It is possible to have a reference point of the element values (the right coordinates for the actual area vector) by doing it this way. The next step is perturbing the actual area values by +/- 10% producing a new vector of areas. Finally, the *G* function is applied over these two vectors (actual and perturbed) to measure their compatibility, obtaining a value of 91.92% (or 8.08% of incompatibility).This result is very close to the standard error deviation calculated as: Σ□*Abs*(perturbed-actual)/actual=10%. This is showing that 90% might represent a good threshold, considering that the difference between both outputs is related with the significant fact that these numbers are not just numbers but weights.

AREA EXAMPLE

	Perturbation 10% up&down	Normalized	Actual Value	Error Deviation	
Circle [A]	0,5225	0,505	0,475	10,00%	0,0475
Square [C]	0,2088	0,202	0,232	10,00%	0,0151
Diamond [D]	0,1661	0,160	0,151	10,00%	0,0093
Rectangle [E]	0,0837	0,081	0,093	10,00%	0,0049
Triangle [B]	0,0539	0,052	0,049	10,00%	0,0232
	1,035	1,000	1		10,00%

INDEX OF COMPATIBILITY (*G*)= 91,92%

INDEX OF INCOMPATIBILITY= 8,08% Compatible



FULL RANDOM AREA

Random values	Normalized	
32,41%	15,29%	
46,31%	21,84%	
28,38%	13,39%	
82,37%	38,85%	
22,55%	10,64%	
2,1202	1,0000	
G (random)=	53,27%	
Incompatibility=	46,73%	>10%

SORTED RANDOM AREA

Random with Order	Normalized
0,700	59,83%
0,210	17,95%
0,150	12,82%
0,060	5,13%
0,050	4,27%
1,1700	1,0000
G (random)=	78,35%
Incompatibil	21,65%

	Compatibility Without Order	Compatibility Keeping Order	N° Experiment
	33,57%	75,25%	1
	65,71%	58,08%	2
	64,54%	70,68%	3
	48,06%	85,37%	4
	52,57%	58,88%	5
	45,54%	82,08%	6
	50,62%	84,40%	7
	48,63%	91,98%	8
	28,44%	86,40%	9
	44,51%	74,99%	10
	59,94%	78,88%	11
	35,71%	87,95%	12
	70,23%	75,33%	13
	43,56%	75,79%	14
	61,07%	77,77%	15
Average	50,18%	77,59%	

Conclusions:

If the average of compatibilities generated by a random function is about 50%, and 78% when considering the order of the figures, then the limit of 90% seems to be a good threshold.

Figure 3. Possible threshold of 10% for *G* function

4) The last way to analyze the correctness of calculating 90% for a threshold involves working with a random function and filling the area vector with random values and calculating *G* for every case. The goal is to generate an average *G* for the case of full random values for the areas (full random means without any previous order among the areas, like figure A is clearly bigger than figure B, and so on), and again producing random values but this time keeping the correct order among the figures (imitating the behavior of a rational DM), then once again generating an average *G* for this case, and comparing both results against actual values.

The average value of *G* for 15 tries in the first case (keeping no order) was around 50% compatibility and 78% for the second case (keeping the order among the 5 figures).Both results show that a limit of 90% might be a good threshold. In the first case, the ratio between threshold and the full random *G* is almost double 1.8 (0.90 over .50), keeping the 0.90 compatibility threshold far from random responses.

In the second case (threshold over sorted figures), the ratio is much closer (as expected) with a value of 1.16 (0.90 over 0.78). This indicates that a previous order may help improve compatibility. This however, is not enough, one needs to consider the weight (not just the preference but the intensity of the preference) which is related to the value of the elements that belong to the vector, as well as the angles of both vectors point to point (geometrically viewed as profiles).

Of course, this test should be carried out for a larger number of experiments in order to have a more reliable response. A second test conducted for 225 experiments (15 people conducting 15 experiments each), showed more or less the same initial results for an average *G* value in both cases, with and without order (±0.78 & ±0.50). Next, Figure 4 shows the meaning of ranges of compatibility in terms of index *G* and its description.

Degree of Compatibility	Compatibility value range (G%)	Description
Very High	≥ 90%	Very high compatibility Compatibility at cardinal level of measurement (Totally compatible)
High	85 – 89.9	High Compatibility (Almost totally compatible)
Moderate	75 – 84.9	Moderate compatibility (compatibility at ordinal level)
Low	65 – 74.9	Low level of compatibility
Very Low	60 – 64.9	very low compatibility (Almost incompatible)
Null (random)	<60%	Random compatibility (Totally incompatible)

Figure 4. Ranges and meaning of compatibilities

Another interesting way to illustrate 90% as a good threshold for compatibility is the pattern recognition issue. Compatibility is the way to measure if a set of data (vector of priorities or profile of behavior) corresponds to a recognized pattern. For instance, in a medical pattern recognition application, the diagnosis profile (the pattern) is built with the intensity values of signs and symptoms that correctly describe the disease, and is then compared with the signs and symptoms gathered from the patient. When these two profiles match at 90% or more, then the physician can be confident that the patient has the disease. When the profiles match at 85-90%, the physician in general agreed with the diagnoses offered by the software, but when the *G* value was below 85% (between 79-84%), the doctor sometimes had trouble discerning if the new signs and symptoms (the new patient's profile) were corresponding to the disease initially presented (non-conclusive information).Finally, when the matching value (the *G* index) was below 75%, the physician wasn't able to clearly recognize in the patient's profile the disease initially offered. Note that the new profiles were built artificially by changing some values of signs and symptoms in an imaginary patient profile. This was done in order to achieve values of 90, 85, 80% and so on with the intention of evaluating when a doctor changed his perception (mostly based on his pattern recognition ability).

Thus, two vectors may be considered compatible with great certainty or confidence (similar or matching patterns) when *G* is greater or equal to 90%. Also, values between

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85-90% in general have a good chance of being correct (have a good level of certainty or approximation).

5. Three simple applications of compatibility index *G*

5.1 Example 1: Is the order of choice a must?

It used to be said that under the same decision problem two compatible people should make similar decisions. But, what do we mean when we say, "two compatible people should make similar decisions" (Garuti, 2014; Garuti, 2007)? Does it mean that they should make the same choice?

Consider to the following case:
Two candidates: A, B are up for an election;
Three people, P1: choose candidate A;
P2 & P3: choose candidate B.

P1 and P2 are moderate people, thus their intensity of preference for the candidates are: for P1: 55-45, for candidate A, and P2: 45-55 for candidate B.
On the other hand, P3 is an extreme person, thus his intensity of preference is 5-95 for candidate B.
Is P3 really more compatible with P2 than P1 just because P3 made the same choice of P2? (Both have the same order of choice of voting for candidate B). It seems that the order of choice is not the complete or final answer. On the other hand, we know that in order topology, a metric of decision means intensity of choice *(*dominance of A over B).So, compatibility is not only related to the simple order of choice, but is something more complex and systemic, it is related to the intensity of choice.

The next numerical example is shown in Figure 5 below. Suppose three people have an equal and different order of choice and its related priority vectors.

Person 1 (P1)		Person 2 (P2)		Person 3 (P3)	
Order of Choice	Intensity of Choice	Order of Choice	Intensity of Choice	Order of Choice	Intensity of Choice
1º	0.364	3º	0.310	º1	0.501
2º	0.325	2º	0.325	2º	0.325
3º	0.311	1º	0.365	3º	0.174
		Order totally Inverted with P1		Order equally with P1	

Figure 5. Comparing intensities and order of choice of 3 people

As seen in Figure 5, the order of P1 is the same as the order of P3 and the inverse of P2.
Order(P1) = Order(P3) ≠ Order(P2) (inverse order actually).

Considering just the above information, may we say that P1-P3 is closer than P1-P2?

Calculating *G*, for both combinations P1-P3 and P1-P2 we found:

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G(P1;P2)= 0.9 (≥90%), which implies that P1 and P2 are compatible choices (very high compatibility)
G(P1;P3)= 0.77 (<90%), which implies that P1 and P2 are non-compatible choices (moderate to low).

This is a very interesting result, considering that P2 has a totally inverted order of choice compared with P1yet, they are compatible people. On the other hand, P1 and P3, which have the same order of choice, are not compatible people. Hence, it is very important to be able to measure the degree of compatibility (alignment) in a reliable way.

Alice in Wonderland of Charles Lutwidge Dodgson (Lewis Carroll) has a very interesting and pertinent phrase saying: "I tell you, sometimes 1-2-3 might look more like 3-2-1 than 1-2-3" (Garuti, 2014; Garuti, 2007).

5.2 Example 2: Mixing consistency and compatibility indices in a metric quality test drive

A different and interesting application of *G* is possible when it is used to check the quality of a metric. When it is possible to compare a metric obtained with some method with the expected or actual metric, then the compatibility index *G* represents a great tool to test and verify the quality of the created metric. Suppose for instance, we want to measure the quality of the following simple example.

5.2.1 Presenting the problem (the criticism):

We will set a hypothetical problem (a criticism made by some person), about the quality of the consistency index in pair comparison matrices (Saaty's Index) (Saaty, 2001, 2010).The hypothetical critic says that the consistency index (Saaty's index) is wrong since it may over look some values (comparisons) that are not acceptable by common sense. To illustrate this criticism, the following simple example is presented. Suppose there are three bars of equal length like in Figure6a.

Figure 6a. Bar length

Of course, the correct matrix comparison for this situation is the following (consistent) comparison matrix.

	A	B	C	
	1	1	1	A
A=	1	1	1	B
	1	1	1	C
w=	0,3333	0,3333	0,3333	

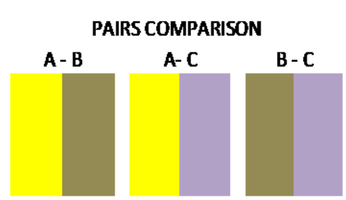

Figure 6b. Bar comparisons case 1

The obvious (correct) priority vector "w" is: 1/3, 1/3, 1/3, with 100% of consistency (CR=0).

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Suppose now that (due to some visualization mistake), the new appreciation about the bars is shown in Figure 6c.

	A	B	C	
	1	1	2	A
A=	1	1	1	B
	1	1	1	C
w*=	0,4126	0,3275	0,2599	

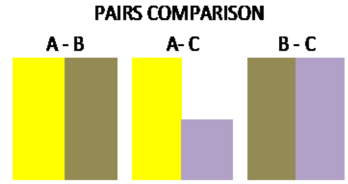

Figure 6c. Bar comparisons case 2

The new (perturbed) priority vector w* is 0.4126, 0.3275, 0,2599, with CR=0,05 (95% of consistency) which according to the theory is the maximum acceptable CR for a 3x3 comparison matrix.

The critic claims that the A-C bar comparison has 100% difference (100% of error) which is not an acceptable/tolerable error (easy to see even with the naked eye). Also, the global error in the priority vectors is15.85%, calculated with the common formula: e=abs(u-v)/v, for each coordinate and then the average of the coordinates is taken. But, Saaty's consistency index says that (CR=95%) which is tolerable for a 3x3 comparison matrix. Hence, the critic claims that Saaty's consistency index is wrong.

5.2.2 The response:

The already described problem has at least two big areas of misunderstanding. First, the CR (the Saaty's index of consistency) comes from the eigenvalue-eigenvector problem, so it is a systemic approach, this means that it is not worried about any specific comparison when building the corresponding metric (Saaty, 2001, 2010). Second, the possible error should be measured by its final result (the resulting metric), not in the prior or middle steps.

5.2.3 The explanation:

The first misunderstanding is explained by itself. For the second one, before any calculation, we need to understand what kind of numbers we are dealing with (in what environment we are working), because is not the same to be close to a big priority as to a small one. This is a weighted environment and the measure of the closeness has to consider this situation. We must work in the order topology domain to correctly measure the proximity in this environment. To do this correctly, two aspects of the information must to be considered. First, the intensity(the weight or priority)and then the degree of deviation between the priority vectors (the projection between the vectors).The index that incorporates these two factors simultaneously is the compatibility index *G*. Summarizing, the vectors of correct and perturbed metric are as follows:

Correct metric (priority vector) :	0.3333	0.3333	0.3333
Perturbed or approximated metric (priority vector) :	0.4126	0.3275	0.2599

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The basic question here is, "how close is the approximated metric to the correct metric?" Evaluating *G*(Correct-Perturbed), the *G* value that is obtained is 85.72%, which in numerical terms represents almost compatible metrics (high compatibility).As explained previously, *G*=90% is a threshold where two priority vectors are considered compatible vectors. Also, *G*=85% is an acceptable lower limit value (high compatibility). Hence**,** the two metrics are relatively close (close enough considering they are not physical measures).

Of course, better consistency can be achieved. The question is, "do we really obtain a better result when being totally consistent?" The answer is, probably no. In real problems, we never have the "real" answer (the true metric to contrast), and experience shows that pursuing consistent metrics *per se*, may provide less sustained results. For instance, in the presented problem one could answer that: A-B=2, A-C=2, B-C=1 and he/she would be totally consistent, but consistently wrong. In this case, the new priority vector would be: w**= (0.5, 0.25, 0.25), with CR=0 (totally consistent), and G= 71.5% which means incompatible vectors (low compatibility). Thus, a totally consistent metric is incompatible with the correct result. So, at the end it is better to be approximately correct than consistently wrong. The consistency index is just a thermometer not a goal[1].

5.3 Example 3: Using compatibility index *G* to measure the comparability of two different metrics

A third example is presented when you have two rules of measurement in the same structure of decision (two different value systems).The problem is then how to combine and or compare the outcome of both.

The first option is to try to reach a verbal consensus (the verbal or psychological option).The second is to use the geometric mean (the numerical or statistical option), and the third option is to measure the compatibility (the topological or closeness option).

The first two options have long been known and are described in the literature (Saaty, 2001, 2010; Garuti, 2014, 2012). The third option is based on the compatibility principle and presents the following five advantages: (Garuti, 2014)

1. Able to find out if the output of one rule is comparable with the other(compatible rules)
2. Able to find out the closeness between the 2 rules (how comparable they are?)
3. Able to find out the criteria responsible for the possible gap (where to act in the most efficient way?)
4. Able to find out the closeness between the initial personal rules with the final group rule. Assessing: G(P1,GM)
5. Able to use the numerical option (geometric mean) when and where it is necessary (modifying values in the places where most necessary in terms of efficiency).

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[1]CR=0.05 is the maximum acceptable value for inconsistency thus, I cannot go further with the bar comparison number (that means I cannot put a 3 instead of 2 in the cell (1,3)).
If metric B is compatible with metric A, then it is possible to use metric B as a good approximation of A. This is a useful property when metric A is not available (most of the time we don't know the correct/exact metric).
The same exercise was performed from 4x4 to 9x9 matrices that is, putting a value (n-1) in the cell-position: (1,n), (n= matrix dimension), obtaining similar results (sometimes even better).

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For a better explanation, we will use an example to illustrate this idea in detail. Suppose a mine company needs to change its shift-work system from the actual 6x1x2x3 (family of shifts, 8 working hours), to the new (desired) shiftwork system 4x4(family of shifts, 12 working hours).6x1x2x3 means a shift of 6 days of work 1 day off, 6 days of work 2 out and 6 days of work 3 out, with 8 hours per day, every week. This nets a total of144 working hours in 24 days of each working cycle. 4x4 means: 4 days of work followed by 4 days off with 12 hours per day every week. This also nets a total of 144 working hours in 24 days of each working cycle.

The question is, "are these shifts equivalent? How do we know what shift-work is better (or less risky, since any shift is bad in essence)?"Even if the total labor hours are the same (144 working hours), the shiftwork are not equivalent (in terms of quality of life and production). This depends on a variety of interdependent variables (number of working hours per day, entry time, number of free days per year, number of complete weekends per year, number of nights per shift, number of changes day/night/day per cycle, number of sleeping hours, opportunity for sleep, among many others. It also depends on how those variables are settled down and, of course, the weight (the importance that each variable has), which in time depends on which people you ask(workers, managers, stakeholders, owners, family or even the people that live around the mine).

Suppose we have two evaluation scenarios. In the first evaluation scenario, the decision rule (DR) of measurement is built with the people that work in the 6x1x2x3 shiftwork. In the second evaluation scenario, the DR of measurement is built with the people that work in the 4x4 shift. In each scenario the workers in that particular shiftwork weight the variables involved in the rule of measurement of their shift, since they know their own shiftwork best. As the process concludes, we end up with two different outputs. For the 6x1x2x3 shift (using the first rule of measurement), we have an impact index of 0.33, while for the second shift with the second rule, we have an impact index of: 0.37. Of course, one cannot just say that shift 6x1x2x3 is better than 4x4 because it has a lower number for the potential negative impact (0.33<0.37), since they were built with different rules of measurement. In the end, we have two DRs for the same problem; of course the question is not what rule is better, but how to make both DRs comparable.

One option could be to agree to use the same DR for both cases (the consensus, or verbal solution).However, this is not an option since the knowledge is located in different groups of people and is specific for each case; also the workers didn't feel comfortable making this agreement. Another option could be to take the geometric average (GM) of both rules and work with it as the final rule. Even if this was a possibility, we really don't know what we are doing when we combine or mix both rules(for example, we can't just combine one rule of measurement in meters with one rule in inches).We first need to know if both DRs are comparable. Figure 7 shows this comparison of two rules of measurement.

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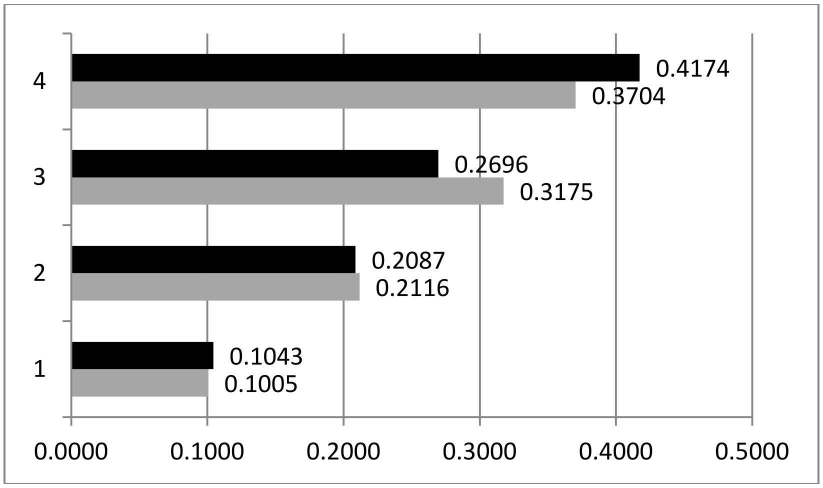

Figure 7.Comparing two rules of measurement

The problem statement can be stated as, "are these two sets of decision criteria (gray and black profiles in Figure 7) comparable (compatible)?"Recall that they represent two different DRs. The black bars show rule one which represents the rule of measurement for 6x1x2x3 shiftwork and is formed by 4 criteria (extracted from the global rule).The gray bars show rule two which represents the rule of measurement for 4x4 shiftwork and is formed by the same 4 criteria, but with different intensities. When we speak of comparable DR, we mean compatible DR (equivalents of measure), that is we can measure the level of risk of the alternatives with any of the two rules described above. To establish if both DRs are equivalent it is necessary to calculate *G* for both profiles:

G(Profile gray – Profile Black)= 0.91 (91%) $\geq$ 90%, which means they are compatible.

Thus, we can use any of the two DRs to measure the effect of the changing shiftwork. Moreover, we now may use the geometric mean (GM) of both DRs as the final rule, but knowing that we are combining rules that are compatible in fact, we are not mixing two far away points of view. This relevant issue is helpful when working with different groups of decision makers.
But, now how are we to make both DRs comparable when they are not compatible? Or what rule is used to measure the alternatives? The following cases make some light on this.

5.3.1 Case to case DR analysis

There are four different cases of compatibility for DR where *G* can be applied:

Case 1: This has already been described where the DRs of each decision making group are compatible. In this case, it is possible to use any of the two rules or (still better) use the geometric mean of both DRs. Figure 8a shows this graphically

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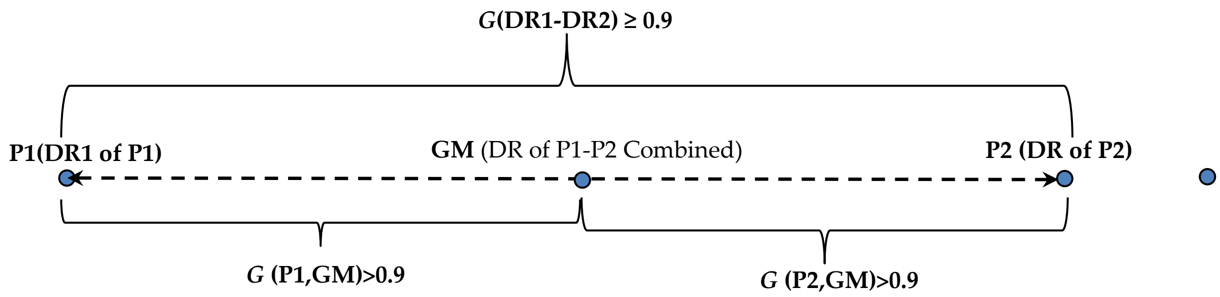

Figure 8a. Compatibility as a concept of distance

Analytically, *G*(DR1,GM) and *G*(DR2,GM) >*G*(DR1,DR2) > 0.90. Thus, the GM rule is clearly (numerically) better than any of the other two. When measuring the alternatives with this final DR, make the results comparable.

Case 2: The DRs of either people (or groups) are not comparable (compatible), but they are compatible with the GM. In this case, take the GM of both rules, and then measure the compatibility of each DR with regards to the GM rule. If both initial DRs are compatible with the GM rule, then you may use the GM rule as the final rule. Figure 8b shows this graphically.

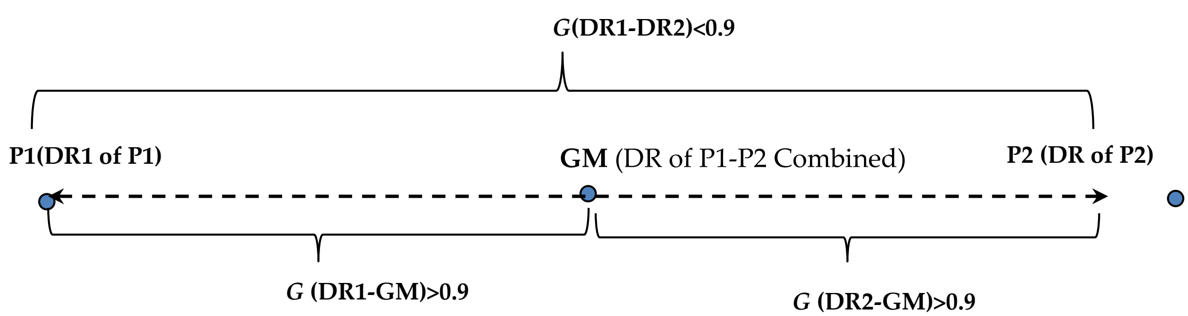

Figure 8b. Compatibility as a concept of distance

Analytically, *G*(DR1,GM) and *G*(DR2,GM)≥0.9>*G*(DR1,DR2). When measuring the alternatives with this final DR, make the results to be comparable.

Case 3: The DRs of both people are not compatible, but one (P1) is compatible with the GM. In this case, look in the compatibility profile of the GM with P2,*G*(GM;P2) for the position with the largest weighted difference and proceed as follows:
1) Check if there is any "entry" error in the calculation process of P2 profile (some large inconsistency, or inverse entry in the comparison matrix).
2) Check if the comparisons in the matrix associated to that position are what P2 really meant to say.
3) Suggest to P2 to test acceptable numbers that produce a bigger *G* (getting closer to GM rule), until you can fall in Case 2.This is shown graphically in Figure 8c.

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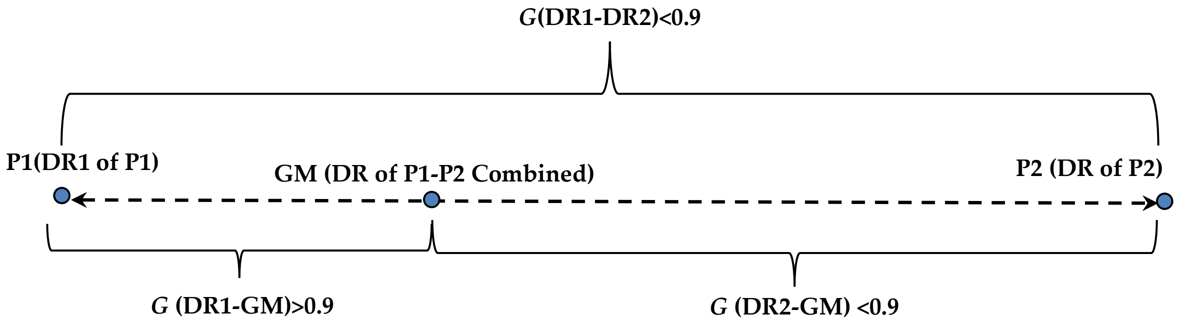

Figure 8c. Compatibility as a concept of distance

Analytically, *G*(GM,P1)>0.9>*G*(P1,P2);*G*(GM,P2)>G(P1,P2)<0.9. When measuring the alternatives with this final DR, make the results to be comparable.

There is also a fourth case, in which the initial DRs are not compatible and none of the DRs are compatible with GM rule. This is the toughest case, since P1 and P2 have very different points of view.
The suggestion for this case is as follows:

- Try to revise the structure of the model to find some lost criterion or border condition not considered.
- The weights of the criteria have to be revisited, and the support information of P1 and P2's opinions revised. Pay special attention to the elements (criteria) where *G* presents a big difference between P1 and P2. Try to negotiate (get closer) on these criteria first. This can be done by changing the weights (plausible changes) of the criteria, the alternatives or both.

If there is no changes with the initial position you may (as a last resource) apply the GM as the final rule, but it is probable that both people (or groups) may feel not fully represented by that imposed DR.

5.3.2 Example of case to case DR analysis:

One, among many, interesting applications of the case discussed previously (case to case DR analysis) is the next extracted example formed by 20 decision makers (DMs). The problem is how to cluster them in the best way in terms of similarity (Meixner et. al, 2016). In brief, the problem consists of 20 priority vectors which represent 20 DMs, of 5 cardinal ranked elements that need to be clustered in the best possible way, in terms of their closeness among each other.

To do this task, we first make the compatibility matrix of the 20 DMs to measure their specific closeness (DM(i) to DM(j)). In doing this, we discovered (once again) the importance of the concept of "weighted environment" when measuring closeness, since the results obtained with *G* (in some specific cases) were very different from those obtained in the original study by Meixner et. al. (2016) using a variation of Norm2, specifically the Square Euclidean Distance. For instance, when searching the minimum distance between the DMs using *G*, the minimum distance was between (DM3-DM18), instead of DM5 with DM15 as in the Meixner et. al. (2016) study which used the

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Euclidean distance. In this study, the distance between (DM3-DM18) is 6 times (600%) bigger (far) than (DM5-DM15) which is a very large difference, *G* shows only 2% (1.02).

The maximum distance is found in (DM9-DM10) as the Meixner et. al (2016) study found, but the min/max ratio is almost 100 times; *G* shows less than 2,5 times for the same ratio. This last difference is due to *G* being based in an absolute ratio scale, not absolute differences as the Euclidean Norm does (ratio scale is in the core of weighted environment measurement and also in the decision making process). The Euclidean distance shows that DM15 is the most aligned (on average) to the rest of the DMs, and the worst is DM10. However, *G* shows that in spite of the fact that DM15 has a very good alignment, DM14 is the best aligned, (slightly better than DM15), and DM9 is the worst aligned, instead of DM10 which, by the way, is also out of line like DM9 (See Figure 9).

The question here is not just about the ranking order, but about the real intensity or "distance" between the DMs preferences measured by their ratio of preference. This measure is quite complicated to interpret if the difference between the ratios of both indices, for some cases, may reach the value of 100 times.

By the way, it is acceptable for the absolute numbers of both indices (*G* and Square Euclidean Distance) to differ even by a large gap. The large difference among its ratios however is not acceptable. This is the main problem of using Euclidean distance (or its derivatives) when working within weighted environments.

Ranking by Closeness to GM		
GM – DM(i)		Alignment Qualification
0,8767	**14**	V.good(Almostfull aligned)
0,8642	**20**	V.good(Almost full aligned)
0,8584	**15**	V.good(Almost full aligned)
0,8463	17	Good
0,8393	4	Good
0,8324	6	Good
0,8081	5	Good
0,8081	2	Good
0,8022	1	Good
0,7986	8	Good
0,7886	19	Regular
0,7816	12	Regular
0,7530	18	Regular
0,7521	13	Regular
0,7475	16	Poor
0,7087	11	Poor
0,7030	7	Poor
0,6132	3	Out of line (Totally misaligned)
0,6046	10	Out of line (Totally misaligned)
0,5929	9	Out of line (Totally misaligned)
0,5929	9	Min. value
0,8767	14	Max. value

Figure 9.Table with level of alignment between GM and DM(i)

The list is obtained by calculating the *G*(GM,DMi) and comparing the result with the different thresholds. The clusters were obtained by measuring the distance of each DM to the GM. This rapid (and easy) process of clustering was possible due to the fact that the *G* index has its own thresholds that make it possible to define where the break points are on the list of DMs as shown in Figure 8.

The two largest distances to the GM are DM9 and DM10, which coincide with the largest distance measured directly between them. The min/max ratio is found for DM9 with DM14 which also coincides with the largest ratio among DMs in the DMs matrix of compatibility, realizing that the clustering process shown in the last table is representative and captures the extreme cases measured for both situations: distance and ratio. With the *G* index the clustering process (using GM) is much easier and direct. For instance, the first cluster is shown in red (very good and good), the second in yellow (regular) and the third in green (bad).There is also a fourth category in black (out of line) which represents those cases totally out of line in terms of alignment. We can also qualify the entire group. For instance, it seems that this group of 20 people, with respect to the GM, to be a group with an important level of agreement. The largest average value is 87.7% which represents, according to *G*, an agreement of almost cardinal level of compatibility.

6. Conclusions

A general conclusion of this paper is that measurements of distance in the decision making domain have to be done with *G* instead of any Euclidean based indices if one wishes to be accurate. Also, *G* is a better index for distance calculation in weighted environments especially when analyzed by single elements. Another conclusion is that the indices based in Euclidean geometry (like square Euclidean distance, Euclidean distance or J) could work in a statistical way of analysis, that is, when analyzing with a global view a large set of elements, but they may present important troubles when being applied to individual's behavior (experiments carried out with those indices have shown some troubles, especially when using with weighted profiles of behavior).

There are several more specific conclusions that can be drawn from this discussion as well. First, we discovered the great importance of the concept of a weighted environment when measuring closeness. Thus, the compatibility index *G* is a necessary index for distance/alignment measurement in weighted environments (order topology domain) in order to correctly (mathematically correct) measure and declare if two profiles of behaviours are really close.

The *G* index makes the following possible:

- A matching analysis process
- Analysis and testing of the quality of the results
- The availability of one more tool for conflict resolution in group decision making to achieve a possible agreement considering that those profiles may represent system values
- A pattern recognition process (assessing how close one pattern is to another)
- Making better benchmarking
- Membership analysis (closeness analysis to estimate if an element belongs to one set of elements or another)

Secondly, this analysis shows that the only compatibility index that performs correctly for every case is the *G* index. This index always keeps the outcome in the 0-100% range, which is an important condition, since any value out of the 0-100% range would be difficult to interpret (and the beginning of a possible divergence). It is also important to note that the *G* and Euclidean outcomes can be close, but the *G* is much more accurate or sensitive to changes because it is not based on absolute differences (Δx_i) like distance, but on relative absolute ratio scales. We have to remember that we are working on ratio scales (absolute ratio scales to be precise). This is the same behaviour detected in the Garuti (2014) and Garuti (2012) where the Euclidean distance calculation shows no difference in the distance of the parallel trend from case 1 to 6. The Euclidean based index cannot detect a bit of difference in the compatibility value among those cases because the absolute difference of the coordinates remains the same. Therefore, with a Euclidian based index one may reach the wrong conclusion that no difference exists for vector compatibility from cases 1 to 6 (the first case study is as incompatible as 2, 3, 4, 5

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or 6), which is not an expected result. This unexpected behaviour occurs because the Euclidean norm is based on differences, and also because it's not concerned about the weights of the coordinates and the projection between the priority vectors. It is important to remember that the numbers inside the priority vector represent preferences. Hence, in terms of proximity it is better to be close to a big preference (big coordinate) than to the small one and this issue is better resolved in ratio scales[2]. Other tests performed in greater spaces (3D to 10D) show the same trend. The bigger the space dimension the greater the likelihood of finding singularity points for the other compatibility indices like Hadamard product (Saaty's compatibility index), inverted dot product, weighted dot product, and Hilbert's index. This problematic behavior occurs in both parallel and perpendicular trends[3].

It is interesting to note that function *G* is not the simple dot product since it depends on two different dimensional factors. On one side, you have the intensity of preference (related with the weight of the element), and on the other side you have the angle of projection between the vectors (the profiles). This means that *G* is a function based on the intensity of preference (I) and the angle of projection (α) between the priority vectors, that is: $G = f(I,\alpha)$. Clearly, the *G* function is not the simple dot product (as normally defined), but a combination of weight (intensity of preference) and projection (angle between vectors), that is, something more complex and rich with information.

It is also important to note that both data (intensity and angle) are normally implicit in the coordinates of the priority vectors and have to be correctly extracted. It also matters if the priorities are presented in relative measurement (RM) or absolute measurement (AM) format. There are huge possible applications of this index in different fields. As an example, Figure 9 lists the possibilities in the field of social and management sciences.

[2] See the complete example in Garuti (2014, 2012)
[3] Details in Garuti (2014, 2012)

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On Medicine	Measuring the degree of matching (proximity) between patient and disease diagnose profiles.
On Buyers-Seller profiles	Measuring the degree of matching between house buyers and sales project.
On Group Decision Making (Conflict Resolution)	Measuring how close are two (or more) different value systems (where they differ and for how much).
On Quality Tests	Measuring what MCDM decision method can builds a better metric.
On Agricultural Production& Supplier Selection	Measuring the proximity between the cultivate plants against a healthy plant (based on its micro & macro nutrients) and selecting the best nutrient seller.
On Shiftwork Prioritization	Measuring how close are the different views among the different stakeholders (Workers view, Company view, Community view).
On Company Social Responsibility (CSR)	Measuring how close are the different views among the different stakeholders (Economic, Environmental and Social view).

Figure 9. Possible applications of index *G*

All the examples mentioned above are from real cases of application.

There are many different applications for index *G*. This is a summary of all possibilities that *G* may have:

- **Compatibility of systems value:**

G is an index able to be used in social and management sciences to measure compatibility of group decision making (DMs) intra and inter groups. The expression of *G* for this case is: *G*(DM1-DM2), which means level of compatibility(closeness) between DM1 and DM2.With DM1 and DM2 being the decision's metric of each decision maker.

- **Compatibility for quality test:**

G can help assess the quality of a built decision metric. As presented in section7.2, *G* may help evaluate the quality of any new metric based in a ratio scale. The result is achieved by comparing the new metric with some standard or with an already known result.

- **Profiles alignment:**

G can help establish if two different profiles are aligned. In general, it is not easy to know if two complex profiles are aligned, especially when the profiles are complex with many

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variables, with different importance and different behavior on each one. This is the case when trying to measure the degree of matching between a medical diagnose and a list of diseases, or the degree of matching between a sale project and its possible buyers and many other similar cases.

- **Compatibility for comparability:**

G can help to establish if two different measures are or are not comparable. One relevant point when comparing numbers from different outcomes is to know if those numbers are comparable or not. For instance, if I know that the impact of strategy A is 0.3 and the impact of strategy B is 0.6, I cannot say that strategy A has twice the impact of strategy B, unless both strategies were measured with exactly the same rule. However, for many reasons, sometimes that is not possible. In that case, I need to know if the rules of measurement are compatible. If so, it is possible to compare both numbers, otherwise it is not possible.

- **Compatibility for sensitive analysis and threshold:**

G can help establish the degree of membership or the trend for membership (tendency) of an alternative. The idea is equivalent to the classic sensitive analysis when making small changes in the variables. The change resulting in the *G* value (before and after the sensitive analysis) would show where the alternative is more likely to belong (trend of belonging).

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