AN INTERPRETATION OF THE AHP EIGENVECTOR SOLUTION FOR THE LAY PERSON[1]

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ABSTRACT

An AHP priority vector represents the importance, preference, or likelihood of its elements with respect to a certain property or criterion and here we examine how that priority vector can be derived through an iterative process applied to the pairwise comparison matrix. Further, we show that the vector obtained in this way satisfies the definition for an eigenvector of the original judgment matrix. Practical managers using AHP in decision making would most likely be better able to appreciate this approach than they would a process phrased in the language of linear algebra. The overall priority vector for the alternatives in a hierarchy and, further, in a network, can be obtained in the same way by applying the iterative process to the supermatrix of the ANP. This claim is examined in depth in a forthcoming paper that will appear in this journal.

Keywords: Analytic Hierarchy Process, AHP, eigenvector
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"Everything is foreseen, yet freedom of choice is granted" - The Wisdom of the Fathers.

1. Introduction

The Analytic Hierarchy Process (AHP) and Analytic Network Process (ANP) are the methodologies and methods of multiple criteria decision making in complex hierarchies and networks, and were originated and developed by Thomas Saaty (1977, 1980, 1994, 1996, 2000, and 2010). The AHP and ANP have been expanded by numerous authors and widely used for solving various multi-criteria decision problems (Saaty and Kearns, 1985; Saaty and Vargas, 1994; Golden, Wasil, and Harker, 1989; Wasil and Golden, 2003). Some works contributed to AHP techniques in relation to optimizing procedures, group decisions, robust and statistical evaluations (for example, Lipovetsky and Tishler, 1999; Lipovetsky and Conklin, 2002; Lipovetsky, 1996, 2005, 2008, 2009a, b; and references within).

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This paper describes how to interpret AHP solutions. In AHP, the elements of the priority vector represent the mean values of the preference of each alternative over the others. In addition to using a regular synthesizing process to obtain the AHP global vector, it is possible to construct it by raising the supermatrix of the hierarchy to powers until it converges, or by finding the principal eigenvector of this matrix. In an ANP model with feedback, the solution can also be interpreted as a mean preference of alternatives weighted by the criteria, and vice versa – the means of the criteria weighted by the alternatives. The ANP solution can be obtained from the limit power of the supermatrix, or by solving the eigenproblem for this matrix. The approach suggested here offers a clear description of complex AHP/ANP interrelations and an explicit interpretation of the priority results for managerial decision making.

In this paper, we offer an interpretation of the priority vector for an AHP pairwise comparison matrix that supports the use of the principal eigenvector as a solution. In the next paper, Part II, which will appear in a later issue of this journal, we will show that similar reasoning can be applied to interpret the priority vectors that are either a global synthesis of the priorities of the alternatives in an AHP model or of an ANP model with feedback derived from the supermatrix.

2. The AHP solution and its interpretation

For each group of compared items, classic AHP considers judgments elicited from an expert arranged in a pairwise comparison matrix as shown below:

(1)
$$A = \begin{pmatrix} 1 & a_{12} & \ldots & a_{1n} \\ a_{21} & 1 & \ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{n1} & a_{n2} & \ldots & 1 \end{pmatrix},$$

where each element a_{ij} is the judge's estimate of the quotient of the i-th and j-th priorities, respectively. The solution of the eigenproblem (see Saaty, 1980, 1994, 1996, 2000, and 2010)

(2)
$$A\alpha = \lambda\alpha$$

for the maximum eigenvalue λ yields the principal eigenvector α which serves as the AHP estimate for the priorities. A numerical estimate of the principal eigenvector can be obtained by iteratively solving the equation:

(3)
$$A\alpha^{(t)} = \lambda^{(t+1)}\alpha^{(t+1)}, \quad \text{with } t = 1, 2, 3\ldots.$$

The initial vector $\alpha^{(1)}$ usually taken as the uniform vector *e* divided by *n*, so multiplying *A* times this initial vector in the first step with t = 1 yields the following:

(4)
$$A \cdot \frac{1}{n}\begin{pmatrix}1\\1\\.\\.\\1\end{pmatrix} = \begin{pmatrix}\frac{1}{n}\sum_{j=1}^{n} a_{1j}\\ \frac{1}{n}\sum_{j=1}^{n} a_{2j}\\ .\\ .\\ \frac{1}{n}\sum_{j=1}^{n} a_{nj}\end{pmatrix} = \begin{pmatrix}m_1\\m_2\\.\\.\\m_n\end{pmatrix},$$

where m_j is the mean of the elements in the *j*th row of the matrix *A*. The vector on the the right-hand side of equation (4) serves as the second approximation vector $\alpha^{(2)}$ in (3), and it is proportional to the simple means above:

(5)
$$\alpha^{(2)} = \left(m_1,\ m_2,...,m_n\right)',$$

where prime denotes transposition. Eigenvectors are usually normalized by the Euclidean metric:

(6)
$$\alpha'\alpha = \sum_{i=1}^{n} \alpha_i^2 = 1.$$

But AHP priority vectors are conventionally normalized so the total equals one:

(7)
$$\alpha' e = \sum_{i=1}^{n} \alpha_i = 1,$$

So the vector (5) becomes proportional to relative mean values,

(8)
$$\alpha^{(2)} = \left(s_1,\ s_2,..., s_n\right)',$$

defined as:

(9)
$$s_j = \frac{m_j}{\sum_{k=1}^{n} m_k}, \qquad \sum_{j=1}^{n} s_j = 1.$$

Using the vector in (8) in place of the initial uniform vector in (3) yields the next approximation (4) to the eigenvector which again equals the weighted mean value:

(10)
$$A\begin{pmatrix} s_1 \\ s_2 \\ . \\ . \\ s_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} s_j a_{1j} \\ \sum_{j=1}^{n} s_j a_{2j} \\ . \\ . \\ \sum_{j=1}^{n} s_j a_{nj} \end{pmatrix} = \begin{pmatrix} m_1 \\ m_2 \\ . \\ . \\ m_n \end{pmatrix}.$$

Now the new m_j are from the weighted means in the rows of the matrix A, and the weights are again proportional to the previous means in equation (9). This process continues until the solutions in two consecutive iterations coincide with the needed precision.

Thus, elements of an AHP eigenvector equal the weighted mean values of the elements of the matrix rows and the weights are proportional to these means themselves. Recalling that the elements in the i-th row of matrix (1) express the prevalence of the i-th item over each of the others, we get the conclusion that **the AHP eigenvector consists of weighted averages of the judgments in each row that express the i-th item's preference over all the others.**

A forthcoming paper will describe how the solution of an ANP supermatrix can be characterized in a similar way.

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