FIVE WAYS TO COMBINE TANGIBLES WITH INTANGIBLES

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ABSTRACT

This paper presents five different ways to establish weights for the criteria that govern making comparisons. Four of these can be done in the context of the AHP, but the fifth and most reliable one is obtained by using the ANP.

Keywords: AHP; ANP; decision making; MCDA; tangibles; intangibles

1. Introduction

In decision making there may well be a mix of tangible and intangible criteria in terms of which the alternatives are evaluated. In some cases one may wish to treat the tangible criteria in the same way as the intangible ones and use judgment to determine their relative importance. However, one often uses the data with known existing measurements to rank the alternatives on these criteria The question then is how to use these data alongside the priorities derived by using judgments for the remaining criteria that are intangible.

There are five different ways that govern making comparisons to establish weights for the criteria. Four of these can be done in the context of the AHP, but the fifth and most reliable one, is obtained by using the ANP.

1. The first is the simplest. When there are tangible measurements in the same unit under the criteria, and the criteria are thought to be as important as the measurements of the elements under them then each criterion inherits the proportion of the sum of the weights under it to the total weights under all the criteria that are evaluated using the same scale, such as money. This procedure is applied to all the criteria with a common measurement scale and the alternatives synthesized for each such scale into a single overall criterion such as an economic criterion with several sub-criteria measured in dollars, a technical criterion with sub-criteria measured in meters, and so on. These aggregate criteria are then compared pairwise as to their relative importance with respect to higher level criteria or goals.

2. The second is when the criteria have importance that is not equal based on experience and determined when they are pairwise compared with respect to higher level criteria or the goal. The resulting top-down priorities for the criteria are multiplied times their inherited proportional weight from the measures of the alternatives under them and the result is normalized to give the final priorities for the criteria. If new alternatives are added or subtracted the proportional priorities of the criteria need to be re-calculated and the multiplication and normalization process repeated.
3. The third way is when there are no measurements; here one simply uses judgments to compare the criteria with respect to higher criteria or the goal with respect to their importance. This is useful when standards are established because of long-term experience and applied to alternatives as they are evaluated.
4. The fourth way to evaluate the criteria is to assume there is inadequate knowledge about them and compare widely ranging alternatives under them and later compare the criteria with respect to higher level criteria or the goal having discovered more about their importance by exploring their occurrence in actual but diverse alternatives.
5. The fifth way is to assume that the criteria depend on the alternatives. There are two ways to include this information. In a straight AHP model, compare the alternatives under each criterion for preference first, and after this education implicitly include the knowledge gained in comparing the criteria with respect to the goal. By turning to the ANP, it is possible to compare the alternatives for preference with respect to the criteria, the usual AHP way, but also compare the criteria with respect to how important they are in each of the alternatives. Note that if the criteria have more importance than simply that obtained from data about the alternatives, they can be compared with respect to a goal as was done before, then multiply those priorities times the priorities obtained from the alternatives.

Note that if one were to use a formula from which the measurements of the alternatives are obtained, the criteria would inherit weights involving such a formula. Similarly, if one were to apply a formula to assign the criteria weights in terms of higher order criteria, then the same procedure would be used in weighting those higher level criteria. This process can be generalized to all the measurements in a hierarchy or a network.

2. Theory/calculation

Theorem: A necessary and sufficient condition that multiple criteria tangibles measured on the same scale satisfy the normalization condition on the priorities of intangibles is that the measurement of each alternative with respect to a criterion is weighted by the sum of the measurements of all the alternatives with respect to that criterion divided by the sum of the measurements with respect to all the tangible criteria that use measurements on that scale.

Proof: Let $M=(w_{ij}^{s})$, I =1,…, n; j=1,…,m; s=1,…,S be the matrix of measurements of tangible i with respect to criterion j with scale s. For simplicity assume that we have a single scale of measurement such as length in meters and thus we let s=1 and avoid using the superscript s. To combine the measurements of alternative for all the criteria for an alternative i, we form the sum $\sum_{j=1}^{m} w_{ij}$. We then normalize by dividing by each such sum by the total sum of all the measurements of the alternatives with respect to all the criteria obtaining $\sum_{j=1}^{m} w_{ij} / \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}$. Now assume that if instead of the forgoing we were to normalize the measurements of each alternative with respect to its criterion measurements we have the ratio $w_{ij} / \sum_{i=1}^{n} w_{ij}$ and if we add these ratios for all the criteria, we obtain $\sum_{j=1}^{n} w_{ij} / \sum_{i=1}^{n} w_{ij}$. Normalizing them for the alternatives we obtain

$$(\sum_{j=1}^{m} \frac{w_{ij}}{\sum_{i=1}^{n} w_{ij}}) / (\sum_{i=1}^{n} (\sum_{j=1}^{m} \frac{w_{ij}}{\sum_{h=1}^{n} w_{hj}})) = (\sum_{j=1}^{m} \frac{w_{ij}}{\sum_{i=1}^{n} w_{ij}}) / (\sum_{j=1}^{m} \frac{\sum_{i=1}^{n} w_{ij}}{\sum_{h=1}^{n} w_{hj}}) \neq \sum_{j=1}^{m} w_{ij} / \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}.$$

For the two results to coincide we must normalize each entry with respect to a criterion by multiplying $w_{hl} / \sum_{j=1}^{m} w_{hj}$ by $\sum_{k=1}^{m} w_{hk} / \sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}$ and then summing over l and we have:

$$\sum_{l=1}^{m} \left(\frac{w_{hl}}{\sum_{j=1}^{m} w_{hj}} \right) \left(\frac{\sum_{k=1}^{m} w_{hk}}{\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}} \right) = \frac{\sum_{l=1}^{m} w_{hl}}{\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}}$$

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We now prove the uniqueness of this solution. The supermatrix corresponding to the theorem is given by:

$$\begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & \frac{w_{11}}{\sum_{j=1}^{m} w_{1j}} & \frac{w_{21}}{\sum_{j=1}^{m} w_{2j}} & \cdots & \frac{w_{n1}}{\sum_{j=1}^{m} w_{nj}} \\ 0 & 0 & \cdots & 0 & \frac{w_{12}}{\sum_{j=1}^{m} w_{1j}} & \frac{w_{22}}{\sum_{j=1}^{m} w_{2j}} & \cdots & \frac{w_{n2}}{\sum_{j=1}^{m} w_{nj}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{w_{1m}}{\sum_{j=1}^{m} w_{1j}} & \frac{w_{2m}}{\sum_{j=1}^{m} w_{2j}} & \cdots & \frac{w_{nm}}{\sum_{j=1}^{m} w_{nj}} \\ \frac{w_{11}}{\sum_{i=1}^{n} w_{i1}} & \frac{w_{12}}{\sum_{i=1}^{n} w_{i2}} & \cdots & \frac{w_{1m}}{\sum_{i=1}^{n} w_{im}} & 0 & 0 & \cdots & 0 \\ \frac{w_{21}}{\sum_{i=1}^{n} w_{i1}} & \frac{w_{22}}{\sum_{i=1}^{n} w_{i2}} & \cdots & \frac{w_{2m}}{\sum_{i=1}^{n} w_{im}} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{w_{n1}}{\sum_{i=1}^{n} w_{i1}} & \frac{w_{n2}}{\sum_{i=1}^{n} w_{i2}} & \cdots & \frac{w_{nm}}{\sum_{i=1}^{n} w_{im}} & 0 & 0 & \cdots & 0 \end{pmatrix}$$

The entries of the block matrix Q at the top right are the normalized values of the *rows* of $M = (w_{ij}^{s})$ and the entries of the bottom left block P are the normalized values of the *columns* of $M = (w_{ij}^{s})$.

Let $$p = \left(\frac{\sum_{i=1}^{n} w_{i1}}{\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}}, \frac{\sum_{i=1}^{n} w_{i2}}{\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}}, \cdots, \frac{\sum_{i=1}^{n} w_{im}}{\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}} \right)^{T}$$ and

$$q = \left(\frac{\sum_{j=1}^{m} w_{1j}}{\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}}, \frac{\sum_{j=1}^{m} w_{2j}}{\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}}, \cdots, \frac{\sum_{j=1}^{m} w_{nj}}{\sum_{i=1}^{n}\sum_{j=1}^{m} w_{ij}} \right)^{T}$$ be the vectors of the normalized rows and

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columns of $M = (w_{ij}^{s})$, respectively. Note that $Pp = q$ and $Qq = p$. Thus, we have $PQq = q$ and $QPp = p$. The vector q is the vector of interest in the theorem. Since the matrices PQ and QP are stochastic the vectors q and p are their principal right eigenvectors and are thus unique to within a multiplicative constant.

The transformation of absolute numbers to relative numbers has little influence over how meaning is assigned to generate priorities on a relative scale whose ratios may not be the same as those of the corresponding absolute numbers. Priorities should not be combined with measurements unless they coincide with them, in which case no difficulties arise. However priorities based on information from different scales are a generalization that requires comparison of the criteria with respect to a higher criterion. For emphasis, note that after absolute numbers are converted to priorities, one cannot take the final scale and treat it arithmetically as if it is still the original scale of absolute numbers.

3. Examples of the five ways to combine

We now illustrate, with examples, how to deal with each of the five ways to combine criteria mentioned above.

3.1 Tangibles as intangibles—first case

Suppose we wish to determine the best value for three vacation sites A, B, C in terms of total travel plus lodging cost as shown in Table 1.

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Table 1
Choosing the best vacation site

Alternatives a_i	Criteria		Travel + Lodging Cost	Priorities (a_i Cost/Total Cost)
	C_1 Travel Costs ($)	C_2 Lodging Costs ($)		
A	50	200	250	0.278
B	100	170	270	0.333
C	150	230	380	0.422
		Total	900	

Site A would be the preferred site as it has the least travel plus lodging cost. Dividing the cost of each alternative by the total cost of $900 yields the fraction of the cost due to the alternative. Or, to put it another way, it is the relative cost of the alternative.

The total of the travel costs, criterion is 50 + 100+ 250, or $300 and the total of the lodging costs, criterion 2, is 200 + 170 + 230, or $600. Thus, travel costs are 300/900 or 1/3 and lodging costs are 600/900 or 2/3 of the measurements in money. These proportions are the priorities of the criteria and sum to 1.0.

The same problem can be studied with a hierarchic interpretation where the relative costs or priorities of travel and lodging for each alternative are multiplied by the weights of the criteria (the total costs of travel and lodging) and summed. We obtain the following results which are the same as the second column on the right in Table 1:

$$\begin{aligned}
\text{Cost A} &= 300 \times \left(\frac{50}{300}\right) + 600 \times \left(\frac{200}{600}\right) = \$250 \\
\text{Cost B} &= 300 \times \left(\frac{100}{300}\right) + 600 \times \left(\frac{170}{600}\right) = \$270 \\
\text{Cost C} &= 300 \times \left(\frac{150}{300}\right) + 600 \times \left(\frac{230}{600}\right) = \$380
\end{aligned} \tag{1}$$

To convert dollar unit measures to priorities divide the cost for each alternative by the sum of the costs for all the alternatives. This converts dollar units to priorities. The sum of a set of priorities should always equal 1. Thus we have for the three alternatives:

$$\text{Priority of A} = \$250/\$900 = 0.278 \tag{1}$$

$$\text{Priority of B} = \$270/\$900 = 0.300 \tag{2}$$

$$\text{Priority of C} = \$380/\$900 = 0.422 \tag{3}$$

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The criteria derived their importance from the alternatives because the units are the same for both criteria, i.e. dollars, so the quantities 300 and 600 can be used to determine the relative importance, or priorities of the criteria C1 and C2. Another way to think of it is, How much money does each criterion control contribute? It is natural to conclude that the more money controlled by a criterion, the more important that criterion is. If we compare these criteria for importance with respect to the goal of selecting the best vacation site, we have:

Table 2
Pairwise comparing criteria for importance

Goal	C_1	C_2	Priorities
C_1	1	1/2	.333
C_2	2	1	.667

3.2 Tangibles as intangibles—second case

To pairwise compare A, B, and C with respect to each criterion, one uses ratios of costs from Table 1 to fill out the pairwise comparison matrix. When A is compared with B for relative cost with respect to travel one has 50/100 = 1/2 placed in the (A, B) cell and so on. The pairwise comparison matrix for the alternatives with respect to travel and lodging respectively are given in Table 3 and Table 4.

Table 3
Relative priorities of the alternatives for travel cost and lodging cost

Travel Cost

Alternatives	A	B	C	Priorities
A	1	1/2	1/3	0.167
B	2	1	2/3	0.333
C	3	3/2	1	0.500

Lodging Cost

Lodging (_C_2)	A	B	C	Priorities
A	1	2/1.7	2/2.3	0.333
B	1.7/2	1	1.7/2.3	0.283
C	2.3/2	2.3/1.73	1	0.383

When weighting the priorities of the travel cost and lodging cost of the alternatives by the priorities of criterion 1 (0.333), and criterion 2 (0.667), and adding, we obtain:

Cost A = .333 x .167 + .667 x .333 = .278
Cost B = .333 x .333 + .667 x .283 = .300
Cost C = .333 x .500 + .667 x .383 = .422

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which is the same as the priorities on the right in Table 1 and Equation (2). Thus, the priorities of the alternatives as obtained by additive hierarchic composition lead to the same solution as an appropriate analysis of the original data using arithmetic.

3.3 Numerical judgments as an approximation

For elements of the same order of magnitude (i.e., they are homogeneous), the paired comparison judgments in the matrices may be approximated by values from the scale 1-9 based on perception. This is useful when there are no known numerical values to form the ratios. Estimating by rounding the ratios to the nearest whole number, the matrices of the above example are shown in Table 4.

Table 4
Estimating relative priorities of alternatives for travel cost using the AHP fundamental scale

Estimating ratios of Travel Cost C_1 for the alternatives

Alternatives	A	B	C	Priorities
A	1	1/2	1/3	0.163
B	2	1	1/2	0.297
C	3	2	1	0.540

Estimating ratios of Lodging Cost C_2 for the alternatives

Alternatives	A	B	C	Priorities
A	1	1	1	.333
B	1	1	1	.333
C	1	1	1	.333

Weighting by the priorities of the criteria, and adding as in (3), we obtain the following results:

Cost A = .333 x .163 + .667 x .333 = .276
Cost B = .333 x .297 + .667 x .333 = .321
Cost C = .333 x .540 + .667 x .333 = .402

A is again the preferred alternative, and the numbers are a little different, being based on estimates, but they are fairly close. The approximation using a 1-9 scale could lead to a different choice than the best one, but there would be no need to approximate if exact numbers are known. However, in general, one needs to compare the dollar values according to the importance of their magnitudes and that depends on the individual or different individuals who may then combine or average their judgments by using the geometric mean which has been proven to be the only way to satisfy the reciprocal relation.

This example demonstrates that when the criteria weights are described in terms of the unit of measurement of the alternatives, the operations of the AHP can be used to duplicate with

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relative numbers the answers one gets with arithmetic. However, this is not the purpose of the AHP. The priorities associated with numbers may not vary linearly or monotonically with those numbers. In fact, for each problem the priorities would satisfy the needs of that problem according to the judgments of the individual or group involved.

3.4 Inadequate knowledge about the criteria

Suppose we have three foods and their content measured in milligrams for the two criteria, vitamin X and vitamin Y. The importance of the criteria is no longer determined by the total or average milligram content of the alternatives as before, but rather by the needs of the body for that vitamin to remain healthy. It may be harmful to get an excessive amount of one vitamin but healthy to get such an amount from the other. We must establish priorities by comparing the criteria with respect to healthful contribution, and the alternative's milligram content for their positive contribution to meet body needs. The actual measurements in small quantities and their totals cannot determine the single best food to eat.

3.5 The case of dependence of the criteria on the alternatives

All very well, one may say when dealing with more than one criterion that are all tangibles measured on the same existing scale. To make the AHP work correctly it is only necessary to observe that the criteria depend on the alternatives and weight the criteria by the proportion of the total property exhibited by the alternatives under it. As new alternatives are added to the model with varying amounts of the property for each criterion, the criteria weights need to be continually re-scaled as the "total" property belonging to each criterion changes. But what does one do with intangibles? Or with properties measured in different scales?

Before answering that question, let us show that the Analytic Network Process automatically accomplishes the same thing as weighting the criteria appropriately in the AHP. To do this, construct an ANP model with the alternatives depending on the criteria and the criteria depending on the alternatives, and enter the known data. Construct the supermatrix by normalizing each set of data. Raise the supermatrix to powers until it converges to the limit supermatrix and use the priorities for the alternatives and of the criteria from that matrix. They will be the same as the priorities from the AHP with the weighted criteria. It is easy to do as one simply enters the data and it does not require any analysis of whether it is necessary to weight the criteria or not.

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