AHP PRIORITIES AND MARKOV-CHAPMAN-KOLMOGOROV STEADY-STATES PROBABILITIES

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ABSTRACT

An AHP matrix of the quotients of the pair comparison priorities is transformed to a matrix of shares of the preferences which can be used in Markov stochastic modeling via the Chapman-Kolmogorov system of equations for the discrete states. It yields a general solution and the steady-state probabilities. The AHP priority vector can be interpreted as these probabilities belonging to the discrete states corresponding to the compared items. The results of stochastic modeling correspond to robust estimations of priority vectors not prone to influence of possible errors among the elements of a pairwise comparison matrix.

Keywords: AHP; Markov stochastic modeling; Chapman-Kolmogorov equations.

1. Introduction

We consider a modified AHP for finding the robust preference estimation by a transformed matrix of the items shares. We show that this approach can be obtained from the Markov Stochastic Processing in the form of Chapman-Kolmogorov equations using the pair comparison data for intensity of transitions among the items in a system of differential equations. Solving this system yields the dynamic and the eventually reached steady-state probabilities. Comparison shows that the results of Markov modeling in the modified AHP and classical AHP priorities are very close. This means that the regular AHP preferences can be interpreted as the probabilities of belonging to the states corresponding to the compared items. The presented techniques have been studied in more detail in several works (Lipovetsky, 1996; Lipovetsky and Tishler, 1999; Lipovetsky and Conklin, 2002, 2003). The considered methods enrich the possibilities of priority estimation and its application for various practical problems, particularly, in the marketing research field.

The approach described in this paper is based on the initial transformation of the pairwise ratios matrix into the matrix of priority shares. This transformation makes the opposite ratios more balanced. For instance the pairwise reciprocal values, 9/1 and 1/9 differ by 81 times, but after the transformation they become 0.9 and 0.1, respectively, so they then differ by 9 times. There is also a problem because of the pairwise ratio'stransitivity and

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the limited scale. The theory says that for consistency in a pairwise ratio matrix the condition $a_{ij}a_{jk}=a_{ik}$ should hold because of the theoretical relation between the priorities w: if $a_{ij}=w_i/w_j$ and $a_{jk}=w_j/w_k$, then $(w_i/w_j)(w_j/w_k)=w_i/w_k=a_{ik}$. But it is often impossible to remain within the bounded scale for the judgments, for instance, if $a_{ij}=3$ and $a_{jk}=5$, then the consistent estimate should be $a_{ik}=15$. However, the scale is limited to a maximum value of 9, so an expert can only assign $a_{ik}=9$, and that would cause inconsistency in the matrix. Thus, consistency can decrease not only because of the actual inconsistency in the expert judgments but also because of the bounded scale. A transformation of the shares, however, makes all the pairwise judgments b_{ij} and b_{ji} to belong to the 0-1 range of values where they are equidistant from the diagonal elements $b_{ii}=0.5$, so that decreases inconsistency.

2. AHP in a modified model

A theoretical Saaty matrix (Saaty, 1980) of pair comparisons for n items defines each ij-th element as a ratio of unknown priorities w_i and w_j:

$$W=\begin{pmatrix} w_1/w_1 & w_1/w_2 & \dots & w_1/w_n \\ ----- & ------- & ---- \\ w_n/w_1 & w_n/w_2 & \dots & w_n/w_n \end{pmatrix}. \qquad (1)$$

Multiplying matrix (1) by the vector $w=(w_1,w_2,....,w_n)'$ we get the identical relation

$$Ww=nw, \qquad (2)$$

Elicited from a judge, an empirical pair comparison matrix of priority ratios is

$$A=\begin{pmatrix} 1 & a_{12} & \dots & a_{1n} \\ a_{21} & 1 & \dots & a_{2n} \\ \dots\dots & \dots\dots & \dots\dots & \dots\dots \\ a_{n1} & a_{n2} & \dots & 1 \end{pmatrix}. \qquad (3)$$

This is a Saaty matrix with transposed-reciprocal elements

$$a_{ij}=a_{ji}^{-1}. \qquad (4)$$

Similarly to (2), priorities in the AHP are estimated by the eigenproblem for a matrix (3):

$$A\alpha=\lambda\alpha, \qquad (5)$$

where the maximum eigenvalue corresponds to the term n in (2), and the principal eigenvector α estimates the vector of priorities w.

Let us introduce a theoretical matrix of shares

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$$U = \begin{pmatrix} w_1/(w_1+w_1) & w_1/(w_1+w_2) & \dots & w_1/(w_1+w_n) \\ ----- & ------- & & ---- \\ w_n/(w_n+w_1) & w_n/(w_n+w_2) & \dots & w_n/(w_n+w_n) \end{pmatrix} \quad (6)$$

Each element u_{ij} is defined as i-th priority in the sum of i-th and j-th theoretical priorities:

$$u_{ij} = \frac{w_i}{w_i + w_j} = \frac{w_i / w_j}{1 + w_i / w_j}. \quad (7)$$

To estimate the priority vector using the matrix (6) we write identical equalities:

$$\begin{cases} \frac{w_1}{w_1+w_1}(w_1+w_1) + \frac{w_1}{w_1+w_2}(w_1+w_2) + \dots + \frac{w_1}{w_1+w_n}(w_1+w_n) = nw_1 \\ - - - - - - - - - - - - - - - - - - \\ \frac{w_n}{w_n+w_1}(w_n+w_1) + \frac{w_n}{w_n+w_2}(w_n+w_2) + \dots + \frac{w_n}{w_n+w_n}(w_n+w_n) = nw_n \end{cases}. \quad (8)$$

Then using notation (7) we present the system (8) as:

$$\begin{cases} (u_{11} + \sum_{j=1}^{n} u_{1j})w_1 + u_{12}w_2 + \dots + u_{1n}w_n = nw_1 \\ - - - - - - - - - - - \\ u_{n1}w_1 + u_{n2}w_2 + \dots + (u_{nn} + \sum_{j=1}^{n} u_{nj})w_n = nw_n \end{cases} \quad (9)$$

In the matrix form the system (9) is:

$$\left(U + diag(Ue)\right)w = nw, \quad (10)$$

where U is the matrix (6), e denotes a uniform vector of n-th order, and $diag(Ue)$ is a diagonal matrix of totals in each row of matrix U. Relations (8)-(10) for the theoretical matrix of shares (6) are derived similarly to the problem (2) for the matrix (1).

In classical AHP, pair ratios w_i/w_j (1) are estimated by elicited values a_{ij} (3). Using a_{ij} in (7) we obtain empirical estimates b_{ij} of the pairs' shares:

$$b_{ij} = \frac{a_{ij}}{1 + a_{ij}} \quad (11)$$

This transformation of the elements of a matrix A (3) yields a pairwise share matrix B with elements (11). The elements of such a matrix (11) are positive, less than one, and have a property of symmetry:

$$b_{ij} + b_{ji} = 1. \quad (12)$$

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This means that the transposed elements b_{ij} and b_{ji} are equidistant from the diagonal elements b_{ii}=0.5, so $b_{ij} - b_{ii} = b_{ii} - b_{ji}$. Elements of a Saaty matrix (3) with large or small values are transformed in (11) to the values closer to one or zero, respectively.

In the AHP, for the empirical Saaty matrix A (3) we have the eigenproblem (5) in place of the theoretical relations (2). By the same pattern, using empirical matrix B (11) in place of theoretical matrix *U*, we represent the system (10) as an eigenproblem

$$(B + diag(Be))\alpha = \lambda\alpha \,, \tag{13}$$

where α as a vector of priority. Multiplying the matrix in (13) by the uniform vector and using (12) shows that this matrix has a property such that the total in its each column equals *n*:

$$(B + diag(Be))' e = B'e + Be = n\,e \,, \tag{14}$$

where prime denotes transposition. Dividing both sides of equations (13) by the term *n*, we obtain an eigenproblem of a positive matrix with totals in the columns equal to one, which is an eigenproblem of the transposed stochastic matrix. Such a matrix has the maximum eigenvalue equal to one. Due to the Perron-Frobenius theory for a positive matrix, its main eigenvector always exists, is a unique one, and has all positive elements. Thus, the maximum eigenvalue in (13) equals *n*, and a solution for the main eigenvector exists and is unique, which ensures in the desired properties of the priority vector.

3. AHP modeling in Markov-Chapman-Kolmogorov equations

The eigenproblem (13) has a matrix that is of a transposed stochastic kind that relates it to matrices known in Markov modeling. Consider a discrete state and continuous time Markov model presented via Chapman-Kolmogorov differential equations describing a stochastic process of transitions among the states. This model is based on properties of a finite set of the elements (alternatives compared within a criterion) that are tied by the constant transition probabilities of each alternative's prevalence over the others. The prevalence of one item over another one in the AHP corresponds to probability of the former item is preferred over the latter one in the eliciting process. The Chapman-Kolmogorov equations express the change in probability to be found in any of *n* states as a linear combination of these probabilities with the coefficients of the transition intensities.

Taking a pair of the elements b_{ij} and b_{ji} of the share matrix (11) we notice that each element can be interpreted in terms of probability to prefer one of the items over another one, due to the meaning of the theoretical shares (7). The preference of an *i*-th item over a *j*-th item corresponds to transition between them with intensity b_{ij}. The share matrix *B* can be presented as a connected oriented graph with *n* nodes of states/alternatives and two edges between each of pair of nodes, one going to state *i* from state *j* corresponding to the transition intensity b_{ij} and the other going from state *i* to state *j* corresponding to the transition intensity b_{ji}.

An example of such a network is shown in Figure 1.

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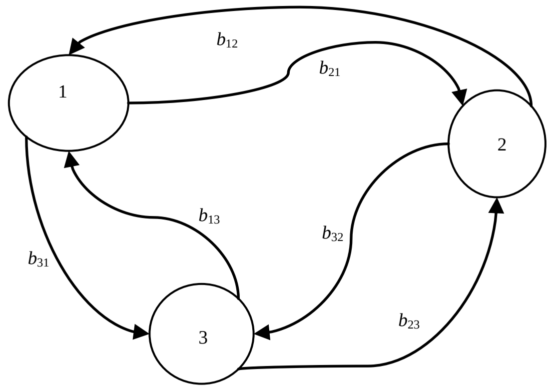

Figure 1. AHP network of the transition shares

The system of Chapman-Kolmogorov equations can be presented as follows:

$$\begin{cases} \dfrac{dp_1}{dt} = (b_{12}p_2 + ... + b_{1n}p_n) - (b_{21}p_1 + ... + b_{n1}p_1) \\ - \ - \ - \ - \ - \ - \ - \ - \ - \ - \ - \\ \dfrac{dp_n}{dt} = (b_{n1}p_1 + ... + b_{n,\,n-1}p_{n-1}) - (b_{1n}p_n + ... + b_{n-1,\,n}p_n) \end{cases} \tag{15}$$

where p_i denotes probability to belong to an i-th state, and coefficients b_{ij} are the values (11). Items with positive signs at the right-hand side (15) define influx to each state from all the others, and those with negative signs define departure from a state to all the other states. If canceling items $0.5p_i$ are added to both positive and negative inputs in each i-th equation (15), this system can be represented in matrix form as:

$$\dot{p} = \left(B - diag(B'e)\right)p\,, \tag{16}$$

where p is a vector of the probabilities p_i for all the states, $\dot{p}$ denotes the vector of their derivatives, B is the same matrix with elements (11), B' is its transposition, and e is the identity vector. Using property (14) that the sum of totals in i-th column and row of the matrix B equals n, we can rewrite (16) as:

$$\dot{p} = \left(B + diag(Be) - nI\right)p\,, \tag{17}$$

where I denotes the identity matrix of the n-th order.

Considering the solution of the Chapman-Kolmogorov equations (17) for the steady-state probabilities when the process is stabilized, we put the derivatives in the left-hand side equal to zero, and (17) reduces to:

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$$\left(B + diag(Be)\right) p = n\, p\,. \tag{18}$$

But (18) is nothing else but the same eigenproblem (13) with the largest eigenvalue $\lambda = n$ and a unique positive main eigenvector, as it was discussed in relation to equality (14). So the results of the AHP priority evaluation (13) can be interpreted from the point of view of the stochastic process steady-state solution (18) as follows: the AHP priority vector corresponds to the eventual probabilities of belonging to the discrete states, or alternatives, and these probabilities define the preferences among the compared items.

Finding a general dynamic solution for system (17) can be also implemented for some problems in the AHP. For instance, a researcher can be interested in differences among the preferences and in their specific behavior (monotonic increase, decrease, or oscillation) before the process stabilizing. The solution of a homogeneous linear system of differential equations with constant coefficients can be presented as:

$$p(t) = P\, diag(\exp(\lambda_j t))\, c\,, \tag{19}$$

where c is a vector of constants, λ_j are the eigenvalues and P is a corresponding matrix of columns p_j of eigenvectors obtained in the problem:

$$\left(B + diag(Be) - nI\right) p = \lambda\, p\,. \tag{20}$$

This is the eigenproblem with the matrix at the right-hand side of Chapman-Kolmogorov system (17), and its solution coincides with the solution for the AHP problem (13) up to reducing the latter eigenvalues by n. For the initial moment t=0 the solution (19) is reducing to $p(0)=Pc$, and solving this linear system with the known vector of initial conditions $p(0)$ yields the vector of constants $c=P^{-1}p(0)$. The general solution of the differential system is

$$p(t) = P\, diag(\exp(\lambda_j t))\, P^{-1} p(0)\,, \tag{21}$$

The expression $P\, diag(\exp(\lambda_j t))\, P^{-1}$ in (21) is known as the matrix exponent. Each component of the vector $p(t)$ is a linear combination of the exponents in (21), and functions $\exp(\lambda_j t)$ behave in accordance with the specific values of λ_j obtained in the eigenproblem (20).

As was mentioned above, the main eigenvalue in (20) is less by n than the main eigenvalue in (13), or (18), so it equals zero, $\lambda_1 = 0$, that corresponds to the constant part of (21) behavior. The other eigenvalues (20) are real numbers or conjugated pairs of complex numbers. As we know from the Perron–Frobenius theory, all other eigenvalues have a less real value than the main eigenvalue, so all real eigenvalues or real parts in complex eigenvalues are negative. Thus, the general behavior of solution (21) is defined by a constant part ($\lambda_1 = 0$), by diminishing exponents (real negative eigenvalues), and by oscillating diminishing exponents (complex eigenvalues giving sine and cosine parts of the functions). There also can be polynomial items corresponding to equal eigenvalues, although in practical numerical evaluations such cases are rare. The eigenvectors p of the complex eigenvalues are complex, but the total expression (21) yields real values.

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4. Numerical comparisons

Table 1 presents an example of a matrix *A* (3) of pair comparison among eight criteria used for the problem of "Choosing the best home" – a classical AHP problem described in several articles (Saaty, 1996; Saaty and Kearns, 1985; Saaty and Vargas, 1994). This matrix was also used for testing some new techniques in Lipovetsky (1996), Lipovetsky and Tishler (1999), Lipovetsky and Conklin (2002). The items of comparison in Table 1 are: 1 – size of house, 2 – location to bus, 3 – neighborhood, 4 – age of house, 5 – yard space, 6 – modern facilities, 7 – general condition, 8 – financing. The transformed matrix *B* (11) is presented in Table 2. This transformation makes all the elements belong to 0-1 interval and diminishes the influence of any possible errors in pair comparisons. The row and column totals in Table 2 correspond to *Be* and *B'e* vectors related due to (14). The grand total can be used for checking – it equals $n^2/2$.

Table 1
Example of AHP pair comparison Matrix A

item	1	2	3	4	5	6	7	8
1	1	5	3	7	6	6	1/3	1/4
2	1/5	1	1/3	5	3	3	1/5	1/7
3	1/3	3	1	6	3	4	6	1/5
4	1/7	1/5	1/6	1	1/3	1/4	1/7	1/8
5	1/6	1/3	1/3	3	1	1/2	1/5	1/6
6	1/6	1/3	1/4	4	2	1	1/5	1/6
7	3	5	1/6	7	5	5	1	1/2
8	4	7	5	8	6	6	2	1

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Table 2
Pair shares Matrix B

Item	1	2	3	4	5	6	7	8	total
1	.500	.833	.750	.875	.857	.857	.250	.200	5.123
2	.167	.500	.250	.833	.750	.750	.167	.125	3.542
3	.250	.750	.500	.857	.750	.800	.857	.167	4.931
4	.125	.167	.143	.500	.250	.200	.125	.111	1.621
5	.143	.250	.250	.750	.500	.333	.167	.143	2.536
6	.143	.250	.200	.800	.667	.500	.167	.143	2.869
7	.750	.833	.143	.875	.833	.833	.500	.333	5.101
8	.800	.875	.833	.889	.857	.857	.667	.500	6.278
Total	2.877	4.458	3.069	6.379	5.464	5.131	2.89	1.722	32

Table 3 presents several methods of priority estimation. Each vector is normalized so that the total equals one. The matrix *A* from Table 1 is used to obtain three vectors shown as "Regular Methods" in Table 3. The first of these vectors is obtained in the classic AHP eigenproblem (5), the next by the Least Squares approach also known in AHP, and the third vector is estimated by the Multiplicative AHP technique (Saaty and Vargas, 1984, 1994; Lootsma, 1999). The last column in Table 3 contains the results of the modified estimation obtained by the transformed matrix *B* from Table 2 in the eigenproblem (13) or the Chapman-Kolmogorov equations for the steady-states (18). The ranks of the items are shown in parentheses after the elements. The solutions yield different priority ordering that could indicate inconsistent relations in the pairwise matrix.

Table 3
Priority Vector in Several Estimations

i	Eigenvector	Least Squares	Multiplicative	Chapman-Kolmogorov
1	.173 (6)	.199 (7)	.175 (7)	.150 (6)
2	.054 (4)	.100 (4)	.063 (4)	.054 (4)
3	.188 (7)	.148 (5)	.149 (5)	.141 (5)
4	.018 (1)	.017 (1)	.019 (1)	.022 (1)
5	.031 (2)	.045 (2)	.035 (2)	.037 (2)
6	.036 (3)	.065 (3)	.042 (3)	.041 (3)
7	.167 (5)	.184 (6)	.167 (6)	.163 (7)
8	.333 (8)	.242 (8)	.350 (8)	.392 (8)

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The eigenvalues in the problem (20) approximately equal the following values:

$\lambda_1 = 0,\ \lambda_2 = -2.0,\ \lambda_{3,4} = -2.9 \pm 0.5i,\ \lambda_5 = -4.0,\ \lambda_6 = -4.8,\ \lambda_7 = -5.3,\ \lambda_8 = -6.1.$

The behavior of the solution (21) is mostly defined by the several first exponents with the bigger absolute value of the eigenvalues – those are the functions $1,\ e^{-2t},\ e^{-2.9t}\cos(0.5t),\ e^{-2.9t}\sin(0.5t),$ and the other functions' decay is much steeper. With equal initial conditions $p_i(0)=1/n=0.125$ in (21), the total behavior of the Chapman-Kolmogorov solution is shown in Figure 2.

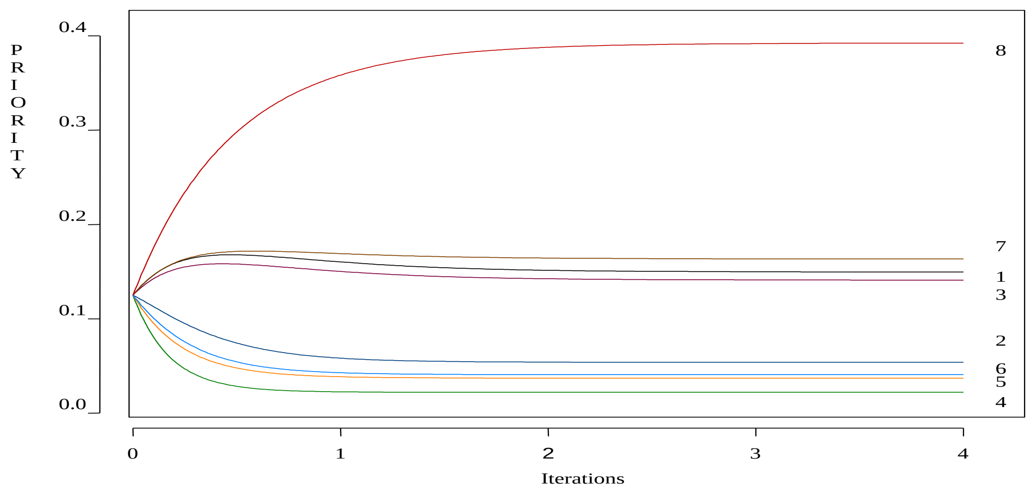

Figure 2. AHP priority in Chapman-Kolmogorov solution

The numbers of the alternatives are given at the right of the curves. Beginning from about the third iteration, the process is stabilized. Eventually, all the curves reach priority levels coinciding with those presented in the Table 3 last column.

5. Inconsistency and robust estimation

As it was shown in Lipovetsky and Conklin (2002) on the same example, the appropriate ranks are given by the results in the last column of Table 3, which presents the robust solution not prone to the possible inconsistencies in the data matrix. Let us describe how to find such inconsistencies. Returning to (1), we see that the theoretical Saaty matrix equals the outer product

$$W = wv', \tag{22}$$

of a vector-column w of priorities

$$w = (w_1, w_2, ..., w_n)' \tag{23}$$

and a vector-row v' of element-wise reciprocal priorities (or anti-priorities):

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$$v' = (w_1^{-1},\ w_2^{-1}, \ldots., w_n^{-1})\ . \qquad (24)$$

Multiplying the matrix (1) by the vector (23) from the right side, or by the vector (4) from the left side, we get identically true relations

$$Ww = nw, \quad W'v = nv \qquad (25)$$

that correspond to the right and the left eigenproblems for the *W* matrix. Notation *W'* is used for the transposed *W* matrix.

Using vectors (23) and (24), consider the theoretical matrix (1)-(2) as the structure in the following Table 2.

Table 4
Theoretical Saaty matrix as a contingency table

Anti-priority / **Priority**	**1**	**2**	**…**	**n**	**Row Total**
1	w_1v_1	w_1v_2	…	w_1v_n	$w_1\Sigma_j v_j$
2	w_2v_1	w_2v_2	…	w_2v_n	$w_2\Sigma_j v_j$
…	-- -- --	-- -- --	-- -- --	-- -- --	-- -- -- --
n	w_nv_1	w_nv_2	…	w_nv_n	$w_n\Sigma_j v_j$
Column Total	$\mathbf{v_1\Sigma_i w_i}$	$\mathbf{v_2\Sigma_i w_i}$	**....**	$\mathbf{v_n\Sigma_i w_i}$	$\mathbf{(\Sigma_i w_i)(\Sigma_j v_j)}$

Each element w_{ij} of this matrix can be seen as product of *i*-th row sum and *j*-th column sum divided by grand total:

$$w_{ij} = w_i\, v_j = \frac{(row\ total)_i\,(column\ total)_j}{grand\ total} \qquad (26)$$

This type of theoretical (or expected) structure of the table is well-known in statistical analysis as a contingency table. AHP theoretical matrix (1) corresponds to an *n* by *n* contingency table of two vectors – priority (in rows) and anti-priority (in columns) (see Table 2). The difference is that the AHP matrix does not describe counts, or frequencies, as a regular contingency table does. However, taking product *p* of all priorities w_i in denominators of theoretical matrix (1)

$$p = w_1\, w_2 \,_{\ldots}\, w_n \qquad (27)$$

and multiplying each element of theoretical Saaty matrix by this term results in integer numbers in place of the pairwise ratios (1):

$$w_{ij} = (w_i\, /\, w_j)\, p \qquad (28)$$

These integer numbers (12) make sense of the proportions which describe how each item is preferred or failed in pairwise comparison with other items in consideration. Relation

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(28) can be seen as a connection between AHP priority ratios and relative proportions of preference among the items.

Similarly to Table 4, an elicited Saaty matrix (3) can be presented as a contingency table in the following Table 5.

Table 5
Empirical Saaty matrix as a contingency table

Anti-priority / Priority	1	2	…	n	Row Total
1	a_{11}	a_{12}	…	a_{1n}	$\Sigma_j a_{1j}$
2	a_{21}	a_{22}	…	a_{2n}	$\Sigma_j a_{2j}$
…	-- -- --	-- -- --	-- -- --	-- -- --	-- -- -- --
n	a_{n1}	a_{n2}	…	a_{nn}	$\Sigma_j a_{nj}$
Column Total	$\Sigma_i a_{i1}$	$\Sigma_i a_{i2}$	….	$\Sigma_i a_{in}$	$\Sigma_i \Sigma_j a_{ij}$

Expected values of elements in this contingency table are defined similarly to the expression (26), using the margins of Table 5:

$$e_{ij} = (\Sigma_k a_{ik})(\Sigma_t a_{tj}) / (\Sigma_k \Sigma_t a_{kt}) \qquad (29)$$

To estimate the agreement between observed (empirical) and expected (theoretical) composition of numbers in cells, we use the Chi-squared objective:

$$\chi^2 = \sum_{i,j=1}^{n} \frac{(a_{ij} - e_{ij})^2}{e_{ij}} \qquad (30)$$

If an elicited data matrix (3) corresponds exactly to the theoretical matrix (1) then all deviations equal zero and the value (30) is zero as well. The bigger the discrepancy between elicited data and theoretical AHP structure (1) the higher the value (30).

For instance, using the same data from the example given in Table 1, the total value (30) equals $\chi^2 = 32.7$. The mean value of these deviations is 0.51, and the standard deviation is 1.51, and all the items of the Chi-squared sum (30) are presented in Table 6.

The biggest deviations of empirical pairwise ratios from their theoretical values identify the coordinates of the inconsistent data in a Saaty matrix. In our example, the Chi-squared item of one element a_{37} equals 11.91 which is far beyond a reasonable confidence interval for a mean value of 0.51 with a standard deviation of 1.51. The reciprocally symmetrical element a_{73} could be also considered as an outlier (both a_{37} and a_{73} are marked by asterisk in Table 6). The input of these two elements among all 64 elements in sum (30) is about 41% of its total value.

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Table 6
Items of the Chi-squared sum

j / i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	0.32	0.13	0.51	0.12	0.15	0.20	1.35	0.12
2	0.44	0.44	0.36	0.55	0.21	0.24	0.53	0.03
3	0.85	0.07	0.25	0.05	0.35	0.00	11.91*	0.11
4	0.00	0.07	0.00	0.18	0.02	0.07	0.00	0.17
5	0.10	0.31	0.01	1.25	0.00	0.25	0.09	0.05
6	0.22	0.63	0.18	1.33	0.20	0.13	0.23	0.01
7	1.14	0.27	1.54*	0.03	0.01	0.02	0.38	0.00
8	1.08	0.25	1.91	0.77	0.14	0.10	0.17	0.15

Criterion (30) uses the expected values estimated as the margins given in (29). More exact estimation of the expected values could be performed using the first pair of the vectors in the spectral decomposition of a matrix A. But a more convenient approach good for all practical needs can be the following one.

When the theoretical Saaty matrix is considered as a contingency table, all the rows are proportional one to another, and the columns as well. This means that for a more consistent empirical matrix the correlations between rows (and between columns) become closer to one. Thus, instead of chi-squared we can find the matrix of correlations between rows and the matrix of correlations between columns – let us denote them as R_r and R_c. For each of these two matrices, calculate vectors of mean correlations in each row, and the results are as follows:

Mean (R_r) = (.78 .85 .57* .80 .80 .83 .73 .73),

Mean (R_c) = (.73 .83 .69 .78 .78 .79 .32* .73).

The asterisk here marks the minimum values among the others, when the average correlation suddenly falls. We see that items 3 and 7 have the lowest mean correlations, so elements a_{37} and a_{73} of the Saaty matrix in Table 1 could be considered outliers.

Instead of averaging the elements in rows of the correlation matrices R_r and R_c we can apply principal component analysis to see the relations between the items. The first vectors of these two matrices are as follows:

PC (R_r) = (.35 .39 .25* .37 .37 .38 .36 .33),

PC (R_c) = (.36 .41 .34 .38 .39 .39 .12* .36),

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where the asterisk denotes positions of elements with the weights noticeably below the mean level of $1/\sqrt{8}=0.35$. Again, the results show that a_{37} and a_{73} are probable outliers in the Saaty matrix in Table 1, and they reduce consistency of the data.

Identification of the outliers can be found by means of the regular AHP measures. Let us take the matrix in Table 1 and solve the AHP problem (5). The maximum eigenvalue there is 9.67, and the consistency index

$$CI=(\lambda-n)/(n-1)=0.24.$$

Excluding one item at a time from the set of eight and solving the AHP problem for each set of seven items, we get the following values of consistency index:

$$CI = (0.20 \quad 0.27 \quad 0.12^* \quad 0.27 \quad 0.28 \quad 0.28 \quad 0.12^* \quad 0.26).$$

We see that without the 3rd and 7th items, the consistency index falls lower, which indicates an improved consistency of the matrix. Again, we localize a_{37} and a_{73} as the probable outliers in the data in Table 1.

In this assumption, we change the value of a_{37} (and take a_{73} as the reciprocal value) in a wide range of possible values, and each time solve the AHP problem (3) for finding maximum eigenvalue and the consistency index *CI*. The results are given in Table 7.

Table 7
Eigenvalue and consistency index of subsequently adjusted matrix

$\mathbf{a_{37}}$	**6**	**4**	**2**	**1**	**1/2**	**1/4**	**1/5**
λ_{max}	9.669	9.369	9.037	8.868	8.812	8.847	8.879
CI	.238	.196	.148	.124	.116	.121	.126

Table 7 shows that the best consistency (the minimum value of CI) is attained for the value a_{37} =1/2 (and a_{73} =2, respectively). Adjusting by these values makes the matrix of Table 1 become highly consistent, so any method yields very similar priority estimations and the same priority ordering. Ranks of priorities for the adjusted matrix coincide with the ranks given in the last column of Table 3.

So we see that the results of the eigenproblem (13) or the Chapman-Kolmogorov equations for the steady-states (18) are very robust to possible inconsistencies in the data. Being applied to the original matrix in Table 1 with the inconsistent elements, the robust methods yield the same priority ordering as estimations by the regular AHP approach applied to the adjusted matrix with the corrected elements. These results were observed in numerous calculations with different data.

Finally, it is interesting to note that the eigenproblem (13) or (18) can be described in terms of the errors in the elicited data as follows. For the elements (11), consider a model with relative errors

$$b_{ij}(\alpha_i+\alpha_j)=\alpha_i(1+\varepsilon_{ij}) \qquad (31)$$

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Summing the relations (31) by index *j* yields a system of equations:

$$(\sum_{j=1}^{n} b_{ij})\alpha_i + \sum_{j=1}^{n} b_{ij}\alpha_j = (n + \sum_{j=1}^{n} \varepsilon_{ij})\alpha_i \quad , \quad i = 1,\ldots,n \ . \qquad (32)$$

In the assumption of equal sums of errors for all *i*, denote

$$\lambda = n + \sum_{j=1}^{n} \varepsilon_{ij} \quad , \quad i = 1,\ldots,n \quad , \qquad (33)$$

then the system (32) reduces to the eigenproblem (13). Thus, the eigenproblem (13) corresponds to the assumptions of multiplicative errors (31) and the equalized total errors in rows (33).

6. Summary

Transformation of a pairwise ratio AHP matrix to the pairwise share matrix and solving the corresponding eigenproblem is considered. This approach can be obtained in Chapman-Kolmogorov modeling of transitions among the discrete states of the alternatives. Coincidence of these results for AHP priority evaluation and of the stochastic steady-state solution suggests a useful interpretation: the AHP priorities have a meaning of the eventual probabilities of belonging to the discrete states of the compared items. The priorities expressed as the choice probabilities are useful in theoretical consideration and practical applications for various multiple criteria decision making problems.

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