THE ANALYTIC HIERARCHY PROCESS WITHOUT THE THEORY OF OSKAR PERRON

Thomas L. Saaty
University of Pittsburgh
Pittsburgh, PA 15260
USA
saaty@katz.pitt.edu

ABSTRACT

It is known and has been mathematically proven that the principal eigenvector is necessary for deriving priorities from judgments in the Analytic Hierarchy Process (AHP). According to the work of Oskar Perron, the principal eigenvector can be obtained as the limiting power of a positive matrix. In this paper we show that the principal eigenvector does not need the theory of Perron for its existence based on the fact that the principal eigenvalue and corresponding principal eigenvector are transparently obtained for a consistent matrix. By perturbation theory the result is obtained for a near consistent matrix.

Keywords: Theory of Perron, AHP eigenvector, pairwise comparison matrix solution
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1. Introduction

We begin by honoring the name of Oskar Perron who proved a very powerful theorem in mathematics about real positive matrices. I regret not meeting Perron (born in 1880) before he died in Munich, Germany, in 1975. My work on the Analytic Hierarchy Process (AHP) with real positive reciprocal matrices leads to the principal eigenvalue and eigenvector without the need for invoking any of Perron's results as will be shown in this paper. He proved that if $A = (a_{ij})$ is an $n \times n$ real positive matrix: $a_{ij} > 0$ for $1 \le i, j \le n$, then there is a positive real number $\lambda_{\max}$, called the Perron root or the Perron–Frobenius eigenvalue, such that $\lambda_{\max}$ is an eigenvalue of A and any other eigenvalue λ (possibly, complex) is strictly smaller than $\lambda_{\max}$ in absolute value, $|\lambda| < \lambda_{\max}$. There exists an eigenvector $w = (w_1,\ldots,w_n)$ of A with eigenvalue $\lambda_{\max}$ such that all components of w are positive: $Aw = \lambda_{\max} w$, $w_i > 0$ for $1 \le i \le n$. There also exists a positive left eigenvector $v : v^T A = \lambda_{\max} v^T$, $v_i > 0$. Perron also proved that the principal eigenvector w corresponding to $\lambda_{\max}$ can be obtained by raising the matrix A to infinite powers.

His theorem was later modified by Frobenius whose theory assures us of a similar result except that he proved the root may no longer be simple if there are zero entries in the matrix.

In the Analytic Hierarchy Process the pairwise comparison judgment matrices are real, positive and reciprocal $(A = (a_{ij}), a_{ji} > 0, a_{ji} = \frac{1}{a_{ij}} \text{ for all } i \text{ and } j)$, and their order is not much larger than 7x7. The entries values lie between 9 and 1/9.

Interestingly, as we shall show here, the principal eigenvalue and its principal eigenvector can be found for a real reciprocal positive matrix of small order without Perron's theory. The principal eigenvalue and eigenvector can be obtained from the solution of a system of equations without using the powers of the matrix as does Perron. We observe that if we know either $\lambda_{\max}$ or w, we also know the other. If, for example, we know $\lambda_{\max}$, we get w by solving, in the familiar way, the homogenous system of linear equations: $\sum_{j=1}^{n} a_{ij} w_j = \lambda_{\max} w_i, i = 1,\ldots,n.$ If we know w then because of the normalization condition $\sum_{i=1}^{n} w_i = 1$ in our case, after taking the sum on both sides of the equation with respect to i and interchanging the sums on the left we obtain: $\sum_{j=1}^{n} w_j \sum_{i=1}^{n} a_{ij} = \lambda_{\max} \sum_{i=1}^{n} w_i = \lambda_{\max}$. In other words we obtain $\lambda_{\max}$ as the scalar product of the vector w with the vector of column sums of the matrix A. If the matrix has real coefficient that are positive, and if w is real and positive then $\lambda_{\max}$ is real and positive. But, we have not yet established that it is a simple eigenvalue and that it dominates all other eigenvalues in modulus.

We follow two routes to obtain the principal eigenvector w: one is by using the general idea of perturbation of the coefficients of a consistent matrix, which also involves reciprocal values in the transpose position, and the other is by considering the graph-theoretic concept of dominance along paths of different lengths leading to Cesaro summability. For us, applied to AHP reciprocal matrices, Cesaro summability says that the average value of the normalized vector of the row sums of the powers of a positive reciprocal matrix is equal to the normalized vector of the row sums of the limiting power of that matrix, which, of course, according to Perron, is the principal eigenvector of that matrix. The latter again gives the same answer as the theory of Perron does without the need for Perron's logic, whether that matrix is consistent (trivial because $A^k = n^{k-1}A$) or inconsistent, by perturbation arguments from the work of J. H. Wilkinson that it yields the principal eigenvalue and eigenvector in the limit.

2. Consistent Positive Reciprocal Matrices

Assume that one is given *n* stones, $A_1,\ldots,A_n$, that we will refer to as alternatives, having known weights $w_i,\ldots,w_n$, respectively. We form a matrix A of pairwise ratios whose *i*th row gives the ratios of the weights of the *i*th stone with respect to all the others.

$$A = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_n \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & \cdots & A_n \end{array} \\ \begin{bmatrix} w_1/w_1 & w_1/w_2 & \cdots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & \cdots & w_2/w_n \\ \vdots & \vdots & \cdots & \vdots \\ w_n/w_1 & w_n/w_2 & \cdots & w_n/w_n \end{bmatrix} \end{array}$$

We note that we can recover the vector of weights $w = w_i,\ldots,w_n$ by solving the system of equations defined below:

$$Aw = \begin{bmatrix} w_1/w_1 & w_1/w_2 & \cdots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & \cdots & w_2/w_n \\ \vdots & \vdots & \cdots & \vdots \\ w_n/w_1 & w_n/w_2 & \cdots & w_n/w_n \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = n \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = nw$$

Solving this homogeneous system of linear equations $Aw = nw$ to find w is a trivial eigenvalue problem, because the existence of a solution depends on whether or not a solution exists if *n* is an eigenvalue of the characteristic equation of *A*. But the matrix *A* has rank one and thus all its eigenvalues but one are equal to zero. The sum of the eigenvalues of a matrix is equal to its trace, the sum of its diagonal elements, which in this case is equal to *n*. Thus *n* is the largest, or the principal, eigenvalue of A and w is its corresponding principal eigenvector and it is positive and unique to within multiplication by a constant, and thus belongs to a ratio scale. We now know what must be done to recover the weights w_i, whether they are known in advance or not.

Definition: An *n* by *n* matrix $A = (a_{ij})$ is consistent if $a_{ij}a_{jk} = a_{ik}\ i, j, k = 1,\ldots,n$ holds among its entries. A consistent matrix always has the form $A = (\frac{w_i}{w_j})$ and we know that $A^k = n^{k-1}A$.

The consistent case has no need for the theorem of Perron to prove the existence of a largest real eigenvalue and its corresponding positive eigenvector, nor to prove that this vector is the limit to which powers of the matrix converge. Of course, real world reciprocal pairwise comparison matrices are very unlikely to be consistent unless they use actual measurement data.

Now, we give a mathematical discussion to show why when a matrix is inconsistent we still need the principal right eigenvector for our priority vector. It is clear that no matter what method we use to derive the weights w_i, we need to get them back so they are proportional to the expression $\sum_{j=1}^{n} a_{ij} w_j \quad i = 1,...,n$, that is, we must solve $\sum_{j=1}^{n} a_{ij} w_j = cw_i$ for $i = 1,...,n$. Otherwise, $\sum_{j=1}^{n} a_{ij} w_j$ for $i = 1,...,n$ would yield another set of different weights and they in turn could be used to form new expressions $\sum_{j=1}^{n} a_{ij} w_j \quad i = 1,...,n$, and so on ad infinitum. Unless we solve the principal eigenvalue problem, our quest for priorities becomes meaningless.

We learn from the consistent case that what we get on the right is proportional to the sum on the left that involves the same scale used to weight the judgments that we are looking for. Thus we have the proportionality constant c. A better way to see this is to use the derived vector of priorities to weight each row of the matrix and take the sum. This yields a new vector of priorities (relative dominance of each element) represented in the comparisons. This vector can again be used to weight the rows and obtain still another vector of priorities. In the limit (if one exists), the limit vector itself can be used to weight the rows and get the limit vector back perhaps proportionately. Our general problem, possibly with inconsistent judgments, takes the form:

$$Aw = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ \frac{1}{a_{12}} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{a_{1n}} & \frac{1}{a_{2n}} & \cdots & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = cw$$

This homogeneous system of linear equations $Aw = cw$ has a solution w if c is the principal eigenvalue of A. That this is the case can be shown using an argument that involves both left and right eigenvectors of A. Two vectors $x = (x_1,...,x_n)$, $y = (y_1,...,y_n)$ are orthogonal if their scalar product $x_1 y_1 + ... + x_n y_n$ is equal to zero. It is known that any left eigenvector of a matrix corresponding to an eigenvalue is orthogonal to any right eigenvector corresponding to a different eigenvalue. This property is known as biorthogonality.

Theorem 1: For a given positive matrix A, the only positive vector w and only positive constant c that satisfy $Aw = cw$, is a vector w that is a positive multiple of the principal eigenvector of A, and that the only such c is the principal eigenvalue of A.

Proof: We know that the right principal eigenvector and the principal eigenvalue satisfy our requirements. We also know that the algebraic multiplicity of the principal eigenvalue is one, and that there is a positive left eigenvector of A (call it z) corresponding to the principal eigenvalue. Suppose there is a positive vector y and a (necessarily positive) scalar d such that $Ay = dy$. If d and c are not equal, then by biorthogonality y is orthogonal to z, which is impossible since both vectors are positive. If d and c are equal, then y and w are dependent since c has algebraic multiplicity one, and y is a positive multiple of w. Thus the proof is complete.

Let a_{ij} be the relative dominance of A_i over A_j. In order to simplify the notation let the matrix corresponding to the reciprocal pairwise relation be denoted by $A = (a_{ij})$. The relative dominance of A_i over A_j along paths of length k is given by

$$\frac{\sum_{j=1}^{n} a_{ij}^{(k)}}{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^{(k)}}$$

where $a_{ij}^{(k)}$ is the (i,j) entry of the k^{th} power of the matrix (a_{ij}). The total dominance $w(A_i)$, of alternative i over all other alternatives along paths of all lengths is given by the infinite series

$$w(A_i) = \sum_{k=1}^{\infty} \frac{\sum_{j=1}^{n} a_{ij}^{(k)}}{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^{(k)}}$$

which coincides with the Cesaro sum in (Equation 1).

(Equation 1)

$$\lim_{x \to \infty} \frac{1}{M} \sum_{k=1}^{M} \frac{\sum_{j=1}^{n} a_{ij}^{(k)}}{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^{(k)}}$$

But this limit of weighted averages (the Cesaro sum) can be evaluated; we have for an n by n consistent matrix $A=(a_{ij})$ where $(a_{ij})=(\frac{w_i}{w_j})$ $i,j=1,\ldots n$, that $A^k = n^{k-1}A$ and the limit in Equation 1 is simply the eigenvector w normalized. In general, it is also true that the Cesaro sum converges to the same limit as does its k^{th} term $A^k e/e^T A^k e$ that yields k step dominance.

Here we see that the requirement for rank takes on the particular form of the principal eigenvector. We will not assume it for the inconsistent case, but will prove its necessity again for that more general case.

We now develop a necessary and sufficient condition for rank preservation. For emphasis, recall from graph theory that an element $a_{ij}^{(m)}$ of A^m gives the cumulative dominance of the ith element over the jth element along all chains of length m. That is precisely how one measures the consistency relation between that row and each column. In fact, when A is consistent we have from $A^m = n^{m-1}A$ that the entries of A^m and those of A differ by a constant thus maintaining consistency. In general we have $A^m = (a_{ij}^{(m)})$.

Theorem 2: *For a positive reciprocal matrix* A

$$\lim_{m\to\infty}\frac{a_{ik}^{(m)}}{\sum_{i-1}^{n} a_{ik}^{(m)}} = \lim_{m\to\infty}\frac{a_{is}^{(m)}}{\sum_{i-1}^{n} a_{is}^{(m)}}, \qquad k,s = 1,2,\ldots,n.$$

Proof. Let $B = NAN^{-1}$ (where N and N^{-1} are non-singular matrices that we will define later) be the Jordan canonical form of A given by:

$$B = \begin{bmatrix} \lambda_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_r \end{bmatrix},$$

where $\lambda_1 \equiv \lambda_{max}$, and B_p , $p = 2,3,\ldots,r$ is the $n_p \times n_p$ Jordan block defined by

$$B_p = \begin{bmatrix} \lambda_p & 0 & 0 & \cdots & 0 & 0 \\ 1 & \lambda_p & 0 & & 0 & 0 \\ 0 & 1 & \lambda_p & & 0 & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \lambda_p \end{bmatrix},$$

where λ_p , $p = 2, \dots, r$ are distinct eigenvalues with multiplicities $n_2, \dots, n_r$ respectively, and $\sum_{p=2}^{r} n_p = n - 1$. We have $A = N^{-1}BN$ and $A^m = N^{-1}B^m B$, where B^m is given by:

$$B^m = \begin{bmatrix} \lambda_1^m & 0 & \cdots & 0 \\ 0 & B_2^m & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & B_r^m \end{bmatrix},$$

Let us denote $N^{-1} \equiv D \; = (d_{ij})$ and $N = (n_{ij})$. We have

$$A^m DB^m N = \begin{bmatrix} n_{11}d_{11}\lambda_1^m + & \cdots & , & n_{12}d_{11}\lambda_1^m + & \cdots & , \dots, & n_{1n}d_{11}\lambda_1^m + & \cdots \\ n_{11}d_{21}\lambda_1^m + & \cdots & , & n_{12}d_{21}\lambda_1^m + & \cdots & , \dots, & n_{1n}d_{21}\lambda_1^m + & \cdots \\ & & \vdots & & \ddots & & \vdots & \\ n_{11}d_{n1}\lambda_1^m + & \cdots & , & n_{12}d_{n1}\lambda_1^m + & \cdots & , \dots, & n_{1n}d_{n1}\lambda_1^m + & \cdots \end{bmatrix}$$

Let $e = (1, 1, \dots, 1)^T = a_1 w_1 + \cdots + a_r w_r$, where w_p is the principal right eigenvector corresponding to λ_p. We have,

$$e^T A^m = \; a_1\lambda_1^m w_1^T + \cdots + a_r \lambda_1^m w_1^T = \left(n_{11} \sum_{i=1}^{n} d_{11}\lambda_1^m + \cdots, \dots, n_{1n} \sum_{i=1}^{n} d_{11}\lambda_1^m + \cdots \right).$$

Given two columns of A, k and s we have,

$$\frac{a_{ik}^{(m)}}{\sum_{i=1}^{n} a_{ik}^{(m)}} = \frac{n_{1k}d_{i1}\lambda_1^m + \cdots}{n_{1k}\sum_{i=1}^{n} d_{i1}\lambda_1^m + \cdots} \qquad \text{and} \qquad \frac{a_{is}^{(m)}}{\sum_{i=1}^{n} a_{is}^{(m)}} = \frac{n_{1s}d_{i1}\lambda_1^m + \cdots}{n_{1s}\sum_{i=1}^{n} d_{i1}\lambda_1^m + \cdots}$$

Since both numerators and denominators are polynomials in $\lambda_p^m, p = 1, 2, \dots, r$, and $\lambda_1 = \; \lambda_{max} > |\lambda_p|, p \; \neq 1$, we have for the *i*th entries of two arbitrary columns *k* and *s*:

$$\lim_{m\to\infty} \frac{a_{ik}^{(m)}}{\sum_{i=1}^{n} a_{ik}^{(m)}} = \lim_{m\to\infty} \frac{a_{is}^{(m)}}{\sum_{i-1}^{n} a_{is}^{(m)}} = \frac{d_{i1}}{\sum_{i-1}^{n} d_{i1}}.$$

Definition: *A positive matrix A is said to be m-dominant if there exists* m_0 *such that* for $m \geqslant \; m_n$ either $a_{ik}^{(m)} \geqslant \; a_{i\prime k}^{(m)}$ or $a_{ik}^{(m)} \leqslant \; a_{i\prime k}^{(m)}$ for all *k* and for any pair *i* and *i'*.

Corollary: *A positive reciprocal matrix is asymptotically m-dominant.*

Proof. We have from Theorem 2 that the normalized columns of A^m are the same in the limit. Since the elements in each row are identical, the result follows by choosing m_0 to be the maximum of its values for each pair of rows.

We now show that the rank of an inconsistent matrix A is determined in terms of the powers of A. To do this, we demonstrate that there is a method of estimating w which coincides with the normalized limiting columns of A. This method is precisely the eigenvalue method.

Theorem 3: $\lim_{m\to\infty}\left(\frac{a_{ik}^{(m)}}{\sum_{i=1}^{n} a_{ik}^{(m)}}\right) = w_i,\ i = 1,2,\dots,n.$

Proof: From $\lim_{m\to\infty}\left(\frac{A^m e}{||A^m||}\right) = w$, we have $w_i = \lim_{m\to\infty}\left(\frac{1}{||A^m||}\right)\sum_{k=1}^{n} a_{ik}^{(m)}$.

Multiplying and dividing $a_{ik}^{(m)}$ by $\sum_{k=1}^{n} a_{ik}^{(m)}$, and then re-grouping the terms, we have, on distributing the limit with respect to the finite sum,

$$w_i = \sum_{k=1}^{n} \lim_{m\to\infty}\left[\frac{a_{ik}^{(m)}}{||A^m||},\frac{\sum_{i=1}^{n} a_{ik}^{(m)}}{\sum_{i=1}^{n} a_{ik}^{(m)}}\right] = \sum_{k=1}^{n}\left[\lim_{m\to\infty}\frac{a_{ik}^{(m)}}{\sum_{i=1}^{n} a_{ik}^{(m)}}\right]\left[\lim_{m\to\infty}\frac{\sum_{i=1}^{n} a_{ik}^{(m)}}{||A^m||}\right].$$

By Theorem 2 $\lim_{m\to\infty}\frac{a_{ik}^{(m)}}{\sum_{i=1}^{n} a_{ik}^{(m)}}$ is the same constant for all k hence we have

$$w_i = \lim_{m\to\infty}\frac{a_{ik}^{(m)}}{\sum_{i=1}^{n} a_{ik}^{(m)}}\sum_{k=1}^{n}\left[\lim_{m\to\infty}\frac{\sum_{i=1}^{n} a_{ik}^{(m)}}{||A^m||}\right].$$

Since $||A^m|| = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}^{(m)}$, the proof is complete.

There is a natural way to derive the rank order of a set of alternatives from a pairwise comparison matrix A. The rank order of each alternative is the relative proportion of its dominance over the other alternatives. This is obtained by adding the elements in each row in A and dividing by the total. However, A only captures the dominance of one alternative over each other in one step. But an alternative can dominate a second by first dominating a third alternative, and then the third dominates the second. Thus, the first alternative dominates the second in two steps (along a path of length two). It is known that the result for dominance in two steps is obtained by squaring the pairwise comparison matrix. Similarly dominance can occur in three steps, four steps and so on, the value of each obtained by raising the matrix to the corresponding power. The rank order of an alternative is the average of the relative values for dominance in one step, two steps, and so on. We show below that when we take this infinite series of dominance along paths of length one, two, three, and so on and calculate its limiting value we obtain precisely the principal right eigenvector of the matrix A. This demonstrates that the eigenvector is derived deductively to obtain a relative scale among n alternatives from their matrix of pairwise comparisons. It is the desired solution because it preserves rank order rather than because it is a convenient criterion introduced for minimization purposes.

We have Theorem 4 below from Saaty and Vargas (1984).

Theorem 4: *The relative dominance of an alternative is given by the solution of the eigenvalue problem* $Aw = \lambda_{max} w$.

Proof: The relative dominance of an alternative along all paths of length $k \leqslant m$ is given by $\frac{1}{m}\sum_{k=1}^{m}\frac{A^k e}{e^T A^k e}$.

Let $s_k = \frac{A^k e}{e^T A^k e}$ and $t_m = \frac{1}{m}\sum_{k=1}^{m} s_k$.

Note that $\lim_{m\to\infty} t_m < \infty$. This a consequence of a theorem due to G. H. Hardy (1949) which gives the necessary and sufficient conditions for a transformation of a convergent sequence to also be convergent. Let T be such a transformation mapping

$$(s_1, \dots, s_m) \to t_m = \sum_{k=1}^{\infty} c_{m,k} s_k.$$

T is regular if $t_m \to s$ as $m \to \infty$ whenever $s_k \to s$ as k $\to \infty$. It is known (Hardy, 1949) that T is regular if and only if the following conditions hold:

(1) $\sum_{k=1}^{\infty}|c_{m,k}| < H$ (independent of m),
(2) $c_{m,k} \to \delta_k$ for each k, when $m \to \infty$,
(3) $\sum_{k=1}^{\infty} c_{m,k} \to \delta$ when $m \to \infty$,
(4) $\delta_k = 0$ for each k,
(5) $\delta = 1$.

Here,

$$c_{m,k} = \begin{cases} \dfrac{1}{m} & for\ 1 \leqslant k \leqslant m \\ 0 & for\ k > m. \end{cases},$$

Thus, we have

(1) $\sum_{k=1}^{\infty}|c_{m,k}| = \sum_{k=1}^{m}\left|\frac{1}{m}\right| = 1,$

(2) $c_{m,k} = \frac{1}{m} \to 0$ as $m \to \infty$. Hence (4) $\delta_k = 0$ for each k,
(3) $\sum_{k=1}^{\infty} c_{m,k} = \sum_{k=1}^{\infty} \left(\frac{1}{m}\right) = 1$ and hence (5) $\delta = 1$.

It follows that T is regular. Since $s_k = \frac{A^k e}{e^T A^k e} \to w$ as $k \to \infty$ (Saaty, 1980), where w is the principal right eigenvector of A we have,

$$t_m = \frac{1}{m} \sum_{k=1}^{m} \frac{A^k e}{e^T A^k e} \to w \text{ as } m \to \infty$$

In input/output analysis in economics, multipliers are traced by raising the input/output matrix to higher and higher powers and taking their sums to obtain the overall impact of each sector of the economy on every other sector.

Still, another argument can be constructed from Theorem 4 because for large m the normalized columns of A^m are the same and converge to the principal eigenvector.

Theorem 5: *The Cesaro sum,* $\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} \frac{A^k e}{e^T A^k e}$ *,is the principal right eigenvector of A.*

Proof: By Theorem 4 of Saaty and Vargas (1984) we know that,
$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} \frac{A^k e}{e^T A^k e} = \lim_{m\to\infty} \sum_{k=1}^{m} \frac{A^k e}{e^T A^k e}$$

Multiplying $\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} \frac{A^k e}{e^T A^k e}$ by A on the left we have,

$$A\left(\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} \frac{A^k e}{e^T A^k e}\right) = A\left(\lim_{k\to\infty} \frac{A^k e}{e^T A^k e}\right) = \left(\lim_{k\to\infty} \frac{e^T A^{k+1} e}{e^T A^k e}\right)\left(\lim_{k\to\infty} \frac{A^{k+1} e}{e^T A^{k+1} e}\right)$$

$$= \left(\lim_{k\to\infty} \frac{e^T A^{k+1} e}{e^T A^k e}\right)\left(\lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} \frac{A^k e}{e^T A^k e}\right)$$

There is a vector $y = \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} \frac{A^k e}{e^T A^k e}$ and a constant $d = \lim_{k\to\infty} \frac{e^T A^{k+1} e}{e^T A^k e}$ such that $Ay = dy$. Under the assumption that A has r distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ with multiplicities $n_1, \ldots, n_r$, respectively, by using the Jordan canonical form of A we can write $A = N^{-1} B N$ where N is an invertible matrix and B is shown below:

$$B = \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_r \end{bmatrix}$$

and

$$B_p = \begin{bmatrix} \lambda_p & 0 & 0 & 0 & \cdots & 0 \\ 1 & \lambda_p & 0 & 0 & \cdots & 0 \\ 0 & 1 & \lambda_p & 0 & \cdots & 0 \\ 0 & 0 & 1 & \lambda_p & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \lambda_p \end{bmatrix}, p = 1, \dots, r.$$

We have:

$$A^k = N^{-1}B^kN, B^k = \begin{bmatrix} B_1^k & 0 & \cdots & 0 \\ 0 & B_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_r^k \end{bmatrix}$$

The matrix B_p^k is shown in Equation 2 below.

Letting $N^{-1} = D = (d_{ij})$ and $N = (n_{ij})$ we write:

$A^k = N^{-1}B^kN = DB^kN.$

The first n_1 columns of A^k are given by $D_{n_1}B_1^k$, as shown in Equation 3 below, and $D_{n_1+1,n_2}B_1^k$ as shown in Equation 4 below.

(Equation 2)

$$B_p^k = \begin{bmatrix} \lambda_p^k & 0 & 0 & 0 & \cdots & 0 \\ k\lambda_p^{k-1} & \lambda_p^k & 0 & 0 & \cdots & 0 \\ \frac{k(k-1)}{2!}\lambda_p^{k-2} & k\lambda_p^{k-1} & \lambda_p^k & 0 & \cdots & 0 \\ \frac{k(k-1)(k-2)}{3!}\lambda_p^{k-3} & \frac{k(k-1)}{2!}\lambda_p^{k-2} & k\lambda_p^{k-1} & \lambda_p^k & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \frac{k(k-1)\cdots(k-n_p+1)}{(n_p-1)!}\lambda_p^{k-n_p+1} & \frac{k(k-1)\cdots(k-n_p+2)}{(n_p-2)!}\lambda_p^{k-n_p+2} & \frac{k(k-1)\cdots(k-n_p+3)}{(n_p-3)!}\lambda_p^{k-n_p+3} & \cdots & k\lambda_p^{k-1} & \lambda_p^k \end{bmatrix}$$

(Equation 3)

$$D_{n_1}B_1^k = \left(\begin{bmatrix} d_{11}\lambda_1^k + d_{12}\binom{k}{1}\lambda_1^{k-1} + \cdots + d_{1n_1}\binom{k}{n_1-1}\lambda_1^{k-n_1+1} \\ d_{21}\lambda_1^k + d_{22}\binom{k}{1}\lambda_1^{k-1} + \cdots + d_{2n_1}\binom{k}{n_1-1}\lambda_1^{k-n_1+1} \\ \vdots \\ d_{n1}\lambda_1^k + d_{n2}\binom{k}{1}\lambda_1^{k-1} + \cdots + d_{nn_1}\binom{k}{n_1-1}\lambda_1^{k-n_1+1} \end{bmatrix} \begin{bmatrix} d_{12}\lambda_1^k + d_{13}\binom{k}{1}\lambda_1^{k-1} + \cdots + d_{1n_1}\binom{k}{n_1-2}\lambda_1^{k-n_1+2} \\ d_{22}\lambda_1^k + d_{23}\binom{k}{1}\lambda_1^{k-1} + \cdots + d_{2n_1}\binom{k}{n_1-2}\lambda_1^{k-n_1+2} \\ \vdots \\ d_{n2}\lambda_1^k + d_{n3}\binom{k}{1}\lambda_1^{k-1} + \cdots + d_{nn_1}\binom{k}{n_1-2}\lambda_1^{k-n_1+2} \end{bmatrix}, \dots, \begin{bmatrix} d_{1n_1}\lambda_1^k \\ d_{2n_1}\lambda_1^k \\ \vdots \\ d_{nn_1}\lambda_1^k \end{bmatrix} \right)$$

(Equation 4)

$D_{n_1+1,n_2}B_1^k =$

$$\left(\begin{bmatrix} d_{11}\lambda_1^k + d_{12}\binom{k}{1}\lambda_1^{k-1} + \cdots + d_{1n_1}\binom{k}{n_1-1}\lambda_1^{k-n_1+1} \\ d_{21}\lambda_1^k + d_{22}\binom{k}{1}\lambda_1^{k-1} + \cdots + d_{2n_1}\binom{k}{n_1-1}\lambda_1^{k-n_1+1} \\ \vdots \\ d_{n1}\lambda_1^k + d_{n2}\binom{k}{1}\lambda_1^{k-1} + \cdots + d_{nn_1}\binom{k}{n_1-1}\lambda_1^{k-n_1+1} \end{bmatrix} \begin{bmatrix} d_{12}\lambda_1^k + d_{13}\binom{k}{1}\lambda_1^{k-1} + \cdots + d_{1n_1}\binom{k}{n_1-2}\lambda_1^{k-n_1+2} \\ d_{22}\lambda_1^k + d_{23}\binom{k}{1}\lambda_1^{k-1} + \cdots + d_{2n_1}\binom{k}{n_1-2}\lambda_1^{k-n_1+2} \\ \vdots \\ d_{n2}\lambda_1^k + d_{n3}\binom{k}{1}\lambda_1^{k-1} + \cdots + d_{nn_1}\binom{k}{n_1-2}\lambda_1^{k-n_1+2} \end{bmatrix}, \dots, \begin{bmatrix} d_{1n_1}\lambda_1^k \\ d_{2n_1}\lambda_1^k \\ \vdots \\ d_{nn_1}\lambda_1^k \end{bmatrix}\right)$$

Thus, finally, we have:

$$A^k = \begin{bmatrix} \sum_{j=1}^{n_1} d_{1j}n_{j1}\lambda_1^k + \sum_{j=n_1+1}^{n_2} d_{1j}n_{j1}\lambda_2^k + \cdots & \cdots & \sum_{j=1}^{n_1} d_{1j}n_{jn}\lambda_1^k + \sum_{j=n_1+1}^{n_2} d_{1j}n_{jn}\lambda_2^k + \cdots \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^{n_1} d_{nj}n_{j1}\lambda_1^k + \sum_{j=n_1+1}^{n_2} d_{nj}n_{j1}\lambda_2^k + \cdots & \cdots & \sum_{j=1}^{n_1} d_{nj}n_{jn}\lambda_1^k + \sum_{j=n_1+1}^{n_2} d_{nj}n_{jn}\lambda_2^k + \cdots \end{bmatrix}.$$

Let us assume $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_r|$. Then we have the ratio:

(Equation 5)

$$\lim_{k\to\infty}\frac{e^T A^{k+1} e}{e^T A^k e} = \lim_{k\to\infty}\frac{\sum_{i,j=1}^{n_1} d_{ij} n_{ij}\lambda_1^{k+1} + \sum_{i,j=n_1+1}^{n_2} d_{ij} n_{ij}\lambda_2^{k+1} + \cdots}{\sum_{i,j=1}^{n_1} d_{ij} n_{ij}\lambda_1^{k} + \sum_{i,j=n_1+1}^{n_2} d_{ij} n_{ij}\lambda_2^{k} + \cdots}$$

Since λ_1 is the principal eigenvector of A and $Ay = \lambda_1 y$, then $y = \lim_{m\to\infty}\frac{1}{m}\sum_{k=1}^{m}\frac{A^k e}{e^T A^k e}$ is the principal right eigenvector of A.

The ratio given in Equation 5 shows how to calculate the principal eigenvalue without solving the characteristic equation or using sophisticated mathematical software. We apply it to the following inconsistent pairwise comparison matrix A:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1/2 & 1 & 5 & 6 \\ 1/3 & 1/5 & 1 & 7 \\ 1/4 & 1/6 & 1/7 & 1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 4 & 9 & 16 \\ 1/4 & 1 & 25 & 36 \\ 1/9 & 1/25 & 1 & 7 \\ 1/16 & 1/36 & 1/49 & 1 \end{pmatrix}$$

$$A^{10} = \begin{pmatrix} 923637. & 1.02324 \times 10^6 & 2.52682 \times 10^6 & 7.73336 \times 10^6 \\ 850356. & 941992. & 2.32613 \times 10^6 & 7.11986 \times 10^6 \\ 369456. & 409271. & 1.01053 \times 10^6 & 3.09298 \times 10^6 \\ 119539. & 132434. & 327011. & 1.00078 \times 10^6 \end{pmatrix}$$

$$A^{11} = \begin{pmatrix} 4.21087 \times 10^6 & 4.66478 \times 10^6 & 1.15187 \times 10^7 & 3.52551 \times 10^7 \\ 3.87669 \times 10^6 & 4.29457 \times 10^6 & 1.06043 \times 10^7 & 3.24561 \times 10^7 \\ 1.68418 \times 10^6 & 1.86579 \times 10^6 & 4.60711 \times 10^6 & 1.41002 \times 10^7 \\ 544954. & 603711. & 1.49077 \times 10^6 & 4.56261 \times 10^6 \end{pmatrix}$$

Computing the ratio below using the 10^{th} and 11^{th} power of the matrix A we obtain a first estimate of the eigenvalue.

$$\frac{e^T A^{11} e}{e^T A^{10} e} = \frac{136340459}{29907404} = 4.558753$$

Going all the way to the 20th and 21st powers of the matrix and forming the ratio as shown below gives us a second somewhat different estimate of the eigenvalue that is quite close to the value computed by a presumably quite accurate commercial mathematical software package.

$$\frac{e^T A^{21} e}{e^T A^{20} e} = \frac{5.28609 \times 10^{14}}{1.15953 \times 10^{14}} = 4.558805$$

The four eigenvalues of A, obtained from the commercial mathematical software, are shown below:

$$\lambda_1 = 4.558805319078529$$
$$\lambda_2 = -0.03996796194816203 + 1.583542975555991\ i$$
$$\lambda_3 = -0.03996796194816203 - 1.583542975555991\ i$$
$$\lambda_4 = -0.4788693951822055$$

We see that our final ratio has converged to the same value, 4.558805, to six decimal places, and we arrived at it using our method of raising the matrix to a sufficiently high power and computing the eigenvalue as a ratio of two successive powers.

3. Conclusion

So, in conclusion, we have shown that for AHP pairwise comparison matrices, which are positive and reciprocal, we do not need to use the beautiful and general theorem of Perron.

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