PRIORITY EIGENVECTORS IN ANALYTIC HIERARCHY/NETWORK PROCESSES WITH OUTER DEPENDENCE BETWEEN ALTERNATIVES AND CRITERIA

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ABSTRACT

This work continues consideration of the relations between Analytic Hierarchy and Network processes. It shows that in the case of a simple network with outer dependence between alternatives and criteria, the priority vectors can be constructed not only via the powered supermatrix but also by the eigenvectors of the supermatrix. The relationship to the AHP least squares approach and other methods of priority estimation are considered as well. An ANP matrix of local eigenvectors includes priorities for all the compared items in the whole network. Here we interpret the complex AHP/ANP connections and show clearly how they result in priority estimations useful for applied decision making.

Keywords: Analytic Hierarchy/Network Process, Outer Dependence, Supermatrix' Eigenvectors.
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"*You can choose where to spend eternity*" - Thomas Saaty,

The Thinking Man's New Millennium Joke Book

1. Introduction

This paper continues the consideration of the relationship between the Analytic Hierarchy Process and the Analytic Network Process started in two previous papers in this journal (Lipovetsky, 2010, 2011). In the current article we consider a simple network with outer dependence between the alternatives compared with respect to the criteria on the one hand and the criteria compared with respect to each of the alternatives on the other hand.

We can show that in this case the priority vectors can be constructed not only via the powered supermatrix approach but also by determining the eigenvectors of the ANP matrix. The relations to the AHP least squares approach and other methods of priority estimation are considered. It is noted that this problem can be related to the system of the so called Geary-Khamis equations known in international statistics for comparing currencies and volumes of various goods produced by different countries (Geary 1958, Khamis 1972, Kravis et al. 1975, Rao and Selvanathan 1992). A similar approach has also been used for finding preferences evaluated both by ranks and rates in marketing research (Lipovetsky 2007).

Starting with the ANP matrix of local eigenvectors, the eigenvectors of the matrix itself give the priorities for all the compared items in the whole network.

2. ANP solution and interpretation

Let us consider the example of the simple network with outer dependence between alternatives compared with respect to the criteria on one hand, and the criteria compared for each of the alternatives on the other hand. For an explicit illustration, we will use a network system with feedback given in Saaty's book (1994) where various water levels in a dam (low, middle, or high) are evaluated for best serving the purposes of flood control, recreation, or power generation. Table 1 shows the local eigenvectors for the three water levels compared with respect to each of the purposes of the dam in the lower left block, and for the purposes compared with respect to each of the levels in the upper right block. The levels might be considered to be the alternatives in this example as the model is for choosing the best one, and the purposes of the dam are the criteria.

Table 1
Local priority vectors for "Management of water reservoir" (Saaty, 1994)

Compared characteristics	Flood Control	Recreation	Electric Power	Low Dam	Middle Dam	High Dam		Overall Priority
Flood Control	0	0	0	.637	.200	.060		.241
Recreation	0	0	0	.258	.600	.231		.374
Electric Power	0	0	0	.105	.200	.709		.385
Low Dam	.722	.072	.058	0	0	0		.223
Middle Dam	.205	.649	.207	0	0	0		.372
High Dam	.073	.279	.735	0	0	0		.405

The last column at the right in Table 1 presents the overall priority vector of the purposes of the dam, and the levels obtained using the standard ANP process of raising the matrix of local eigenvectors to powers until it converges. That is, the elements of the overall priority vectors obtained by summing each row and normalizing the results are sufficiently close for two successive powers of the matrix. Note that the local eigenvectors, the criteria part of the overall vector and the alternatives part of the overall vector are normalized to one; that is, the totals equal one in every case.

Consider such an outer dependent model for the general case of m alternatives and n criteria. Let the local eigenvectors of alternatives and of criteria be stacked into two matrices A and B, of the order m by n, and n by m, respectively. The supermatrix consists of the non-diagonal blocks A (vectors of the alternatives under each criterion) and B (vectors of the criteria compared for each alternative), and two diagonal zero-blocks of n-th and m-th orders, as shown in Saaty (1994):

$$\begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix} \tag{1}$$

Instead of raising the matrix to powers to find the overall priority vector, the same solution can be obtained by solving the eigenproblem for the combined vector $(\alpha \quad \beta)'$ of criteria and alternatives overall priorities:

$$\begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{2}$$

Multiplying the supermatrix in (1) by the equation (2) yields the following eigenproblem for the squared supermatrix:

$$\begin{pmatrix} BA & 0 \\ 0 & AB \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda^2 \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{3}$$

where *BA* and *AB* are square blocks of *n*-th and *m*-th order, respectively.

In the supermatrices in both (2) and (3) each column equals one, so they are column-stochastic matrices. The main eigenvectors in (2) have eigenvalues with an absolute value of one. The eigenproblems (2) and (3) produce essentially the same main eigenvectors as becomes clear after the overall priorities for the criteria and alternatives are normalized to one. The supermatrix in Table 1 gives the main solutions of the eigenproblem (2) and (3) shown in Table 2.

Table 2
Local and global priorities obtained as the main eigenvectors for the supermatrix in the "Management of water reservoir" example

	Supermatrix Eigenproblem (2)				Supermatrix Eigenproblem (3)			
	Original Eigenvectors		Each Part Normalized		Original Eigenvectors		Each Part Normalized	
Compared characteristics	vector 1	vector 2	vector 1	vector 2	vector 1	vector 2	vector 1	vector 2
Flood control	.393	.385	.241	.241	0	.497		.241
Recreation	.611	.598	.374	.374	0	.773		.374
Electric power	.628	.615	.385	.385	0	.794		.385
Low Dam	-.364	.357	.223	.223	.476	0	.223	
Middle Dam	-.607	.595	.372	.372	.794	0	.372	
High Dam	-.661	.647	.405	.405	.864	0	.405	
Eigenvalue λ	-1	1			1	1		

The original eigenvectors (2) and (3) in Table 2 seem to be rather different, but after normalizing to one each segment of the vector separately for the alternatives and criteria, the results obtained from Table 2 coincide with the overall priority shown in the Table 1.

Let us consider how to interpret the eigenproblem solutions for equations (2) and (3). The matrix equation in (2) can be represented by the two equations:

$$B\beta = \lambda\alpha\,, \quad A\alpha = \lambda\beta \tag{4}$$

Substituting one of these equations into another yields two separate eigenproblems:

$$BA\alpha = \lambda^2\alpha\,, \quad AB\beta = \lambda^2\beta \tag{5}$$

which can be expressed by the combined eigenproblem in (3). All the matrices in equations (4) and (5) are column-stochastic, so the absolute value of all the main eigenvalues equals one. Such kinds of equations had been also considered in relation to Bayes theorem and independence effects in a network in Saaty (1994), and for the multidimensional matrices eigenproblem in Lipovetsky and Tishler's article (1994).

The relations (4) show clearly that using an initial identity vectors leads to the next approximation for vector β as the average of the elements in the rows of the *A* matrix of alternatives, and for α the average of the elements in the rows of the *B* matrix of criteria. Eventually the converged solution for the system (4), or the solution for the supermatrix (2), can be interpreted explicitly as follows: **the vector of alternative priorities β is the average of the alternatives with weights of the mean criteria;** and vice versa **– the vector of criteria priorities α is the average of the criteria with weights of the mean alternatives.**

This simple and evident interpretation for a supermatrix solution can be completed by the following observation. For the matrices *A* and *B* of the alternatives and criteria priority vectors (as in Table 1 or in the problem (2)) consider a system of the following equations:

$$\beta_k = \frac{\frac{1}{n}\sum_{j=1}^{n} A_{kj}\alpha_j}{\frac{1}{n}\sum_{i=1}^{n} B_{ik}}\,, \qquad \alpha_q = \frac{\frac{1}{m}\sum_{j=1}^{m} B_{qj}\beta_j}{\frac{1}{m}\sum_{i=1}^{m} A_{iq}}\,. \tag{6}$$

The elements β_k of the alternatives' preference are defined by the quotient of the alternatives averaged with weights of the criteria (the first numerator) and the mean level of each alternative in the criteria matrix (in the first denominator). Similarly, the elements α_q of the criteria preference are defined by the quotient of the criteria averaged with weights of the alternatives (the second numerator) and the mean level of each criterion in the alternatives matrix (in the second denominator). All the local priority vectors are normalized so their total equals one, thus the denominators in the formulae (6) equal 1/*n* or 1/*m*, respectively. Then, in matrix form, the relations (6) are similar to the equations in (4).

It is interesting to note that the system (6) of mutual weighting of two data matrices is known from the so- called Geary-Khamis equations used in international statistics for comparisons of the currencies and volumes of various goods produced by different countries (see the references in the Introduction). For transposed stochastic matrices connecting both vectors, the main eigenvalues equal one and the corresponding vectors give the shares of priorities.

Another AHP technique can be mentioned in relation to the priority estimation in equations (2) to (5) for the ANP with outer dependence. It is based on the least squares (LS) approximation method of finding the priority vector solution of a matrix. Let us assume there is such a solution, and name it α_i. Since such a vector exists, we can form a consistent matrix from ratios of its entries. Since it is a consistent matrix we have not only the priority vector α_i (which can be obtained in a number of equivalent ways: summing across the rows and normalizing the result, summing down the columns and normalizing the result, using least squares, or by finding its principal eigenvector) but also an anti-priority vector β_i which we can obtain from the vector α_i by inverting it. To invert a priority vector replace each entry α_i by $1/\alpha_i$, sum the entries and divide each entry by the sum. Since we have a consistent matrix the assumed solution vector α_i is a right eigenvector and the anti-priority vector β_i is a left eigenvector. These are still unknown vectors, so we need to solve for them, and we shall do this using the method of least squares.

The least squares approximation of the matrix of priority vectors in (1) by two vectors can be written as follows:

$$LS = \sum_{ij} (a_{ij} - \lambda \alpha_i \beta_j)^2 \,, \tag{7}$$

where α is a vector of priorities, β can be understood as a vector of reciprocal priorities, or anti-priorities, and their outer product corresponds to pairwise ratio elements of the matrix (1). Differentiating the LS objective in (7) by the vectors' elements yields a system of equations:

$$\frac{\partial LS}{\partial \alpha_i} = (-2\lambda) \sum_j (a_{ij} - \lambda \alpha_i \beta_j) \beta_j = 0, \quad \frac{\partial LS}{\partial \beta_j} = (-2\lambda) \sum_i (a_{ij} - \lambda \alpha_i \beta_j) \alpha_i = 0 \,. \tag{8}$$

Normalizing the eigenvectors by their Euclidean norm,

$$\sum_j \beta_j^2 = 1, \quad \sum_i \alpha_i^2 = 1 \tag{9}$$

we can represent the system (8) as follows:

$$\sum_j a_{ij} \beta_j = \lambda \alpha_i \,, \quad \sum_i a_{ij} \alpha_i = \lambda \beta_j \,. \tag{10}$$

In matrix form the equations (10) are:

$$A\beta = \lambda \alpha \,, \quad A' \alpha = \lambda \beta \tag{11}$$

and substituting one of these relations into another yields two eigenproblems:

$$AA'\alpha = \lambda^2\alpha\,, \quad A'A\beta = \lambda^2\beta \tag{12}$$

The vectors corresponding to the maximum eigenvalue λ^2 define the priorities α and anti-priorities β of the compared items. The equations in (7) to (12) describe the singular value decomposition (SVD) for the matrix (Eckart and Young 1936; Lipovetsky and Tishler 1994; Gass and Rapcsak 2004). The system of equations in (11) can be represented in a block supermatrix

$$\begin{pmatrix} 0 & A \\ A' & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{13}$$

with the combined α and β vector. Multiplying the matrix in (13) by this relation itself yields the eigenproblem:

$$\begin{pmatrix} AA' & 0 \\ 0 & A'A \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda^2\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{14}$$

which represents the equations in (12) in one combined matrix form.

Comparison of the AHP problem given in (14) with the ANP problem given in (3) shows that the ANP is a generalization of the AHP problem to the case of two matrices in place of only one. But essentially both problems produce the same interpretation: **the priority vectors represent the weighted averages of each item giving its preference over the other items under consideration.**

4. Summary

This work has shown that the priority vector for a simple network with outer dependence between the alternatives and the criteria can be represented in the standard framework of the supermatrix eigenproblem solution which has the main eigenvectors related to the largest by modulo eigenvalues. The relations with other methods are also considered such as the least squares approach applied in AHP and the system so-called Geary-Khamis equations used in international statistics for currency rates, and for finding preferences evaluated both by ranks and ratios. The elements of the ANP priority eigenvector have the same meaning as they do in the AHP – they give the mean preferences of each alternative or criterion (that is, element in the problem) over all the others involved in pairwise comparisons. The interpretation of AHP and ANP priority vectors as representing the mean prevalence among the compared elements is useful for practical purposes by managers and decision makers.

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