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CYCLIC CONTRACTION ON S- METRIC SPACE

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Abstract. In this paper we introduced the concepts of cyclic contraction on S- metric space and proved some fixed point theorems on S- metric space. Our presented results are proper generalization of Sedghi et al. [14]. We also give an example in support of our theorem.

1. Introduction and preliminaries

Metric space is one of the most useful and important space in mathematics. Its wide area provides a powerful tool to the study of variational inequalities, optimization and approximation theory, computer sciences and so many. Recently the study of fixed point theory in metric space is very interesting and attract many researchers to investigated different results on it.

On the other hand, some authors are interested and have tried to give generalizations of metric spaces in different ways. In 1963 Gahler [3] gave the concepts of $2-$ metric space further in 1992 Dhage [2] modified the concept of $2-$ metric space and introduced the concepts of $D-$ metric space but in 2005 Mustafa and Sims [4] pointed out that these attempts are not valid and introduced the concepts of $G-$ metric space and proved fixed point theorems in $G-$ metric space. Many authors proved different fixed point theorems in $G-$ metric space in different ways see in [13] and references theirin. Sedghi et al. [12] modified the concepts of $D-$ metric space and introduced the concepts of D^*- metric space also proved a common fixed point theorems in D^*- metric space.

Recently, Sedghi et al [14] introduced the concept of S- metric space which is different from other space and proved fixed point theorems in S-metric space. They also gives some examples of S- metric spaces which shows that S- metric space is different form other spaces. In fact they gives following concepts of S- metric space.

Definition 1. *Let X be a nonempty set. An S- metric space on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,*

(1) $S(x, y, z) \geq 0$,
(2) $S(x, y, z) = 0$ *if and only if* $x = y = z$,
(3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an S- metric space.

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Examples of such S - metric space are as follows,

Example 2. *Let $X = \mathbb{R}^n$ and $\parallel . \parallel$ a norm on X, then $S(x, y, z) =\parallel y + z - 2x \parallel + \parallel y - z \parallel$ is an S- metric on X.*

Example 3. *Let $X = \mathbb{R}^n$ and $\parallel . \parallel$ a norm on X, then $S(x, y, z) =\parallel x - z \parallel + \parallel y - z \parallel$ is an S- metric on X.*

Example 4. *Let X be a nonempty set, d is ordinary metric on X, then $S(x, y, z) = d(x, z) + d(y, z)$ is an S- metric on X.*

Lemma 5. *Let (X, S) be an S- metric space, then we have,*

$$S(x, x, y) = S(y, y, x)$$

.

Proof. By the third condition of S- metric, we have

$$S(x, x, y) \leq S(x, x, x) + S(x, x, x) + S(y, y, x)$$

and similarly

$$S(y, y, x) \leq S(y, y, y) + S(y, y, y) + S(x, x, y)$$

which implies that

$$S(x, x, y) = S(y, y, x)$$

. □

Definition 6. *Let (X, S) be an S- metric space.*

(1) *A sequence $\{x_n\}$ in X is said to be converges to x if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is for each $\epsilon > 0$ there exists $n_0 \in N$ such that for all $n \geq n_0$, $S(x_n, x_n, x) < \epsilon$ and we denote this by $lim_{n\to\infty} x_n = x$.*
(2) *A sequence $\{x_n\}$ in X is said to be Cauchy sequence if and only if $S(x_n, x_m, x) \to 0$ as $n, m \to \infty$. That is for each $\epsilon > 0$ there exists $n_0 \in N$ such that for all $n, m \geq n_0$, $S(x_n, x_m, x) < \epsilon$.*

Definition 7. *The S- metric space (X, S) is said to be complete if every Cauchy sequence is convergent.*

Every S- metric on X defines a metric d_S on X by

$$d_S(x, y) = S(x, x, y) + S(y, y, x) \quad \forall x, y \in X. \tag{1.1}$$

Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exists $r > 0$ such that $B_S(x, r) \subset A$. Then τ is a topology on X. Also, nonempty subset A in the S-metric space (X, S) is S- closed if $\bar{A} = A$.

Lemma 8. *Let (X, S) be a S- metric space and A is a nonempty subset of X. A is said S- closed iof for any sequence $\{x_n\}$ is A such that $x_n \to x$ as $n \to \infty$, then $x \in A$.*

2. Main results

In this article we introduce the concept of cyclic contraction in S- metric space and proved some fixed point theorems in S- metric space.

Definition 9. *Denote by Φ the set of functions $\phi : [0,\infty) \to [0,\infty)$ satisfying,*

(1) *ϕ is non-decreasing,*
(2) *there exist $k_0 \in N$, $a \in (0,1)$ and a convergent series of nonnegative terms $\Sigma_{k=1}^{\infty} v_k$ such that*
$$\phi^{k+1}(t) \leq a\phi^k(t) + v_k$$
for $k \geq k_0$ and any $t > 0$.

Then $\phi \in \Phi$ is called a (c)- comparison function.

Lemma 10. *If $\phi \in \Phi$, then the following properties hold:*

(1) *$(\phi^n(t))_{n\in N}$ converges to 0 as $n \to \infty$, for all $t > 0$,*
(2) *$\phi(t) < t$ for any $t > 0$,*
(3) *ϕ is continuous at 0,*
(4) *the series $\Sigma_{k=0}^{\infty} \phi^k(t)$ converges for any $t > 0$.*

Lemma 11. *If $\phi \in \Phi$, then the function $p : (0,\infty) \to (0,\infty)$ defined by*
$$p(t) = \Sigma_{k=0}^{\infty} \phi^k(t), \quad t > 0, \tag{2.1}$$
is non decreasing and continuous at 0.

First, we consider the Picard iteration $\{x_n\}$ defined by
$$x_{n+1} = Tx_n, \quad \forall n \geq 0. \tag{2.2}$$

Our first result is the following.

Theorem 12. *Let (X,S) be a S- complete S- metric space. Let $\{A_i\}_{iC1}^{m}$ be a family of non empty S- closed subsets of X, m a positive integer and $Y = \cup_{i=i}^{m} A_i$. Let $T : Y \to Y$ be a mapping such that*
$$T(A_i) \subseteq A_{i+1} \ \ \forall \quad i = 1,2....m \quad with \ \ A_{i+1} = A_i \tag{2.3}$$

Suppose also that there exists $\phi \in \Phi$ such that
$$S(Tx,Ty,Tz) \leq \phi(S(x,y,z)) \tag{2.4}$$

For all $(x,y,z) \in A_i \times A_i \times A_{i+1}$ for all $i = 1,2....m$. Then

(I) *T has a unique fixed point, say u, that belongs to $\cap_{i=i}^{m} A_i$,*
(II) *the following estimates hold:*
$$S(x_n,x_n,u) \leq p(\phi^n(S(x_0,x_0,x_1))), \quad n \geq 1, \tag{2.5}$$
$$S(x_n,x_n,u) \leq p(S(x_n,x_n,x_{n+1})), \quad n \geq 1, \tag{2.6}$$
(III) *for any $x \in Y$,*
$$S(x,x,u) \leq p(S(x,x,Tx)), \tag{2.7}$$
where p is given in 2.1 in Lemma 11.

Proof. Let $x_0 \in Y = \cup_{i=i}^{m} A_i,$. Without loss of generality, let $x_0 \in A_1$. Consider the Picard iteration $\{x_n\}$ defined by 2.2 and starting from x_0.

If for some integer k, $x_k = x_{k+1}$, so $\{x_n\}$ is constant for any $n \geq k$ then $\{x_n\}$ is S-Cauchy in (X, S).

Suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. For any $n \geq 0$, there us $i_n \in \{1, 2....m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_n+1}$. By 2.4, we have

$$S(x_{n+1}, x_{n+1}, x_{n+2}) = S(Tx_n, Tx_n, Tx_{n+1}) \leq \phi(S(x_n, x_n, x_{n+1})) \tag{2.8}$$

The function ϕ is non decreasing, so by induction

$$S(x_n, x_n, x_{n+1}) \leq \phi^n(S(x_0, x_0, x_1)) \forall \quad n \geq 0. \tag{2.9}$$

By rectangle inequality and 2.9 , for $r \geq 1$

$$\begin{aligned} S(x_n, x_n, x_{n+r}) &\leq S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_{n+2}) \\ &+ .... + S(x_{n+r-1}, x_{n+r-1}, x_{n+r}) \\ &\leq \phi^n(S(x_0, x_0, x_1)) + \phi^{n+1}(S(x_0, x_0, x_1)) \\ &+ .... + \phi^{n+r-1}(S(x_0, x_0, x_1)) \end{aligned}$$

Denote

$$\delta_n = \Sigma_{k=0}^{n} \phi^k(S(x_0, x_0, x_1)), \quad n \geq 0$$

Therefore

$$S(x_n, x_n, x_{n+r}) \leq \delta_{n+p-1} - \delta_{n-1} \tag{2.10}$$

Since the function $\phi \in \Phi$ and $S(x_0, x_0, x_1) > 0$, so by (4) of Lemma 10, we get that

$$\Sigma_{k=0}^{\infty} \phi^k(S(x_0, x_0, x_1)) < \infty,$$

Which implies that there exists a positive real S such that $lim_{n\to\infty} \delta_n = \delta$. Thus, from 2.10 we have

$$lim_{n\to\infty} S(x_n, x_n, x_{n+r}) = 0$$

This yields that $\{x_n\}$ is S-Cauchy sequence in (X, S).

Since (X, S) is S- complete, hence there exists $u \in X$ such that

$$lim_{n\to\infty} x_n = u \tag{2.11}$$

We shall prove that

$$u \in \cap_{i=i}^{m} A_i. \tag{2.12}$$

Since $x_0 \in A_1$, we have $\{x_{n_p}\}_{n\geq 0} \in A_1$. Since A_1 is S- closed and 2.11, by Lemma 8, we have $u \in A_1$. Again, $\{x_{n_p+1}\}_{n\geq 0} \in A_2$. Since A_2 is S- closed and

2.11, by Lemma 8, we have $u \in A_2$. Continuing this process, we obtain 2.12.

We claim that u is a fixed point of T. We have that for any $n \geq 0$ there exists us $i_n \in \{1, 2....m\}$ such that $x_n \in A_{i_n}$. Also form 2.12, $u \in A_{i_n+1}$, so applying 2.4 for $x = y = x_n$ and $z = u$, we get that

$$S(x_{n+1}, x_{n+1}, u) = S(Tx_n, Tx_n, Tu) \leq \phi(S(x_n, x_n, u)) \tag{2.13}$$

Since ϕ is continuous at 0 and $lim_{n\to\infty} S(x_n, x_n, u) = 0$, so

$$lim_{n\to\infty} S(x_{n+1}, x_{n+1}, u) \leq \phi(0).$$

But, since $\phi(t) < t$ for all $t > 0$ and again ϕ is continuous at 0, hence we get that $\phi(0) = 0$. We deduce from the above inequality, $x_{n+1} \to Tu$ as $n \to \infty$. By uniqueness of limit, it follows that $Tu = u$.

Now, we prove that u is the unique fixed point of T. Assume that v us another fixed point of T, that is, $Tv = v$. We have $v \in \cap_{i=1}^{m} A_i$. Suppose that $u \neq v$, so $S(u, u, v) > 0$. Taking $x = y = u$ and $z = v$ in 2.4, we get that

$$0 < S(u, u, v) = S(Tu, Tu, Tv) \leq \phi(S(u, u, v)) \leq S(u, u, v),$$

Which is a contradiction. We deduce u is the unique fixed point of T. This completes the proof of (I).

We shall prove (II). From 2.10, we have

$$S(x_n, x_n, x_{n+r}) \leq \Sigma_{k=n}^{n+r-1} \phi^k(S(x_0, x_0, x_1))$$

Letting $r \to \infty$ in above inequality, we get the estimate 2.5.

For $n \geq 0$ and $k \geq 1$, we have

$$S(x_{n+k}, x_{n+k}, x_{n+k+1}) = S(Tx_{n+k-1}, Tx_{n+k-1}, Tx_{n+k}) \leq \phi(S(x_{n+k-1}, x_{n+k-1}, x_{n+k})) \tag{2.14}$$

And for $k \geq 2$,

$$S(x_{n+k-1}, x_{n+k-1}, x_{n+k}) = S(Tx_{n+k-2}, Tx_{n+k-2}, Tx_{n+k-1}) \leq \phi(S(x_{n+k-2}, x_{n+k-2}, x_{n+k-1})) \tag{2.15}$$

By monotonicity of ϕ, 2.14 and 2.15 imply that

$$S(x_{n+k}, x_{n+k}, x_{n+k+1}) \leq \phi^2(S(x_{n+k-2}, x_{n+k-2}, x_{n+k-1})), \quad n \geq 0, \quad k \geq 2. \tag{2.16}$$

By induction we get that

$$S(x_{n+k}, x_{n+k}, x_{n+k+1}) \leq \phi^k(S(x_n, x_n, x_{n+1})), \quad n \geq 0, \quad k \geq 0. \tag{2.17}$$

But by rectangle inequality

$$S(x_n, x_n, x_{n+r}) \leq S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_{n+2}) + ... + S(x_{n+r-1}, x_{n+r-1}, x_{n+r})$$

Hence, form 2.17, we have

$$S(x_n, x_n, x_{n+r}) \le \Sigma_{k=0}^{n+r-1} \phi^k (S(x_n, x_n, x_{n+1}))$$

Letting $r \to \infty$ in the above inequality, we get that

$$S(x_n, x_n, u) \le \Sigma_{k=0}^{\infty} \phi^k (S(x_n, x_n, x_{n+1})) = p(S(x_n, x_n, x_{n+1})) \tag{2.18}$$

This yields (II).

Now we will prove (III). Let $x \in Y$. Form 2.18, for $x_0 = x$, we have

$$S(x, x, u) \le \Sigma_{k=0}^{\infty} \phi^k (S(x, x, Tx)) = p(S(x, x, Tx))$$

Which is the estimate 2.7. □

As consequences of Theorem 12, we have the following results.

Theorem 13. *Let $T : Y \to Y$ be defined as Theorem 12. Then*

$$\Sigma_{n=0}^{\infty} S(T^n x, T^n x, T^{n+1}) < \infty, \quad \forall x \in Y, \tag{2.19}$$

That is, T is a good Picard operator.

Proof. Let $x = x_0 \in Y$. If for some integer k, $T^k x_0 = T^{k+1} x_0$ so the sequence $\{T^n x_0\}$ is constant for all $n \ge k$, hence obiviosly 2.19 holds. Otherwise, assume that $T^k x_0 \ne T^{k+1} x_0$ for all $n \ge 0$. By 2.9 in the proof of Theorem 12, we know that

$$S(T^n x, T^n x, T^{n+1} x) = S(x_n, x_n, x_{n+1}) \le \phi(S(x_0, x_0, x_1)), \quad \forall n \ge 0.$$

Then

$$\Sigma_{n=0}^{\infty} S(T^n x, T^n x, T^{n+1} x) \le \Sigma_{n=0}^{\infty} \phi(S(x_0, x_0, x_1)) = p(S(x_0, x_0, x_1)).$$

By Lemma 11, it follows that $\Sigma_{n=0}^{\infty} S(T^n x, T^n x, T^{n+1} x) < \infty$, so T is a good Picard operator. □

Theorem 14. *Let $T : Y \to Y$ be defined as in Theorem 12. Then*

$$S(T^n x, T^n x, u) = S(x_n, x_n, x_{n+1}) \le \phi(S(x_0, x_0, x_1)), \quad \forall n \ge 0. \tag{2.20}$$

That is, T is a special Picard operator.

Proof. If $x = u$, then cleary 2.20 is true. Suppose $x \ne u$ and $x \in Y$. We rewrite 2.13 with $Tu = u$.

$$S(T^{n+1} x, T^{n+1} x, u) = S(T^{n+1} x, T^{n+1} x, Tu) \le \phi(S(x_n, x_n, u))$$

By induction and considering the monotonicity of ϕ, we obtain

$$S(T^n x, T^n x, u) \le \phi^n (S(x, x, u)), \quad \forall n \ge 0.$$

Therefore

$$\Sigma_{n=0}^{\infty} S(T^n x, T^n x, u) \leq \Sigma_{n=0}^{\infty} \phi^n(S(x,x,u)) = p(S(x,x,u)),$$

Consequently, $\Sigma_{n=0}^{\infty} S(T^n x, T^n x, u) \leq \infty$, so T is a special Picard operator. □

Definition 15. *Let X be a nonempty set. A fixed point problem of a given mapping $f : X \to X$ on X is called well-posed if $F(f)$ is a singleton and for any sequence $\{a_n\}$ in X with $x^* \in F(f)$ and $lim_{n\to infty} S(a_n, a_n, fa_n)$ implies $x^* = lim_{n\to\infty} a_n$.*

Theorem 16. *Let $f : Y \to Y$ be defined as in Theorem 12. Then the fixed point problem for T is well posed that is, assuming that there exists $\{z_n\} \in Y$, $n \in N$ such that $lim_{n\to\infty} S(z_n, z_n, fz_n)$ implies $z = lim_{n\to infty} z_n$.*

Proof. Let $\{z_n\} \in Y$, $n \in N$ such that $lim_{n\to\infty} S(z_n, z_n, Tz_n) = 0$. Applying 2.7 for $x = z_n$ and $u = z$ then we have

$$S(z_n, z_n, z) \leq p(S(z_n, z_n, Tz_n)). \tag{2.21}$$

Having the mind from Lemma 11 that p is continuous at 0, so letting $n \to \infty$ in 2.21, we have

$$lim_{n\to\infty} S(z_n, z_n, z) = 0,$$

So $z = lim_{n\to\infty} z_n$. Hence the fixed point problem for T is well posed. □

Theorem 17. *Let $T : Y \to Y$ be defined as in Theorem 12. Let $f : Y \to Y$ such that*

1. *F has at least one fixed point, say $z_f \in F(f)$*
2. *there esists $v > 0$ such that*

$$S(fx, fx, Tx) \leq y, \quad \forall x \in Y. \tag{2.22}$$

Then $S(z_f, z_f, z_T) \leq s(v)$ where $F(T) = z_T$.

Proof. Assume $z_f \neq Z_Y$. Otherwise the proof is completed. We apply 2.7 from Theorem 12 for $x = x_f$ to have

$$S(z_f, z_f, z_T) \leq p(S(z_f, z_f, Tz_f) = p(S(fz_f, fz_f, Tz_f) \tag{2.23}$$

By Lemma 11, then function p is non decreasing, so by **??**ith $x = z_f$, it follows that

$$S(z_f, z_f, z_T) \leq s(v). \tag{2.24}$$

□

3. Cyclic $(\psi - \phi)$ - contraction on S- metric space

Denote by Ψ the set of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying

(ψ_1) ψ is continuous,
(ψ_2) ψ is non decreasing,
(ψ_3) $\psi(t) = 0$ if and only if $t = 0$.

Also, denote by Φ the set of functions $\phi : [0,\infty) \to [0,\infty)$ satisfying

(ϕ_1) ϕ is lower semi- continuous,

(ϕ_2) $\phi(t) = 0$ if and only if $t = 0$

The object of this section is to give some more general classes of mappings involving cyclic $(\psi - \phi)$- contractions. Note that, in our result the monotony property of the function ϕ is omitted and the continuity property of ϕ is replaced by lower semi-continuity.

The main result of this section is the following.

Theorem 18. *Let (X, S) be a S- complete S- metric space. Let $\{A_i\}_{iC1}^m$ be a family of non empty S- closed subsets of X, m a positive integer and $Y = \cup_{i=i}^m A_i$. Let $T : Y \to Y$ be a mapping such that*

$$T(A_i) \subseteq A_{i+1} \quad \forall i = 1, 2....m \quad with A_{i+1} = A_i \tag{3.1}$$

Suppose also that there exists $\phi \in \Phi$ such that

$$\psi(S(Tx, Ty, Tz)) \leq \psi(S(x, y, z)) - \phi(S(x, y, z)), \quad \forall (x, y, z) \in A_i \times A_i \times A_{i+1} \tag{3.2}$$

For $i = 1, 2......m$. Then T has a unique fixed point that belongs to $\cap_{i=1}^m A_i$.

Proof. Let $x_0 \in A_1$. Consider the Picard iteration $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geq 0$.
If for some integer k, $x_k = x_{k+1}$, so $\{x_n\}$ is constant for any $n \geq k$, then $\{x_n\}$ is S- Cauchy sequence in (X, S).

Suppose that $x_n \neq x_{n+1}$ for all $n \geq 0$. For any $n \geq 0$, there is $i_n \in \{1, 2, ...m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} \in A_{i_n+1}$. By 3.2, we have

$$\begin{aligned} \psi(S(x_{n+1}, x_{n+1}, x_{n+2})) &= \psi(S(Tx_n, Tx_n, Tx_{n+1})) \\ &\leq \psi(S(x_n, x_n, x_{n+1})) - \phi(S(x_n, x_n, x_{n+1})) \end{aligned}$$

$$\psi(S(x_{n+1}, x_{n+1}, x_{n+2})) \leq \psi(S(x_n, x_n, x_{n+1})) \tag{3.3}$$

The function ψ is non-decreasing, so we have

$$S(x_{n+1}, x_{n+1}, x_{n+2}) \leq S(x_n, x_n, x_{n+1}), \quad \forall n \geq 0. \tag{3.4}$$

Therefore the sequence $\{S(x_n, x_n, x_{n+1})\}$ is non-increasing, so it converges to some real $r \geq 0$. Letting $n \to \infty$ in 3.3, using the continuity of ψ and the lower semi-continuity of ϕ, we get that

$$\psi(r) \leq \psi(r) - \phi(r).$$

which implies that $\phi(r) = 0$. By (ϕ_2), we have $r = 0$, that is,

$$lim_{n\to\infty} S(x_n, x_n, x_{n+1}) = 0. \tag{3.5}$$

Since $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$, hence by 3.5, we have

$$Lim_{n\to\infty} S(x_{n+1}, x_{n+1}, x_n) = 0. \tag{3.6}$$

Now, we prove that $\{x_n\}$ is a S- Cauchy sequence. We argue by contradiction. Assume that for $\{x_n\}$ is not a S- Cauchy sequence. Then, following Definition 6, there exists $\epsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) > k$ such that

$$S(x_{n(k)}, x_{n(k)}, x_{m(k)}) \geq \epsilon \tag{3.7}$$

Further corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k) > k$ and satisfying 3.7. Then

$$S(x_{n(k)-1}, x_{n(k)-1}, x_{m(k)}) < \epsilon \tag{3.8}$$

Using 3.8 and property of S- metric space we have

$$\epsilon \leq S(x_{n(k)}, x_{n(k)}, x_{m(k)}) \leq S(x_{n(k)}, x_{n(k)}, x_{n(k)-1}) + S(x_{n(k)}, x_{n(k)}, x_{(k)-1}) + S(x_{m(k)}, x_{m(k)}, x_{n(k)-1}) \tag{3.9}$$

$$\epsilon \leq S(x_{n(k)}, x_{n(k)}, x_{m(k)}) \leq \epsilon + 2S(x_{n(k)}, x_{n(k)}, x_{n(k)-1})$$

Letting $k \to \infty$ in 3.9 and using 3.6, we find

$$lim_{k\to\infty} S(x_{n(k)}, x_{n(k)}, x_{n(k)-1}) = \epsilon \tag{3.10}$$

On the other hand, for all k, there exists $j(k)$, $0 \leq j(k) \leq m$, such that $n(k) - m(k) + j(k) = 1(q)$. Then $x_{m(k)Cj(k)}$ (for k large enough, $m(k) > jCk)$) and $x_{n(k)}$ lie in different adjacently labeled sets A_i and A_{i+1} for certaing $i = 1, 2, ...m$. From 3.2, we have

$$\psi(S(x_{n(k)+1}, x_{n(k)+1}, x_{m(k)Cj(k)+1})) = \psi(S(Tx_{n(k)}, Tx_{n(k)}, Tx_{m(k)Cj(k)}))$$

$$\psi(S(Tx_{n(k)}, Tx_{n(k)}, Tx_{m(k)Cj(k)})) \leq \psi(S(x_{n(k)}, x_{n(k)}, x_{m(k)-j(k)})) - \phi(S(x_{n(k)}, x_{n(k)}, x_{m(k)-j(k)})) \tag{3.11}$$

By using the property of S- metric space and as $n \to \infty$ we have

$$lim_{k\to\infty} S(x_{n(k)}, x_{n(k)}, x_{m(k)-j(k)}) = \epsilon. \tag{3.12}$$

Similarly by using the property of S- metric space , 3.6, 3.7, 3.12 and as $k \to \infty$ we find

$$lim_{k\to\infty} S(x_{n(k)+1}, x_{n(k)+1}, x_{m(k)Cj(k)+1}) = \epsilon. \tag{3.13}$$

Now letting $k \to \infty$ in 3.11 and using 3.12, 3.13 we get that

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon) \tag{3.14}$$

Which yields that $\epsilon = 0$, a contradiction.

This shows that $\{x_n\}$ is S- Cauchy sequence in (X, S).

Since (X, S) is S-complete, hence there exists $u \in X$ such that

$$lim_{k\to\infty}x_n = u. \tag{3.15}$$

We shall prove that

$$u \in \cap_{i=1}^{m} A_i \tag{3.16}$$

Since $x_0 \in A_1$, we have $\{x_{nl}\}_{n\geq 0}A_1$. The fact that A_1 is S- closed and 2.11 yield that $u \in A_1$. Again, $\{x_{nl+1}\}_{n\geq 0}A_2$. Since A_2 is S- closed and 3.15 yield that $u \in A_2$. Continuing this process, we obtain 3.16.

We claim that u is a fixed point of T. We have in mind that for any $n \geq 0$, there exists $i_n \in \{1, 2, ...m\}$ such that $x_n \in A_{i_n}$. Also, form 3.16, $u \in A_{i_n+1}$ so applying 3.2 for $x = y = x_n$ and $z = u$, we get that

$$\begin{aligned}\psi(S(x_{n+1}, x_{n+1}, Tu)) = \psi(S(Tx_n, Tx_n, Tu))&\\ \leq \quad \psi(S(x_n, x_n, u)) - \psi(S(x_n, x_n, u))&\end{aligned}$$

Letting $n \to \infty$ in above inequality, we obtain

$$\psi(S(u, u, Tu)) \leq \psi(o) - \phi(o)$$

Which implies that $\psi(S(u, u, Tu)) = 0$, so $S(u, u, Tu) = 0$. It follows that $Tu = u$.

Now, we prove that u is the unique fixed point of T. Assume that v is another fixed point of T, that is $Tv = v$. We have $v \in \cap_{i=1}^{m} A_i$. Taking $x = y = u$ and $z = v$ in 3.2, we get that

$$\psi(S(Tu, Tu, Tv)) \leq \psi(S(u, u, v)) - \phi(S(u, u, v)), \tag{3.17}$$

So that $\phi(S(u, u, v)) = 0$ that is $u = v$. □

Example 19. *Let $X = [0, \infty)$ be equipped with the S- metric space S given as follows*

$$S(x, y, z) =\mid x - z \mid + \mid y - z \mid$$

$(X.S)$ is S- complete metric space. Consider $A_1 = \{0, 1\}$, $A_2 = \{1, 4\}$ and $Y = A_1 \cup A_2$. It is obvious that A_1 and A_2 are S- closed subsets of (X, S). We define $T : Y \to Y$ by

$$T0 = 1, T1 = 1 \;\; and \;\; T4 = 0$$

We have $T(A_1) \subseteq A_2$ and $T(A_2) \subseteq A_1$. Define $\psi(t) = t$ and $\phi = \frac{2}{3}t$. We shall prove that $(x, y, z) \times A_1 \times A_1 \times A_2$ and $(x, y, z) \times A_2 \times A_2 \times A_1$. To check this we have following conditions:

(1) *If $(x, y, z) \times A_1 \times A_1 \times A_2$ then,*

***Case - 1**: If $x = y = 0$ and $z = 1$ in this case*

$$S(Tx, Ty, Tz) = 0.$$

Case - 2: *If* $x = 0, y = 1$ *and* $z = 4$ *or* $x = 1, y = 0$ *and* $z = 4$ *in this case 18 true and from 3.2 we have*

$$S(Tx, Ty, Tz) = 2 \leq \frac{1}{2}S(x, y, z)$$

which is true.

Case - 3: *If* $x = y = z = 1$ *in this case 18 true and from 3.2 we have*

$$S(Tx, Ty, Tz) = 0.$$

Case - 4: *If* $x = y = 0$ *and* $z = 4$ *in this case 18 true and from 3.2 we have*

$$S(Tx, Ty, Tz) = 2 \leq \frac{1}{3}S(x, y, z)$$

Case - 5: *If* $x = y = 1$ *and* $z = 4$ *in this case 18 true and from 3.2 we have*

$$S(Tx, Ty, Tz) = 2 \leq \frac{1}{3}S(x, y, z)$$

Case - 6: *If* $x = y = 4$ *and* $z = 1$ *in this case 18 true and from 3.2 we have*

$$S(Tx, Ty, Tz) = 2 \leq \frac{1}{3}S(x, y, z)$$

(2) *If* $(x, y, z) \times A_2 \times A_2 \times A_1$ *then,*

Case - 7: *If* $x = y = 1$ *and* $z = 0$ *in this case*

$$S(Tx, Ty, Tz) = 0 < 2 = S(x, y, z).$$

Case - 8: *If* $x = 1, y = 4$ *and* $z = 0$ *or* $x = 4, y = 1$ *and* $z = 0$ *in this case 18 true and from 3.2 we have*

$$S(Tx, Ty, Tz) = 1 = \frac{1}{5}S(x, y, z)$$

which is true.

Case - 9: *If* $x = y = 4$ *and* $z = 0$ *in this case 18 true and from 3.2 we have*

$$S(Tx, Ty, Tz) = 2 \leq \frac{1}{4}S(x, y, z)$$

Case - 10: *If* $x = y = 4$ *and* $z = 1$ *in this case 18 true and from 3.2 we have*

$$S(Tx, Ty, Tz) = 2 \leq \frac{1}{3}S(x, y, z)$$

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References

[1] R. Chugh, T. Kadian, A. Rani, B.E. Rhoades, Property P in G-metric spaces, Fixed Point Theory Appl. Vol. 2010, Article ID 401684.

[2] B.C. Dhage, Generalized metric spaces mappings with fixed point, Bull. Calcutta Math. Soc. 84 (1992), 329-336.

[3] S. Gahler, 2-metrische Raume und iher topoloische Struktur, Math. Nachr. 26 (1963), 115-148.

[4] Z. Mustafa, B. Sims, A new approach to generalized metric spaces, J. Nonlinear Convex Anal. 7 (2006), 289C297.

[5] Z. Mustafa, H. Obiedat, F. Awawdeh, Some common fixed point theorems for mapping on complete G-metric spaces, Fixed Point Theory Appl. Vol. 2008, Article ID 189870.

[6] Z. Mustafa, A New Structure for Generalized Metric Spaces with Applications to Fixed Point Theory, Ph. D. Thesis, The University of Newcastle, Callaghan, Australia, 2005.

[7] Z. Mustafa, B. Sims, Some results concerning D-metric spaces, Proc. Internat. Conf. Fixed Point Theory and Applications, pp. 189-198, Valencia, Spain, 2003.

[8] S.V.R. Naidu , K.P.R. Rao, N. Srinivasa Rao, On the topology of D-metric spaces and the generation of D-metric spaces from metric spaces, Internat. J. Math. Math. Sci. 2004 (2004), No. 51, 2719-2740.

[9] S.V.R. Naidu, K.P.R. Rao, N. Srinivasa Rao, On the concepts of balls in a D-metric space, Internat. J. Math. Math. Sci. 2005 (2005), 133-141.

[10] S.V.R. Naidu, K.P.R. Rao, N. Srinivasa Rao, On convergent sequences and fixed point theorems in D-metric spaces, Internat. J. Math. Math. Sci. 2005 (2005), 1969-1988.

[11] S. Sedghi, K.P.R. Rao, N. Shobe, Common fixed point theorems for six weakly compatible mappings in D^*-metric spaces, Internat. J. Math. Math. Sci. 6 (2007), 225-237.

[12] S. Sedghi, N. Shobe, H. Zhou, A common fixed point theorem in D^*-metric spaces, Fixed Point Theory Appl. Vol. 2007, Article ID 27906, 13 pages.

[13] W. Shatanawi, Fixed point theory for contractive mappings satisfying ψ-maps in G-metric spaces, Fixed Point Theory Appl. Vol. 2010, Article ID 181650.

[14] S. Sedghi, N. Shobe, A. Aliouche, A generalization of fixed point theorem in S-metric spaces,Mat. Vesnik 64 (2012), 258-266.

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