International Journal of Analysis and Applications
ISSN 2291-8639
Volume 12, Number 1 (2016), 15-21
http://www.etamaths.com

SOME FIXED POINT RESULTS FOR CARISTI TYPE MAPPINGS IN MODULAR METRIC SPACES WITH AN APPLICATION

DURAN TURKOGLU[1] AND EMINE KILINÇ[2,*]

Abstract. In this paper we give Caristi type fixed point theorem in complete modular metric spaces. Moreover we give a theorem which can be derived from Caristi type. Also an application for the bounded solution of funcional equations is investigated.

1. Introduction

Fixed point theory is one of the very popular tools in various fields. Since Banach intoruced this theory in 1922[6], ıt has been extended and generalized by several authors. Caristi type fixed point theorem is one of these genealizations. It is a modification of $\varepsilon-$variational principle of Ekeland[13].It is crucial in nonlinear analysis, in particular, optimization, variational inequalites, differantial equations and contol theory.

The notion of modular space was introduced by Nakano [20] and was intensively developed by Koshi, Shimogaki, Yamamuro (see [18, 21]) and others. A lot of mathematicians are interested in fixed point of modular space. In 2008, Chistyakov introduced the notion of modular metric space generated by F-modular and developed the theory of this space [8], on the same idea was defined the notion of a modular on an arbitrary set and developed the theory of metric space generated by modular such that called the modular metric spaces in 2010 [9]. Afrah A. N. Abdou [1] studied and proved some new fixed points theorems for pointwise and asymptotic pointwise contraction mappings in modular metric spaces. Azadifer et. al. [3] introduced the notion of modular G-metric spaces.Azadifer et. al. [5] proved the existence and uniqueness of a common fixed point of compatible mappings of integral type in this space. Kılınç and Alaca [14] defined $(\varepsilon, k)-$uniformly locally contractive mappings and η-chainable concept and proved a fixed point theorem for these concepts in a complete modular metric spaces. Kılınç and Alaca [15] proved that two main fixed point theorems for commuting mappings in modular metric spaces. Recently, many authors [4, 7, 10, 11, 12, 19] studied on different fixed point results for modular metric spaces. In 2014 Khamsi and Abdou investigated Hausdorff modular metric in modular metric spaces[16], and proved fixed point theorem for multivalued mappings.

In this paper we investigate Caristi type fixed point theorems for multivalued mappings in modular metric spaces, which is more general than the results of Khojasteh, Karapinar and Khandani[17]. And we also give an application of results for functional equations.

2. Preliminaries

In this section, we will give some basic concepts and definitions about modular metric spaces.

Definition 2.1 [[9], Definition 2.1] Let X be a nonempty set, a function $w:(0,\infty)\times X\times X\to[0,\infty]$ is said to be a metric modular on X if satisfying, for all $x,y,z\in X$ the following condition holds:

(i) $w_\lambda(x,y)=0$ for all $\lambda>0 \Leftrightarrow x=y$;
(ii) $w_\lambda(x,y)=w_\lambda(y,x)$ for all $\lambda>0$;
(iii) $w_{\lambda+\mu}(x,y)\le w_\lambda(x,z)+w_\mu(z,y)$ for all $\lambda,\mu>0$.

If instead of (i), we have only the condition

$(\acute{i})$ $w_\lambda(x,x)=0$ for all $\lambda>0$, then w is said to be a (metric) pseudomodular on X.

2010 *Mathematics Subject Classification.* 46A80, 47H10, 54E35.
Key words and phrases. modular metric spaces; fixed point; Hausdorff metric.

The main property of a metric modular [[9]] w on a set X is the following: given $x, y \in X$, the function $0 < \lambda \mapsto w_\lambda(x, y) \in [0, \infty]$ is nonincreasing on $(0, \infty)$. In fact, if $0 < \mu < \lambda$, then (iii), $(\acute{i})$ and (ii) imply

$$w_\lambda(x, y) \leq w_{\lambda-\mu}(x, x) + w_\mu(x, y) = w_\mu(x, y).$$

It follows that at each point $\lambda > 0$ the right limit $w_{\lambda+0}(x, y) = \lim\limits_{\mu \to \lambda+0} w_\mu(x, y)$ and the left limit $w_{\lambda-0}(x, y) = \lim\limits_{\varepsilon \to +0} w_{\lambda-\varepsilon}(x, y)$ exist in $[0, \infty]$ and the following two inequalities hold:

$$w_{\lambda+0}(x, y) \leq w_\lambda(x, y) \leq w_{\lambda-0}(x, y).$$

Theorem 2.1 [[19]] Let X_w be a complete modular metric space and T a contraction on X_w. Then, the sequence $(T^n x)_{n\in\mathbb{N}}$ converges to the unique fixed point of T in X_w for any initial $x \in X_w$.

Now we give some definitions, which are useful for our main results.

Definition 2.2 Let X_w be a modular metric space. Then following definitions exist:

(1) The sequence $(x_n)_{n\in\mathbb{N}}$ in X_w is said to be convergent to $x \in X_w$ if $w_1(x_n, x) \to 0$, as $n \to \infty$
(2) The sequence $(x_n)_{n\in\mathbb{N}}$ in X_w is said to be Cauchy if $w_1(x_m, x_n) \to 0$, as $m, n \to \infty$
(3) A subset C of X_w is said to be closed if the limit of a convergent sequence of C always belong to C.
(4) A subset C of X_w is said to be complete if any Cauchy sequence in C is a convergent sequence and its limit is in C.
(5) A subset C of X_w is said to be $w-$bounded if

$$\delta_w(C) = \sup\{w_1(x, y); x, y \in C\} < \infty.$$

(6) A subset C of X_w is said to be $w-$compact if for any (x_n) in C there exists a subset sequence (x_{n_k}) and $x \in C$ such that $w_1(x_{n_k}, x) \to 0$
(7) w is said to satisfy the Fatou property if and only if for any sequence $(x_n)_{n\in\mathbb{N}}$ in X_w $w-$convergent to x, we have

$$w_1(x, y) \leq \liminf_{n\to\infty} w_1(x_n, y),$$

for any $y \in X_w$.

Now we will give some basic properties and notions of multivalued mappings in modular metric spaces which was given in [2] For a subset M of modular metric space X_w set

(i) $\mathcal{CB}(M) = \{C : C \text{ is nonempty } w-\text{closed and } w-\text{bounded subset of } M\}$;
(ii) $K(M) = \{C : C \text{ is nonempty } w-\text{compact subset of } M\}$;
(iii) the Haussdorf modular metric is defined on$\mathcal{CB}(M)$ by

$$H_w(A, B) = \max\left\{\sup_{x\in A} w_1(x, B), \sup_{y\in B} w_1(y, A)\right\},$$

where $w_1(x, B) = \inf\limits_{y\in B} w_1(x, y)$.

Definition 2.3 Let X_w be a complete modular metric space and M be a nonempty subset of X_w. A mapping $T : M \to \mathcal{CB}(M)$ is called a multivalued Lipschitzian mapping, if there exists a constant $k > 0$ such that

$$H_w(Tx, Ty) \leq k w_1(x, y),$$

for any $x, y \in M$.

A point $x \in M$ is called fixed point of T whenever $x \in Tx$.The set of fixed points of T will be denoted by Fix(T)

It was shown in[2] that Definition 2.4 is more general than Theorem 2.1.

3. MAIN RESULTS

In this section we will give a fixed point theorem for Caristi type mappings and a generalization of the theorem in modular metrc spaces. This works are more general than the results of [17].

Theorem Let X_w be a complete modular metric space and $T : X_w \to \mathcal{CB}(X_w)$ be a nonexpansive mapping such that for each $x \in X_w$ and for all $y \in Tx$, there exists $z \in Ty$ such that

$$w_\lambda(x, y) \leq \frac{1}{\lambda}(\varphi_w(x, y) - \varphi_w(y, z)) \tag{3.1}$$

specificially for $\lambda = 1$ we can write

$$w_1(x, y) \leq \varphi_w(x, y) - \varphi_w(y, z)$$

where $\varphi : X_w \times X_w \to [0, \infty]$ is lower semicontinuous with respect to the firs variable. Then for $w_1(x_n, x_{n+1}) < \infty$, T has a fixed point.

Proof. Let $x_0 \in X_w$ and $x_1 \in Tx_0$. If $x_1 = x_0$, then x_0 is a fixed point and theorem is satisfied. Otherwise, let $x_0 \neq x_1$. By assumption there exists $x_2 \in Tx_1$ such that

$$w_1(x_0, x_1) \leq \varphi_w(x_0, x_1) - \varphi_w(x_1, x_2)$$

Alternatively, one can choose $x_n \in Tx_{n-1}$ such that $x_n \neq x_{n-1}$ and find $x_{n+1} \in Tx_n$ such that

$$0 < w_1(x_{n-1}, x_n) \leq \varphi_w(x_{n-1}, x_n) - \varphi_w(x_n, x_{n+1}), \tag{3.2}$$

which means that $(\varphi_w(x_{n-1}, x_n))_n$ is a nonincreasing sequence in fact

$$\begin{aligned} 0 &< w_1(x_{n-1}, x_n) + \varphi_w(x_n, x_{n+1}) \leq \varphi_w(x_{n-1}, x_n) \\ 0 &< \varphi_w(x_n, x_{n+1}) \leq \varphi_w(x_{n-1}, x_n) \end{aligned}$$

Thus it is bounded below, so it converges to some $r \geq 0$. By taking the limit on both sides of (3.2) we get

$$\begin{aligned} \lim_{n\to\infty} w_1(x_{n-1}, x_n) &\leq \lim_{n\to\infty} (\varphi_w(x_{n-1}, x_n) - \varphi_w(x_n, x_{n+1})) \\ \lim_{n\to\infty} w_1(x_{n-1}, x_n) &\leq \lim_{n\to\infty} \varphi_w(x_{n-1}, x_n) - \lim_{n\to\infty} \varphi_w(x_n, x_{n+1}) \\ \lim_{n\to\infty} w_1(x_{n-1}, x_n) &= r - r \\ \lim_{n\to\infty} w_1(x_{n-1}, x_n) &= 0 \end{aligned}$$

Now we show that $w_1(x_n, x_{m-n})$ is Cauchy sequence . For all $m, n \in \mathbb{N}$ with $m > n$,

$$w_1(x_n, x_m) \leq \sum_{i=n+1}^{m} w_{\frac{1}{m-n}}(x_{i-1}, x_i) \tag{3.3}$$

On the other hand from the main property of metric modular one can choose that

$$\begin{aligned} w_1(x_{n-1}, x_n) &\leq w_{\frac{1}{m-n}}(x_{n-1}, x_n) \\ w_{\frac{1}{m-n}}(x_{n-1}, x_n) &\leq \frac{1}{(m-n)}\varphi_w(t) \end{aligned}$$

when we write this in (3.3) we get

$$\begin{aligned} w_1(x_n, x_m) &\leq \frac{1}{m-n}(\varphi_w(x_{n-1}, x_n) - \cdots - \varphi_w(x_m, x_{m+1})) \\ w_1(x_n, x_m) &\leq \frac{1}{m-n}(\varphi_w(x_{n-1}, x_n) - \varphi_w(x_m, x_{m+1})) \end{aligned}$$

When we take the limsup on both sides of the inequalities above, we have

$$\begin{aligned}
\lim_{n\to\infty}(\sup\{w_1(x_n,x_m):m>n\} &\leq \lim_{n\to\infty}(\sup\frac{1}{(m-n)}(\varphi_w(x_{n-1},x_n)-\varphi_w(x_m,x_{m+1}))) \\
\lim_{n\to\infty}(\sup\{w_1(x_n,x_m):m>n\} &\leq \lim_{n\to\infty}(\sup\frac{1}{(m-n)})(r-r) \\
\lim_{n\to\infty}(\sup\{w_1(x_n,x_m):m>n\} &= 0
\end{aligned}$$

Thus (x_n) is a Cauchy sequence. Since X_w is complete it converges to $u \in X_w$. Now we show that u is a fixed point of T.We have

$$\begin{aligned}
w_1(u,Tu) &\leq w_{\frac{1}{2}}(u,x_{n+1})+w_{\frac{1}{2}}(Tu,x_{n+1}) \\
&\leq w_{\frac{1}{2}}(u,x_{n+1})+H_w(Tu,Tx_n) \\
&\leq w_{\frac{1}{2}}(u,x_{n+1})+w_{\frac{1}{2}}(u,x_n)
\end{aligned}$$

When we take the limit on both sides of the inequalities above, we get

$$\begin{aligned}
\lim_{n\to\infty} w_1(u,Tu) &\leq \lim_{n\to\infty} w_{\frac{1}{2}}(u,x_{n+1})+\lim_{n\to\infty} w_{\frac{1}{2}}(u,x_n) \\
\lim_{n\to\infty} w_1(u,Tu) &\leq w_{\frac{1}{2}}(u,u)+w_{\frac{1}{2}}(u,u) \\
\lim_{n\to\infty} w_1(u,Tu) &= 0
\end{aligned}$$

Hence we get $u \in Tu$. Therefore u is fixed point of T. □

Now let us give the theorem which is a generalized version of the theorem above.

Theorem Let X_w be a complete modular metric space and $T : X_w \to \mathcal{CB}(X_w)$ be a multivalued mapping such that

$$H_w(Tx,Ty) \leq \eta(w_1(x,y))$$

for all $x,y \in X_w$; where $\eta : [0,\infty] \to [0,\infty]$ is a lower semicontunious map such that $\eta(t) < t$ for all $t \in [0,\infty]$, $\frac{\eta(t)}{t}$ is nondecreasing. Then T has a fixed point.

Proof. Let $x \in X_w$ and $y \in Tx$. If $x = y$ then T has a fixed point and the proof is complete, so we suppose that $x \neq y$. Let define that

$$\theta(t) = \frac{\eta(t)+t}{2}, \text{for all } t \in [0,\infty]$$

Then we have

$$H_w(Tx,Ty) \leq \eta(w_1(x,y))$$

Since η is nondecreasing and $\eta(t) < t$ we get

$$\begin{aligned}
\eta(t) &< t \Leftrightarrow \eta(w_1(x,y)) \leq w_1(x,y) \\
\theta(w_1(x,y)) &= \frac{\eta(w_1(x,y))+w_1(x,y)}{2} \leq w_1(x,y) \\
\theta(w_1(x,y)) &= \frac{\eta(w_1(x,y))+w_1(x,y)}{2} \geq \eta(w_1(x,y))
\end{aligned}$$

Then we get

$$H_w(Tx,Ty) \leq \eta(w_1(x,y)) \leq \theta(w_1(x,y)) \leq w_1(x,y) \tag{3.4}$$

Thus there exists $\epsilon_0 > 0$ such that $\theta(w_1(x,y) = H_w(Tx,Ty)+ \epsilon_0$. So there exists $z \in Ty$ such that

$$w_1(y,z) \leq H_w(Tx,Ty) + \epsilon_0 = \theta(w_1(x,y)) \leq w_1(x,y) \tag{3.5}$$

Again suppose that $y \neq z$.Then

$$w_1(x,y) - \theta(w_1(x,y)) \leq w_1(x,y) - w_1(y,z)$$

or we can rewrite this inequality as

$$w_1(x,y) \leq \frac{w_1(x,y)}{1 - \frac{\theta(w_1(x,y))}{w_1(x,y)}} - \frac{w_1(y,z)}{1 - \frac{\theta(w_1(x,y))}{w_1(x,y)}} \tag{3.6}$$

Since $\frac{\theta(t)}{t}$ is nondecreasing and $w_1(y,z) < w_1(x,y)$ we rewrite the (3.5) as

$$w_1(x,y) \leq \frac{w_1(x,y)}{1 - \frac{\theta(w_1(x,y))}{w_1(x,y)}} - \frac{w_1(y,z)}{1 - \frac{\theta(w_1(y,z))}{w_1(y,z)}}$$

Define

$$\phi_w(x,y) = \begin{cases} \frac{w_1(x,y)}{1-\frac{\theta(w_1(x,y))}{w_1(x,y)}} & ,x \neq y \\ 0 & ,x = y \end{cases}$$

Therefore T satisfies (3.1) of Theorem 1, so we conclude that T has a fixed point $u \in X_w$ and the proof is complete. □

4. AN APPLICATION TO FUNCTIONAL EQUATIONS

We use fixed point theory in many fields of mathematics. One of theese fields is mathematical optimization.Dynamic programing is useful for mathematical optimizationand it is related to a multistage process reduces to solving functional equation $p(x)$. In this section we try to give an application of Theorem 2 to a functional equation defined as

$$p(x) = \sup_{y \in T} \quad \{f(x,y) + \Im(x,y,p(\eta(x,y)))\}, \quad x \in Z, \tag{4.1}$$

where $\eta : \mathcal{Z} \times \mathcal{T} \to \mathcal{Z},\ f : \mathcal{Z} \times \mathcal{T} \to \mathbb{R}$, and $\Im : Z \times T \times \mathbb{R} \to \mathbb{R}$. We assume that M and N are Banach spaces$\mathcal{Z} \subset \mathcal{M}$and $\mathcal{T} \subset \mathcal{N}$

Now we study the existence of the bounded solution of the functional equation (4.1). Let $\mathcal{B}(\mathcal{Z})$ denote the set of all bounded real-valued functions on Z, and for an arbitrary $h \in \mathcal{B}(\mathcal{Z})$, define $\|h\| = \sup_{x \in Z} |h(x)|$. Clearly, $(\mathcal{B}(\mathcal{Z}), \|.\|)$ endowed with the metric modular

$$w_\lambda(h,k) = \frac{1}{1+\lambda} \sup_{x \in Z} |h(x) - k(x)| \tag{4.2}$$

specificially for $\lambda = 1$ writen as

$$w_1(h,k) = \frac{1}{2} \sup_{x \in Z} |h(x) - k(x)|$$

for all $h, k \in \mathcal{B}(\mathcal{Z})$, is Banach space, so the convergence in this space according to $w_1(h,k)$ is uniform which means a Cauchy sequence in $\mathcal{B}(\mathcal{Z})$ is uniformly convergent to a function say h_*, that is bounded and so $h_* \in \mathcal{B}(\mathcal{Z})$

Now we will give the theorem that gives the solution of the functional equation defined in (4.1). To give the solution let us define an operator as follows:

$$S(h)(x) = \sup_{y \in T} \quad \{f(x,y) + \Im(x,y,h(\eta(x,y)))\} \tag{4.3}$$

for all $h \in \mathcal{B}(\mathcal{Z})$ and $x \in \mathcal{Z}$

Now we give the theorem for the bounded solution of functional equation given in (4.4)

Theorem Let $S : \mathcal{B}(\mathcal{Z}) \to \mathcal{B}(\mathcal{Z})$be an upper semicontinuous operator defined by (4.4) and assume that the following conditions are satisfied:
(i) f :$\mathcal{Z} \times \mathcal{T} \to \mathbb{R}$, and $\Im : \mathcal{Z} \times \mathcal{T} \times \mathbb{R} \to \mathbb{R}$ are continuous and bounded;
(ii) for all $h, k \in \mathcal{B}(\mathcal{Z})$, if

$$\begin{aligned} 0 &< w_1(h,k) < 1 \\ & \text{implies} \\ \frac{1}{2}\left|\Im(x,y,h(\eta(x,y))) - \Im(x,y,k(\eta(x,y)))\right| &\leq w_1^2(h,k) \\ w_1(h,k) &\geq \\ & \text{implies} \\ \frac{1}{2}\left|\Im(x,y,h(\eta(x,y))) - \Im(x,y,k(\eta(x,y)))\right| &\leq w_1(h,k) \end{aligned}$$

.where $x \in \mathcal{Z}$ and $y \in \mathcal{T}$. Then the functional equation (4.1) has a bounded solution.

Proof. Clearly$\mathcal{B}(\mathcal{Z})$ is a complete modular metric space .Let $\mu > 0$ be arbitrary, $x \in \mathcal{Z}$, and $h_1, h_2 \in \mathcal{B}(\mathcal{Z})$, then there exists $y_1, y_2 \in \mathcal{T}$ such that

$$\begin{aligned} S(h_1)(x) &< f(x,y) + \Im(x,y,h_1(\eta(x,y))) + \mu \\ S(h_2)(x) &< f(x,y) + \Im(x,y,h_2(\eta(x,y))) + \mu \\ S(h_1)(x) &\geq f(x,y) + \Im(x,y,h_1(\eta(x,y))) \\ S(h_2)(x) &\geq f(x,y) + \Im(x,y,h_2(\eta(x,y))) \end{aligned}$$

Let $\varrho : [0,\infty] \to [0,\infty]$ be defined as

$$\varrho(t) = \begin{cases} t^2, & 0 < t < 1 \\ t, & t \geq 1 \end{cases}$$

Then we get

$$\frac{1}{2}\left|\Im(x,y,h(\eta(x,y))) - \Im(x,y,k(\eta(x,y)))\right| \leq \varrho(w_1(h,k))$$

for all $h, k \in \mathcal{B}(\mathcal{Z})$.It is clear that $\varrho(t) < t$ for all $t > 0$ and $\frac{\varrho(t)}{t}$ is nondecreasing function. Therefore when we use the inequalities above we get

$$\begin{aligned} \frac{1}{2}(S(h_1)(x) - S(h_2)(x)) &< \frac{1}{2}\left\{\Im(x,y,h_1(\eta(x,y))) - \Im(x,y,h_2(\eta(x,y)))\right\} + \mu \\ &\leq \frac{1}{2}\left|\Im(x,y,h_1(\eta(x,y))) - \Im(x,y,h_2(\eta(x,y)))\right| + \mu \\ &\leq \varrho(w_1(h_1,h_2)) + \mu \end{aligned}$$

Then inequality turns into

$$\frac{1}{2}\left\{S(h_1)(x) - S(h_2)(x)\right\} < \varrho(w_1(h_1,h_2)) + \mu$$

Analogously we get

$$\frac{1}{2}\left\{S(h_2)(x) - S(h_1)(x)\right\} < \varrho(w_1(h_1,h_2)) + \mu$$

Hence theese inequalities equal to

$$\frac{1}{2}\left|S(h_1)(x) - S(h_2)(x)\right| < \varrho(w_1(h_1,h_2)) + \mu$$

which means that

$$w_1(S(h_1)(x), S(h_2)(x)) < \varrho(w_1(h_1,h_2)) + \mu$$

Since the above inequality turns into

$$w_1(S(h_1)(x), S(h_2)(x)) \leq \varrho(w_1(h_1, h_2)),$$

hence we get that S is a $\varrho-$contraction. And S satisfies the conditions in Theorem 2 and S has a fixed point say $h_* \in \mathcal{B}(\mathcal{Z})$, which means is a bounded solution of the functional equation (4.1). □

References

[1] Afrah A.N. Abdou, *On asymptotic pointwise contractions in modular metric spaces*, Abstract and Applied Analysis 2013(2013), Art. ID 50163.

[2] Afrah A.N Abdou, Mohamed Khams, *Fixed points of multivalued contraction mappings in modular metric spaces,*Fixed Point Theory and Applications 2014(2014), Art. ID 249.

[3] B. Azadifar, M. Maramaei, Gh. Sadeghi, *On the modular G-metric spaces and fixed point theorems*, Journal of Nonlinear Sciences and Applications, 6(2013), 293-304.

[4] B. Azadifar, M. Maramaei, Gh. Sadeghi, *Common fixed point theorems in modular G-metric spaces*, Journal Nonlinear Analysis and Application, 2013(2013), Art. ID jnaa-00175.

[5] B. Azadifar, Gh. Sadeghi, R. Saadati, C. Park, *Integral type contractions in modular metric spaces*, Journal of Inequalities and Applications, 2013(2013), Art. ID 483.

[6] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., 3(1922), 133-181.

[7] P. Chaipunya, Y.J. Cho, P. Kumam, *Geraghty-type theorems in modular metric spaces with an application to partial differential equation*, Advances in Difference Equations, 2012(2012), Art. ID 83.

[8] V.V. Chistyakov, *Modular metric spaces generated by F-modulars*, Folia Math., 14(2008), 3-25.

[9] V.V. Chistyakov, *Modular metric spaces I. basic conceps*, Nonlinear Anal., 72(2010), 1-14.

[10] V.V. Chistyakov, *Fixed points of modular contractive maps*, Doklady Mathematics, 86(1)(2012), 515-518.

[11] Y.J. Cho, R. Saadati, Gh. Sadeghi, *Quasi-contractive mappings in modular metric spaces*, Journal of Applied Mathematics, 2012(2012), Art. ID 907951.

[12] H. Dehghan, M. Eshaghi Gordji, A. Ebadian, *Comment on 'Fixed point theorems for contraction mappings in modular metric spaces*, Fixed Point Theory and Applications, 2012(2012), Art. ID 144.

[13] I. Ekeland, *On the variational principle,*J. Math. Anal. Appl.47(1974),324-353.

[14] E. Kılınç, C. Alaca, *A fixed point theorem in modular metric spaces*, Advances in Fixed Point Theory, 4(2)(2014), 199-206.

[15] E. Kılınç, C. Alaca, *Fixed point results for commuting mappings in modular metric spaces*, Journal of Applied Functional Analysis,(10)(2015), 204-210

[16] Afrah AN Abdou, M. Khamsi, *Fixed points of multivalued contraction mappings in modular metric spaces,* Fixed Point Theory and Applications, 2014(2014), Art. ID 249.

[17] F. Khojasten, E. Karapınar, H. Khandani, *Some applications of Caristi's fixed point theorem in metric spaces,*Fixed Pont Theory and Applications, (2016)2016, Art. ID 16.

[18] S. Koshi, T. Shimogaki, *On F-norms of quasi-modular space*, J. Fac. Sci. Hokkaido Univ. Ser 1., 15(3-4)(1961), 202-218.

[19] C. Mongkolkeha, W. Sintunavarat, P. Kumam, *Fixed point theorems for contraction mappings in modular metric spaces*, Fixed Point Theory and Applications, 2011(2011), Art. ID 93.

[20] H. Nakano, *Modulared semi-ordered linear space*, In Tokyo Math. Book Ser, vol.1, Maruzen Co, Tokyo (1950).

[21] S. Yamammuro, *On conjugate space of Nakano space*, Trans. Amer. Math. Soc., 90(1959), 291-311.

[1]Department of Mathematics, Faculty of Science, Gazi University, Ankara, Turkey

[2]Department of Mathematics, Institute of Natural and Applied Science, Gazi University, Ankara, Turkey

*Corresponding author: eklnc07@gmail.com