International Journal of Analysis and Applications
ISSN 2291-8639
Volume 3, Number 2 (2013), 68-80
http://www.etamaths.com

POLYNOMIAL APPROXIMATION ON UNBOUNDED SUBSETS AND THE MARKOV MOMENT PROBLEM

OCTAV OLTEANU

ABSTRACT. We start this review paper by recalling some known and relatively recent results in polynomial approximation on unbounded subsets. These results allow approximation of nonnegative continuous functions with compact support contained in the first quadrant by sums of tensor products of positive polynomials in each separate variable, on the positive semiaxes. Consequently, we characterize the existence of solution of a two dimensional Markov moment problem in terms of products of quadratic forms. Secondly, one proves some applications of abstract results on the extension of linear operators with two constraints to the Markov moment problem. Two applications related to this last part are considered.

1. Introduction

Using polynomial decomposition and approximation in existence, uniqueness and construction of the solution of the classical moment problem is a well known technique [1]-[3], [5]-[20]. Another general method is to endow different concrete spaces (including spaces of analytic functions) with a natural linear order relation. On such spaces, the abstract results from [4], [12] and many other works can be applied. For the construction and the uniqueness of the solution, L^2 – approximation is usually sufficient [9], [14],

---

2010 Mathematics Subject Classification. 47A57, 47A50, 41A10
Key words and phrases. Markov moment problem, approximation

but not necessary [2]. For the characterization of the existence of the solution, L^1 – approximation is necessary, and sometimes is also sufficient. The idea is to approximate nonnegative continuous functions with compact support by sums of tensor products of positive polynomials in each separate variable, which for the expression in terms of sums of squares is well known. This leads to characterization of the existence of the solution in terms of products of quadratic forms or mappings [16]-[20]. A similar idea in solving complex moment problems, by other techniques, appears in [8].

The first aim of this paper is to illustrate these methods. Secondly, we apply general extension theorems for linear operators, with two constraints, to the moment problem. Theorems 4.2 and 4.4 are such applications. For uniqueness of the solution see [1], [2]. For other aspects of the moment problem see [3], [5]. The background of this work is partially contained in [1], [4], [12].

The rest of this work is organized in the following way. Section 2 contains polynomial approximation results on unbounded closed subsets.

In Section 3, an application of these results to the real multidimensional moment problem is illustrated. Section 4 is devoted to some applications of earlier theorems of extension of linear operators to the Markov moment problem. A common point of all these considerations is the Hahn-Banach principle and its generalizations.

2. Approximation on unbounded subsets

The following results were published in [10], [11], being applied in solving moment problems from [11], recently recalled in [14]-[15]. For the multidimensional moment problem on unbounded subsets, Stone-Weierstrass and Luzin's theorems are used too.

Lemma 2.1. *For any $x \in R$, we have*

$$\exp(x) - \left(1 + \frac{x}{1!} + \ldots + \frac{x^m}{m!}\right) = \frac{\exp(x)}{m!} \cdot \int_0^x \exp(-t) \cdot t^m \, dt, \; m \in \mathrm{N}.$$

The proof is quite standard. Multiplication by $\exp(-x)$ followed by derivation-operation leads to the equality of the derivatives. Then the conclusion follows easily.

Remark 2.1. The statement remains true when we replace $x \in R$ with $z \in C$, by analytic continuation.

Corollary 2.1. *For all* $\alpha > 0, k \in \mathrm{N}$ *we have*

$$\exp(-\alpha t) \leq 1 - \frac{\alpha \cdot t}{1!} + \frac{\alpha^2 \cdot t^2}{2!} - \cdots + \frac{\alpha^{2k} t^{2k}}{(2k)!}, \ t \geq 0,$$

$$\exp(-\alpha t) \geq 1 - \frac{\alpha \cdot t}{1!} + \frac{\alpha^2 \cdot t^2}{2!} - \cdots + \frac{\alpha^{2k} t^{2k}}{(2k)!} - \frac{\alpha^{2k+1} t^{2k+1}}{(2k+1)!}, \ t \geq 0.$$

Corollary 2.2. *Let* $\varphi_k(t) = \exp(-kt), t \geq 0, k \in \mathrm{N}$, *and* ψ *an element of the linear subspace generated by* $\{\varphi_k ; k \in \mathrm{N}\}$. *Then there exists a sequence of polynomials*

$$(p_l)_{l \in \mathrm{N}}, \ p_l(t) > \psi(t) \ \forall t \geq 0,$$

and $\lim p_l = \psi$ *uniformly on the compact subsets* $K \subset [0, \infty)$ *and in* $L^1_\nu([0, \infty))$, *for any positive regular determinate Borel measure* ν *on* $[0, \infty)$.

Theorem 2.1. *Let* $\psi : [0, \infty) \to R_+$ *be a continuous function, such that* $\lim_{t \to \infty} \psi(t) \in R_+$ *exists. Then there is a decreasing sequence* $(h_l)_l$ *in the linear hull of the functions* $\varphi_k, k \in \mathrm{N}$ *defined above, such that* $h_l(t) > \psi(t), t \geq 0, l \in \mathrm{N}, \lim h_l = \psi$ *uniformly on* $[0, \infty)$. *There exists a sequence* $(\tilde{p}_l)_l, \tilde{p}_l \geq h_l > \psi, \forall l \in \mathrm{N}, \lim \tilde{p}_l = \psi$ *uniformly on compact subsets of* $[0, \infty)$.

The idea of the proof is to join the infinity point and to apply Stone-Weierstrass theorem for the subalgebra generated by the functions $\exp(-n \cdot t), n \in \mathrm{N}, t \geq 0$. Then one approximates the exponentials by partial sums-polynomials. The proofs of the preceding results are published in [10] and [11].

Theorem 2.2. (see [11], [16]-[20]). *Let* $A \subset R^n$ *be an unbounded closed subset and* ν *a positive regular Borel measure on* A, *with finite moments of all orders. Then for any* $\psi \in (C_0(A))_+$, *there is a sequence* $(p_m)_m$ *of polynomials on* A, $p_m \geq \psi$, $p_m \to \psi$ *in* $L^1_\nu(A)$. *We have*

$$\lim \int_A p_m \, d\nu = \int_A \psi \, d\nu ,$$

the cone P_+ *of positive polynomials is dense in* $\left(L^1_\nu(A)\right)_+$ *and* P *is dense in* $L^1_\nu(A)$.
An improved proof of this lemma will appear soon.

3. Approximation and the Markov moment problem

Let

$$x_{j,k}(t_1, t_2) = t_1^j t_2^k, \ (j,k) \in \mathrm{N}^2, \ t_l \geq 0, \ l = 1,2.$$

Using the form of positive polynomials on R_+ [1] in terms of squares:

$$p_l(t_l) = p_{l,1}^2(t_l) + t_l \, p_{l,2}^2(t_l), \ t_l \geq 0, \ l = 1,2,$$

and the above approximation results, one obtains the following result.

Let H be a Hilbert space, A_1, A_2 two positive commuting selfadjoint operators acting on H, with spectrums $\sigma(A_j)$, $j = 1,2$. We introduce the commutative algebra $Y = Y(A_1, A_2)$ of selfadjoint operators ([4], [7]), which is also an order-complete vector lattice:

$$Y_1 = \{T \in \mathrm{A}(H); TA_j = A_j T, \ j = 1,2\},$$
$$Y = \{U \in Y_1; UT = TU \quad \forall \, T \in Y_1\}.$$

We denote by X the space of all continuous functions $x : [0,\infty)^2 \to R$, with the modulus dominated by a polynomial at each point of $[0,\infty)^2$, and by

$x_{(j,k)}$ the elements of the base of polynomials, namely $x_{(j,k)}(t_1,t_2)=t_1^j t_2^k,\ t_j\geq 0$. Let $K=\sigma(A_1)\times\sigma(A_2)$, $X_1=C(K)$.

Theorem 3.1 (see also [11], [15]) *Let* $\left(B_{(j,k)}\right)_{(j,k)\in Z_+^2}\subset Y$. *The following assertions are equivalent:*

(a) *there is a linear operator* $F\in L(X_1,Y)$ *such that:*

$$F\left(x_{(j,k)}\right)=B_{(j,k)}\ \forall (j,k)\in Z_+^2,\ \ 0\leq F(x)\leq \int_{\sigma(A_1)\times\sigma(A_2)} x(t_1,t_2)\,dE_{A_1}\,dE_{A_2}\ \ \forall x\in X_{1,+},$$

$$\left|F(\varphi)\right|\leq \int_{\sigma(A_1)\times\sigma(A_2)}\left|\varphi(t_1,t_2)\right|dE_{A_1}\,dE_{A_2}\ \ \forall \varphi\in X_1,\ \|F\|\leq 1;$$

(b) *for any finite subsets* $J_1,J_2\subset Z_+$ *and any* $\{\alpha_j\}_{j\in J_1}$, $\{\beta_k\}_{k\in J_2}$, *we have*:

$$0\leq \sum_{i,j\in J_1,k,l\in J_2}\alpha_i\alpha_j\beta_k\beta_l B_{(i+j,k+l)}\leq \sum_{i,j\in J_1,k,l\in J_2}\alpha_i\alpha_j\beta_k\beta_l A_1^{i+j}A_2^{k+l},$$

$$0\leq \sum_{i,j\in J_1,k,l\in J_2}\alpha_i\alpha_j\beta_k\beta_l B_{(i+j+1,k+l)}\leq \sum_{i,j\in J_1,k,l\in J_2}\alpha_i\alpha_j\beta_k\beta_l A_1^{i+j+1}A_2^{k+l},$$

$$0\leq \sum_{i,j\in J_1,k,l\in J_2}\alpha_i\alpha_j\beta_k\beta_l B_{(i+j,k+l+1)}\leq \sum_{i,j\in J_1,k,l\in J_2}\alpha_i\alpha_j\beta_k\beta_l A_1^{i+j}A_2^{k+l+1},$$

$$0\leq \sum_{i,j\in J_1,k,l\in J_2}\alpha_i\alpha_j\beta_k\beta_l B_{(i+j+1,k+l+1)}\leq \sum_{i,j\in J_1,k,l\in J_2}\alpha_i\alpha_j\beta_k\beta_l A_1^{i+j+1}A_2^{k+l+1}.$$

Proof. Let x be a nonnegative continuous function defined on $\sigma(A_1)\times\sigma(A_2)$. Then x is approximated by means of a sequence formed by sums of tensor products of positive functions from $(C_c([0,\infty)))_+\otimes(C_c([0,\infty)))_+$. To this end, one applies firstly Luzin's theorem and Bernstein approximation theorem for a rectangle

$$[a_1, b_1] \times [a_2, b_2] \supset \sigma(A_1) \times \sigma(A_2).$$

Extend each factor of any of the terms of this sum with zero value outside the intervals $[a_j, b_j]$, $j = 1,2$ and then apply Luzin's Theorem in each separate variable. Let $x_1 \otimes x_2$, $x_j \geq 0$, $j = 1,2$ be such a tensor product. For each of the separate variables t_1, t_2, applying Theorem 2.1, there is a sequence of positive polynomials on the nonnegative semi axes such that:

$$p_{m,j} > x_j \geq 0, \forall m \in Z_+, \ p_{m,j} \to x_j, \ m \to \infty, \ j = 1,2,$$

the convergence being uniform on compact subsets. Because the polynomials involved are positive on the whole interval $[0, \infty)$, from [1] we know their form:

$$p_{m,j}(t_j) = q_{m,j}^2(t_j) + t_j r_{m,j}^2(t_j), \ j = 1,2, \ m \in Z_+.$$

We define a linear operator on the subspace $P_1 \otimes P_2$ generated by the products of polynomials in separate variables, such that the moment conditions from the statement holds:

$$f\left(\sum_{j,k} \alpha_j \beta_k x_{(j,k)}\right) = \sum_{j,k} \alpha_j \beta_k B_{(j,k)} \Rightarrow f\left(x_{(j,k)}\right) = B_{(j,k)} \ \forall (j,k) \in Z_+^2.$$

We have already seen that every continuous function with compact support can be approximated by elements from $P_1 \otimes P_2$. Using the above arguments, the assertion (b) says that we have:

$$0 \leq f(p_1 \otimes p_2) \leq \iint_{\sigma(A_1) \times \sigma(A_2)} p_1(t_1) p_2(t_2) dE_{A_1} dE_{A_2}, \ \forall p_j \in (R[t_j])_+, \ j = 1,2.$$

An application of the majorizing subspace lemma ([4] p. 160), leads to the existence of a positive linear (hence bounded on $X_1 = C(\sigma(A_1) \times \sigma(A_2))$) extension $F : X_1 \to Y$ of f. Using the uniform convergence of the special polynomials on the product of spectrums, we obtain:

$$F(x) = \lim_m F\left(\sum_{j=1}^{k(m)} p_{m,1,j} \otimes p_{m,2,j}\right) = \lim_m f\left(\sum_{j=1}^{k(m)} p_{m,1,j} \otimes p_{m,2,j}\right) \leq$$

$$\leq \lim_m \iint_{\sigma(A_1)\times\sigma(A_2)} \left(\sum_{j=1}^{k(m)} p_{m,1,j} \otimes p_{m,2,j}\right) dE_{A_1} dE_{A_2} =$$

$$\iint_{\sigma(A_1)\times\sigma(A_2)} x(t_1, t_2)\, dE_{A_1} dE_{A_2}, \ \forall x \in X_{1,+}.$$

This relation leads to the last conclusion (a), thanks to the following relation:

$$\varphi \in X \Rightarrow |F(\varphi)| \leq F(\varphi^+) + F(\varphi^-) \leq \iint_{\sigma(A_1)\times\sigma(A_2)} |\varphi(t_1, t_2)| dE_{A_1} dE_{A_2} \Rightarrow$$

$$\|F(\varphi)\| \leq \|\varphi\|_\infty, \ \forall \varphi \in X_1 \Rightarrow \|F\| \leq 1.$$

Thus the proof of $(b) \Rightarrow (a)$ is finished. Since the converse is obvious, the proof of the theorem is complete. □

4. Extension of linear operators and the moment problem

If V is a convex neighborhood of the origin in a locally convex space, we denote by p_V the gauge attached to V. This section is based on some results of [6] and on previous results (see the references there).

Theorem 4.1. *.Let X be a locally convex space, Y an order complete vector lattice with strong order unit u_0 and $S \subset X$ a vector subspace. Let $A \subset X$ be a convex subset with the following qualities:*

(i) *there exists a neighborhood V of the origin such that $(S+V)\cap A = \Phi$* (*A and S are distanced*);

(ii) *A is bounded.*

Then for any equicontinuous family of linear operators $\{f_j\}_{j\in J} \subset L(S,Y)$ and for any $\tilde{y} \in Y_+ \setminus \{0\}$, there exists an equicontinuous family $\{F_j\}_{j\in J} \subset L(X,Y)$ such that

$$F_j|_S = f_j \text{ and } F_j|_A \geq \tilde{y}, \forall j \in J.$$

Moreover, if V is a neighborhood of the origin such that

$$f_j(V\cap S) \subset [-u_0, u_0], \quad (S+V)\cap A = \Phi,$$

$$0 < \alpha \in R \ s.t. \ p_V|_A \leq \alpha, \quad \alpha_1 > 0 \ s.t. \ \tilde{y} \leq \alpha_1 u_0,$$

then the following relations hold

$$F_j(x) \leq (1+\alpha+\alpha_1)p_V(x)\cdot u_0, \quad x \in X, \ j \in J.$$

The last relation of Theorem 4.1 gives a relationship between an upper bound and the lower bound of F_j, $j \in J$. The next result is a consequence of the preceding one, in terms of the moment problem. In oder to apply the above general theorem to concrete spaces, let $X = \overline{H}$ be the space of all continuous functions in the polydisc $\overline{D} = \prod_{j=1}^{n} \{|z_j| \leq 1\}$, which can be writen as power series with real coefficients, centered at $(0,\dots,0)$, in the open polydisc D. The order relation on X is given by the positive cone of all power series with nonnegative coefficients. Let

$$\varphi_j(z_1,\dots,z_n) = z_1^{j_1}\cdots z_n^{j_n}, \ j = (j_k)_{k=1}^n, |j| = \sum_{k=1}^{n} j_k \geq 1$$

and A_k, $k = 1,\dots,n$ linear positive selfadjoint commuting operators on H, such that $\|A_k\| < 1$, $k = 1,\dots,n$. We denote:

$$Y_1 = \{U \in A(H);\ A_k U = UA_k,\ k = 1,\dots, n\},\ Y = \{V \in Y_1;\ VU = UV\ \ \forall U \in Y_1\}.$$

Then Y is an order complete Banach lattice and a commutative Banach algebra of selfadjoint operators [4], [7]. Let $\{B_k\}_{k=1}^n \subset Y_+,\ \|B_k\| < 1,\ k = 1,\dots, n.$

Theorem 4.2. *Let* $(U_j)_{j\in N^n},\ |j| \geq 1$ *be sequence in* Y, *such that*

$$|U_j| \leq \rho \cdot A_1^{j_1} \cdots A_n^{j_n} + \delta \cdot B_1^{j_1} \cdots B_n^{j_n},\ \ \forall j = (j_1,\dots, j_n) \in N^n,\ |j| \geq 1.$$

Let $\tilde{B} \in Y_+,\ \{\psi_j\}_{j\in N^n} \subset X,\ \psi_j(0,\dots,0) = 1,\ \|\psi_j\|_\infty \leq 1\ \ \forall j \in N^n$. *Then there is a linear bounded operator* $F \in B(X, Y)$ *such that*

$$F(\varphi_j) = U_j,\ |j| \geq 1,\ F(\psi_j) \geq \tilde{B},\ j \in N^n,$$

$$|F(\varphi)| \leq \left\{ 2 + \|\tilde{B}\| \cdot \left(\rho \Big/ \left(\prod_{k=1}^{n} (1 - \|A_k\|) \right) + \delta \Big/ \left(\prod_{k=1}^{n} (1 - \|B_k\|) \right) \right)^{-1} \right\} \cdot \|\varphi\|_\infty \cdot u_0,$$

$$u_0 := \left(\rho \Big/ \left(\prod_{k=1}^{n} (1 - \|A_k\|) \right) + \delta \Big/ \left(\prod_{k=1}^{n} (1 - \|B_k\|) \right) \right) \cdot I,\quad \varphi \in X.$$

Proof. We apply Theorem 4.1 to $S = Sp\{\varphi_j;\ j \in N^n,\ |j| \geq 1\},\ A = co(\{\psi_j;\ j \in N^n\})$. Conditions imposed on the values at $(0,\cdots,0)$ and on the norms of the functions ψ_j lead to

$$\|\varphi_j - \psi_m\| \geq |\varphi_j(0,\dots,0) - \psi_m(0,\dots,0)| = 1 \Rightarrow$$
$$(S + B(0,1)) \cap A = \Phi \Rightarrow p_V(\cdot) = \|\cdot\| \Rightarrow p_{V|A} \leq 1 = \alpha.$$

On the other hand, for

$$s \in S \cap B(0,1),\ f(s) = f\left(\sum_{j\in J_0} \lambda_j \varphi_j \right) = \sum_{j\in J_0} \lambda_j U_j,$$

Cauchy inequalities for s and the hypothesis on the operators $|U_j|$ yield

$$|f(s)| = \left| \sum_{j \in J_0} \lambda_j U_j \right| \le \sum_{j \in J_0} |\lambda_j| \cdot |U_j| \le$$

$$\|s\| \cdot \left(\rho \cdot \sum_{j \in \mathbb{N}^n} A_1^{j_1} \cdots A_n^{j_n} + \delta \cdot \sum_{j \in \mathbb{N}^n} B_1^{j_1} \cdots B_n^{j_n} \right) \le$$

$$\rho \cdot \left(\sum_{j_1 \in \mathbb{N}} \|A_1\|^{j_1} \right) \cdots \left(\sum_{j_n \in \mathbb{N}} \|A_n\|^{j_n} \right) \cdot \mathrm{I} + \delta \cdot \left(\sum_{j_1 \in \mathbb{N}} \|B_1\|^{j_1} \right) \cdots \left(\sum_{j_n \in \mathbb{N}} \|B_n\|^{j_n} \right) \cdot I =$$

$$= \left(\frac{\rho}{\prod_{k=1}^{n} (1 - \|A_k\|)} + \frac{\delta}{\prod_{k=1}^{n} (1 - \|B_k\|)} \right) \cdot I = u_0 \Rightarrow -u_0 \le f(s) \le u_0, \quad \forall s \in S \cap B(0,1).$$

We also have

$$\tilde{B} \le \|\tilde{B}\| \cdot \mathrm{I} = \left(\rho / \left(\prod_{k=1}^{n} (1 - \|A_k\|) \right) + \delta / \left(\prod_{k=1}^{n} (1 - \|B_k\|) \right) \right)^{-1} \cdot \|\tilde{B}\| \cdot u_0 .$$

Now all conditions from the hypothesis of theorem 4.1 are accomplished and the conclusion follows.
□

We recall the following result [12] on the abstract Markov moment problem, as an extension with two constraints theorem for linear operators.

Theorem 4.3. *Let X be an ordered vector space, Y an order complete vector lattice, $\{x_j\}_{j \in J} \subset X$, $\{y_j\}_{j \in J} \subset Y$ given families and $F_1, F_2 \in L(X,Y)$ two linear operators. The following statements are equivalent:*

(a) *there is a linear operator $F \in L(X,Y)$ such that*

$$F_1(x) \le F(x) \le F_2(x) \, \forall x \in X_+, \, F(x_j) = y_j \; \forall j \in J;$$

(b) *for any finite subset* $J_0 \subset J$ *and any* $\{\lambda_j\}_{j \in J_0} \subset R$, *we have:*

$$\left(\sum_{j \in J_0} \lambda_j x_j = \psi_2 - \psi_1, \psi_1, \psi_2 \in X_+ \right) \Rightarrow \sum_{j \in J_0} \lambda_j y_j \le F_2(\psi_2) - F_1(\psi_1).$$

From Theorem 4.3 we deduce the following result.

Theorem 4.4. *With the notations and using the assumptions preceding Theorem 4.2, the following statements are equivalent:*

(a) *there exists* $F \in B(X, Y)$ *such that*

$$F(\varphi_j) = U_j,\ j \in \mathrm{N}^n,\ 0 \le F(\psi) \le \rho \cdot \psi(A_1, \ldots, A_n) + \delta \cdot \psi(B_1, \ldots, B_n),\ \psi \in X_+,$$
$$\|F\| \le \rho + \delta;$$

(b) *we have:* $0 \le U_j \le \rho \cdot A_1^{j_1} \cdots A_n^{j_n} + \delta \cdot B_1^{j_1} \cdots B_n^{j_n},\ j = (j_1, \ldots, j_n) \in \mathrm{N}^n.$

Proof. The implication $(a) \Rightarrow (b)$ is obvious, because of the relations

$$\varphi_j \in X_+ \Rightarrow U_j = F(\varphi_j) \in [0, \rho \cdot \varphi_j(A_1, \ldots, A_n) + \delta \cdot \varphi_j(B_1, \ldots, B_n)] =$$
$$\left[0, \rho \cdot A_1^{j_1} \cdots A_n^{j_n} + \delta \cdot B_1^{j_1} \cdots B_n^{j_n}\right], \quad j \in \mathrm{N}^n.$$

Conversely, assume that (b) holds. We verify the implication in (b), Theorem 4.3. Namely, we have:

$$\sum_{j \in J_0} \lambda_j \varphi_j = \psi_2 - \psi_1 = \sum_{m \in \mathrm{N}^n} \alpha_m \varphi_m - \sum_{m \in \mathrm{N}^n} \beta_m \varphi_m,\ \ \alpha_m, \beta_m \ge 0,\ m \in \mathrm{N}^n \Rightarrow$$
$$\sum_{j \in J_0} \lambda_j U_j \le \sum_{j \in J_0^+} \lambda_j U_j \le \sum_{j \in \mathrm{N}^n} \alpha_j U_j \le \sum_{j \in \mathrm{N}^n} \alpha_j \left(\rho \cdot A_1^{j_1} \cdots A_n^{j_n} + \delta \cdot B_1^{j_1} \cdots B_n^{j_n}\right) =$$
$$= \rho \cdot \psi_2(A_1, \ldots, A_n) + \delta \cdot \psi_2(B_1, \ldots, B_n) = F_2(\psi_2) - F_1(\psi_1),\ F_1 := 0,$$
$$J_0^+ = \{j \in J_0;\ \lambda_j \ge 0\}.$$

A direct application of Theorem 4.3 leads to the existence of a linear operator $F \in L(X, Y)$, such that

$$0 \leq F(\psi) \leq \rho \cdot \psi(A_1, \ldots, A_n) + \delta \cdot \psi(B_1, \ldots, B_n), \forall \psi \in X_+.$$

For an arbitrary $\varphi \in X$, one obtains:

$$|F(\varphi)| \leq F(\varphi^+) + F(\varphi^-) \leq \rho \cdot |\varphi|(A_1, \ldots, A_n) + \delta \cdot |\varphi|(B_1, \ldots, B_n) \leq$$
$$[\rho \cdot |\varphi|(\|A_1\|, \ldots, \|A_n\|) + \delta \cdot |\varphi|(\|B_1\|, \ldots, \|B_n\|)] \cdot I \Rightarrow \|F(\varphi)\| \leq (\rho + \delta) \cdot \|\varphi\|,$$
$$\forall \varphi \in X \Rightarrow \|F\| \leq \rho + \delta.$$

This concludes the proof.
□

REFERENCES

[1] Akhiezer, N. I., The Classical Moment Problem and some related Questions in Analysis, Oliver and Boyd, Edinburgh-London, 1965.
[2] Berg, C. & Thill, M., Rotation invariant moment problems, Acta Mathematica, 11 (1991), 207-227.
[3] Choquet, G., Le problème des moments, Séminaire d'Initiation à l'Analise, Institut H. Poincaré, Paris 1962.
[4] Cristescu, R., Ordered Vector Spaces and Linear Operators, Academiei, Bucharest, and Abacus Press, Tunbridge Wells, Kent, 1976.
[5] Krein M. G. & Nudelman, A. A., Markov Moment Problem and Extremal Problems, American Mathematical Society, Providence, RI, 1977.
[6] Lemnete Ninulescu & Olteanu, O., Extension of linear operators, distanced convex sets and the moment problem, Mathematica, 46(69), 1 (2004), 81-88.
[7] Lemnete Ninulescu, L., Using the solution of an abstract moment problem to solve some classical complex moment problems, Revue Roumaine de Mathématiques Pures et Appliquées., 51 (2006), 703-711.
[8] Lemnete-Ninulescu, L. & Zlătescu, A., Some new aspects of the L moment problem, Revue Roumaine de Mathématiques Pures et Appliquées, 55, 3 (2010), 197-204.
[9] Mihăilă, J. M., Olteanu, O. & Udrişte, C., Markov-type and operator-valued multidimensional moment problems, with some applications, Revue Roumaine de Mathématiques Pures et Appliquées, 52, 4 (2007), 405-428.

[10] Mihăilă, J. M., Olteanu, O. & Udrişte, C., Markov-type moment problems for arbitrary compact and for some non-compact Borel subsets of R^n, Revue Roumaine de Mathématiques Pures et Appliquées, 52, 6 (2007), 655-664.
[11] Olteanu, A. & Olteanu, O., Some unexpected problems of the Moment Problem, "Proc. of the Sixth Congress of Romanian Mathematicians", Academiei, Vol I, Bucharest, (2009), 347-355.
[12] Olteanu, O., Application de théorèmes de prolongement d'opérateurs linéaires au problème des moments et à une generalisation d'un théorème de Mazur-Orlicz, Comptes Rendus de l'Académie des Sciences Paris, Série I, Vol. 313 (1991), 739-742.
[13] Olteanu, O., Geometric aspects in operator theory and applications, Lambert Academic Publishing, Saarbrücken, 2012.
[14] Olteanu, O., New results on Markov moment problem, International Journal of Analysis, Vol. 2013, Article ID 901318 (2013), 17 pages.
[15] Olteanu, O., Moment problems on bounded and unbounded domains, International Journal of Analysis, Vol. 2013, Article ID 524264 (2013), 7 pages.
[16] Olteanu, O., Markov moment problem and related approximation, (submitted, under review).
[17] Olteanu, O., Markov moment problems, related approximation and the Hahn-Banach principle, (submitted, under review).
[18] Olteanu, O., Approximation and Markov moment problem on concrete spaces, (submitted, under review).
[19] Olteanu, O., Markov moment problem and approximation (submitted, under review).
[20] Polynomial Approximation on Unbounded Sets and the Multidimensional Markov Moment Problem (submitted, under review).

DEPARTMENT OF MATHEMATICS-INFORMATICS, UNIVERSITY POLITEHNICA OF BUCHAREST, SPLAIUL INDEPENDENŢEI 313, 060042 BUCHAREST, ROMANIA