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HERMITE-HADAMARD TYPE INEQUALITIES FOR p-CONVEX FUNCTIONS

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Abstract. In this paper, the author establishes some new Hermite-Hadamard type inequalities for p-convex functions. Some natural applications to special means of real numbers are also given.

1. Introduction

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. Note that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if f is concave. For some results which generalize, improve and extend the inequalities (1.1) we refer the reader to the recent papers (see [2, 3, 5, 6, 8, 9, 12]).

In [3], Dragomir gave the following Lemma:

Lemma 1. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t)\, f'\left(ta+(1-t)b\right) dt. \tag{1.2}$$

By using this Lemma, Dragomir obtained the following Hermite-Hadamard type inequalities for convex functions:

Theorem 1. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)\left(|f'(a)|+|f'(b)|\right)}{8}. \tag{1.3}$$

Theorem 2. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$ and $p > 1$. If the new mapping $|f'|^q$ is convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{2(p+1)^{1/p}}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{1/q}, \tag{1.4}$$

where $1/p + 1/q = 1$.

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Let $A(a,b;t)=ta+(1-t)b$, $G(a,b;t)=a^{t}b^{1-t}$, $H(a,b;t)=ab/(ta+(1-t)b)$ and $M_{p}(a,b;t)=(ta^{p}+(1-t)b^{p})^{1/p}$, $p\in\mathbb{R}\backslash\{0\}$, be the weighted arithmetic, geometric, harmonic , power of order p means of two positive real numbers a and b with $a\neq b$ for $t\in[0,1]$, respectively. $M_{p}(a,b;t)$ is continuous and strictly increasing with respect to $t\in\mathbb{R}$ for fixed $p\in\mathbb{R}\backslash\{0\}$ and $a,b>0$ with $a>b$. See [13, 7] for some kinds of convexity obtained by using weighted means.

In [7], the author, gave definition Harmonically convex and concave functions as follow.

Definition 1. *Let $I\subset\mathbb{R}\backslash\{0\}$ be a real interval. A function $f:I\rightarrow\mathbb{R}$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right)\leq tf(y)+(1-t)f(x) \tag{1.5}$$

for all $x,y\in I$ and $t\in[0,1]$. If the inequality in (1.5) is reversed, then f is said to be harmonically concave.

The following result of the Hermite-Hadamard type holds for harmonically convex functions.

Theorem 3 ([7])**.** *Let $f:I\subset\mathbb{R}\backslash\{0\}\rightarrow\mathbb{R}$ be a harmonically convex function and $a,b\in I$ with $a<b$. If $f\in L[a,b]$ then the following inequalities hold*

$$f\left(\frac{2ab}{a+b}\right)\leq\frac{ab}{b-a}\int_{a}^{b}\frac{f(x)}{x^{2}}dx\leq\frac{f(a)+f(b)}{2}. \tag{1.6}$$

The above inequalities are sharp.

Lemma 2 ([7])**.** *Let $f:I\subset\mathbb{R}\backslash\{0\}\rightarrow\mathbb{R}$ be a differentiable function on I° and $a,b\in I$ with $a<b$. If $f'\in L[a,b]$ then*

$$\begin{aligned}&\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_{a}^{b}\frac{f(x)}{x^{2}}dx\\ &=\frac{ab(b-a)}{2}\int_{0}^{1}\frac{1-2t}{(tb+(1-t)a)^{2}}f'\left(\frac{ab}{tb+(1-t)a}\right)dt.\end{aligned} \tag{1.7}$$

Using this Lemma, the following inequalities hold.

Theorem 4 ([7])**.** *Let $f:I\subset(0,\infty)\rightarrow\mathbb{R}$ be a differentiable function on I°, $a,b\in I$ with $a<b$, and $f'\in L[a,b]$. If $|f'|^{q}$ is harmonically convex on $[a,b]$ for $q\geq1$, then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_{a}^{b}\frac{f(x)}{x^{2}}dx\right|\\ &\leq\frac{ab(b-a)}{2}\lambda_{1}^{1-\frac{1}{q}}\left[\lambda_{2}\left|f'(a)\right|^{q}+\lambda_{3}\left|f'(b)\right|^{q}\right]^{\frac{1}{q}},\end{aligned} \tag{1.8}$$

where

$$\begin{aligned}\lambda_{1}&=\frac{1}{ab}-\frac{2}{(b-a)^{2}}\ln\left(\frac{(a+b)^{2}}{4ab}\right),\\ \lambda_{2}&=\frac{-1}{b(b-a)}+\frac{3a+b}{(b-a)^{3}}\ln\left(\frac{(a+b)^{2}}{4ab}\right),\\ \lambda_{3}&=\frac{1}{a(b-a)}-\frac{3b+a}{(b-a)^{3}}\ln\left(\frac{(a+b)^{2}}{4ab}\right)\\ &=\lambda_{1}-\lambda_{2}.\end{aligned}$$

Theorem 5 ([7]). *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I$ with $a<b$, and $f' \in L[a,b]$. If $|f'|^q$ is harmonically convex on $[a,b]$ for $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \le \frac{ab\,(b-a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\mu_1\left|f'(a)\right|^q+\mu_2\left|f'(b)\right|^q\right)^{\frac{1}{q}}, \tag{1.9}$$

where

$$\begin{aligned}\mu_1 &= \frac{\left[a^{2-2q}+b^{1-2q}\left[(b-a)(1-2q)-a\right]\right]}{2\,(b-a)^2(1-q)(1-2q)},\\ \mu_2 &= \frac{\left[b^{2-2q}-a^{1-2q}\left[(b-a)(1-2q)+b\right]\right]}{2\,(b-a)^2(1-q)(1-2q)}.\end{aligned}$$

In [16], Zhang and Wan gave definition of p-convex function as follow:

Definition 2. *Let I be a p-convex set. A function $f : I \to \mathbb{R}$ is said to be a p-convex function or belongs to the class PC(I), if*

$$f\left(\left[\alpha x^p+(1-\alpha)y^p\right]^{1/p}\right) \le \alpha f(x)+(1-\alpha)f(y)$$

for all $x,y \in I$ and $\alpha \in [0,1]$.

Remark 1 ([16]). *An interval I is said to be a p-convex set if $\left[\alpha x^p+(1-\alpha)y^p\right]^{1/p} \in I$ for all $x,y \in I$ and $\alpha \in [0,1]$, where $p=2k+1$ or $p=n/m$, $n=2r+1$, $m=2t+1$ and $k,r,t \in \mathbb{N}$.*

Remark 2 ([10]). *If $I \subset (0,\infty)$ be a real interval and $p \in \mathbb{R}\backslash\{0\}$, then*
$\left[\alpha x^p+(1-\alpha)y^p\right]^{1/p} \in I$ for all $x,y \in I$ and $\alpha \in [0,1]$.

According to Remark 2, we can give a different version of the definition of p-convex function as follow:

Definition 3 ([10]). *Let $I \subset (0,\infty)$ be a real interval and $p \in \mathbb{R}\backslash\{0\}$. A function $f : I \to \mathbb{R}$ is said to be a p-convex function, if*

$$f\left(\left[\alpha x^p+(1-\alpha)y^p\right]^{1/p}\right) \le \alpha f(x)+(1-\alpha)f(y) \tag{1.10}$$

for all $x,y \in I$ and $\alpha \in [0,1]$. If the inequality in (1.10) is reversed, then f is said to be p-concave.

According to Definition 3, It can be easily seen that for $p=1$ and $p=-1$, p-convexity reduces to ordinary convexity and harmonically convexity of functions defined on $I \subset (0,\infty)$, respectively.

Example 1. *Let $f : (0,\infty) \to \mathbb{R}$, $f(x)=x^p, p \neq 0$, and $g : (0,\infty) \to \mathbb{R}$, $g(x)=c$, $c \in \mathbb{R}$, then f and g are both p-convex and p-concave functions.*

In [4, Theorem 5], if we take $I \subset (0,\infty)$, $p \in \mathbb{R}\backslash\{0\}$ and $h(t)=t$, then we have the following Theorem.

Theorem 6. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a p-convex function, $p \in \mathbb{R}\backslash\{0\}$, and $a,b \in I$ with $a<b$. If $f \in L[a,b]$ then we have*

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) \le \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx \le \frac{f(a)+f(b)}{2}. \tag{1.11}$$

For some results related to p-convex functions and its generalizations, we refer the reader to see [4, 10, 15, 14, 16].

In [15, Lemma 2.4], if we take $I \subset (0,\infty)$ and $p \in \mathbb{R}\backslash\{0\}$, then we have the following Lemma.

Lemma 3. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I° and $a,b \in I$ with $a<b$ and $p \in \mathbb{R}\setminus\{0\}$. If $f' \in L[a,b]$ then*

$$\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx$$
$$(1.12)\qquad = \frac{b^p-a^p}{2p}\int_0^1 \frac{1-2t}{\left[ta^p+(1-t)b^p\right]^{1-1/p}} f'\left(\left[ta^p+(1-t)b^p\right]^{1/p}\right)dt.$$

Remark 3. *In Lemma 3,*
(i) If we take $p=1$, then we have inequality (1.2) in Lemma 1.
(ii) If we take $p=-1$, then we have inequality (1.7) in Lemma 2.

For finding some new inequalities of Hermite-Hadamard type for functions whose derivatives are p-convex, we need Lemma 3.

We recall the following special functions

(1) The Beta function:

$$\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\int_0^1 t^{x-1}(1-t)^{y-1}dt,\quad x,y>0,$$

(2) The hypergeometric function:

$${}_2F_1(a,b;c;z)=\frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt,\ c>b>0,\ |z|<1 \text{ (see [11])}.$$

The main purpose of this paper is to establish some new results connected with the right-hand side of the inequalities (1.11) for p-convex functions.

2. Main Results

We obtain the another version of [15, Theorem 3.2] as follow:

Theorem 7. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I^\circ$ with $a<b$, $p \in \mathbb{R}\setminus\{0\}$ and $f' \in L[a,b]$. If $|f'|^q$ is p-convex on $[a,b]$ for $q \geq 1$, then*

$$(2.1)\qquad \left|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx\right| \leq \frac{b^p-a^p}{2p}C_1^{1-\frac{1}{q}}\left[C_2\left|f'(a)\right|^q+C_3\left|f'(b)\right|^q\right]^{\frac{1}{q}},$$

where

$$\begin{aligned} C_1 &= C_1(a,b;p)=\frac{1}{4}\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}-1} \\ &\quad\times\left[{}_2F_1\left(1-\frac{1}{p},2;3;\frac{a^p-b^p}{a^p+b^p}\right)+{}_2F_1\left(1-\frac{1}{p},2;3;\frac{b^p-a^p}{a^p+b^p}\right)\right], \end{aligned}$$

$$\begin{aligned} C_2 &= C_2(a,b;p)=\frac{1}{24}\left(\frac{a^p+b^p}{2}\right)^{\frac{1}{p}-1}\left[{}_2F_1\left(1-\frac{1}{p},2;4;\frac{a^p-b^p}{a^p+b^p}\right)\right. \\ &\quad\left.+6.{}_2F_1\left(1-\frac{1}{p},2;3;\frac{b^p-a^p}{a^p+b^p}\right)+{}_2F_1\left(1-\frac{1}{p},2;4;\frac{b^p-a^p}{a^p+b^p}\right)\right], \end{aligned}$$

$$C_3=C_3(a,b;p)=C_1-C_2,$$

Proof. From Lemma 3 and using the Power mean integral inequality, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx\right| \\
\leq\ & \frac{b^p-a^p}{2p}\int_0^1 \left|\frac{1-2t}{\left[ta^p+(1-t)b^p\right]^{1-1/p}}\right| \left|f'\left(\left[ta^p+(1-t)b^p\right]^{1/p}\right)\right| dt \\
\leq\ & \frac{b^p-a^p}{2p}\left(\int_0^1 \frac{|1-2t|}{\left[ta^p+(1-t)b^p\right]^{1-1/p}}dt\right)^{1-\frac{1}{q}} \\
& \times\left(\int_0^1 \frac{|1-2t|}{\left[ta^p+(1-t)b^p\right]^{1-1/p}}\left|f'\left(\left[ta^p+(1-t)b^p\right]^{1/p}\right)\right|^q dt\right)^{\frac{1}{q}}.
\end{aligned}$$

Hence, by p-convexity of $|f'|^q$ on $[a,b]$, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx\right| \\
\leq\ & \frac{b^p-a^p}{2p}\left(\int_0^1 \frac{|1-2t|}{\left[ta^p+(1-t)b^p\right]^{1-1/p}}dt\right)^{1-\frac{1}{q}} \\
& \times\left(\int_0^1 \frac{||1-2t||\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]}{\left[ta^p+(1-t)b^p\right]^{1-1/p}}dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{b^p-a^p}{2p}C_1^{1-\frac{1}{q}}\left[C_2\left|f'(a)\right|^q+C_3\left|f'(b)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

It is easily check that

$$\int_0^1 \frac{|1-2t|}{\left[ta^p+(1-t)b^p\right]^{1-1/p}}dt = C_1(a,b;p),$$

$$\int_0^1 \frac{|1-2t|\,t}{\left[ta^p+(1-t)b^p\right]^{1-1/p}}dt = C_2(a,b;p),$$

$$\int_0^1 \frac{|1-2t|\,(1-t)}{\left[ta^p+(1-t)b^p\right]^{1-1/p}}dt = C_1(a,b;p)-C_2(a,b;p).$$

□

Remark 4. *If we take $p=-1$ in Theorem 7, then we have inequality (1.8) in Theorem 4.*

If we take $q=1$ in Theorem 7, then we have the following corollary.

Corollary 1. *Let $f: I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I^\circ$ with $a<b$, $p\in\mathbb{R}\backslash\{0\}$ and $f'\in L[a,b]$. If $|f'|$ is p-convex on $[a,b]$, then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx\right| \\
\leq\ & \frac{b^p-a^p}{2p}\left[C_2\left|f'(a)\right|+C_3\left|f'(b)\right|\right],
\end{aligned}$$

where C_2 and C_3 are defined as in Theorem 7.

Remark 5. *If we take $p=1$ in Corollary 1, then we have inequality (1.3) in Theorem 1.*

Theorem 8. *Let $f: I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I$ with $a<b$, $p \in \mathbb{R}\setminus\{0\}$ and $f' \in L[a,b]$. If $|f'|^q$ is p-convex on $[a,b]$ for $q>1$, $\frac{1}{r}+\frac{1}{q}=1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx\right| \le \frac{b^p-a^p}{2p}\left(\frac{1}{r+1}\right)^{\frac{1}{r}}\left(C_4\left|f'(a)\right|^q+C_5\left|f'(b)\right|^q\right)^{\frac{1}{q}}, \tag{2.2}$$

where

$$C_4 = C_4(a,b;p;q) = \begin{cases} \frac{1}{2a^{qp-q}}\cdot {}_2F_1\left(q-\frac{q}{p},1;3;1-\left(\frac{b}{a}\right)^p\right), & p<0 \\ \frac{1}{2b^{qp-q}}\cdot {}_2F_1\left(q-\frac{q}{p},2;3;1-\left(\frac{a}{b}\right)^p\right), & p>0 \end{cases},$$

$$C_5 = C_5(a,b;p;q) = \begin{cases} \frac{1}{2a^{qp-q}}\cdot {}_2F_1\left(q-\frac{q}{p},2;3;1-\left(\frac{b}{a}\right)^p\right), & p<0 \\ \frac{1}{2b^{qp-q}}\cdot {}_2F_1\left(q-\frac{q}{p},1;3;1-\left(\frac{a}{b}\right)^p\right), & p>0 \end{cases}.$$

Proof. From Lemma 3, Hölder's inequality and the p-convexity of $|f'|^q$ on $[a,b]$,we have,

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx\right| \\
\le\ & \frac{b^p-a^p}{2p}\left(\int_0^1 |1-2t|^r\,dt\right)^{\frac{1}{r}} \\
&\times\left(\int_0^1 \frac{1}{[ta^p+(1-t)b^p]^{q-q/p}}\left|f'\left([ta^p+(1-t)b^p]^{1/p}\right)\right|^q dt\right)^{\frac{1}{q}} \\
\le\ & \frac{b^p-a^p}{2p}\left(\frac{1}{r+1}\right)^{\frac{1}{r}} \\
&\times\left(\int_0^1 \frac{t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q}{[ta^p+(1-t)b^p]^{q-q/p}}dt\right)^{\frac{1}{q}},
\end{aligned}$$

where an easy calculation gives

$$\int_0^1 \frac{t}{[ta^p+(1-t)b^p]^{q-q/p}}dt = \begin{cases} \frac{1}{2a^{qp-q}}\cdot {}_2F_1\left(q-\frac{q}{p},1;3;1-\left(\frac{b}{a}\right)^p\right), & p<0 \\ \frac{1}{2b^{qp-q}}\cdot {}_2F_1\left(q-\frac{q}{p},2;3;1-\left(\frac{a}{b}\right)^p\right), & p>0 \end{cases} \tag{2.3}$$

and

$$\int_0^1 \frac{1-t}{[ta^p+(1-t)b^p]^{q-q/p}}dt \tag{2.4}$$
$$= \begin{cases} \frac{1}{2a^{qp-q}} \cdot {}_2F_1\left(q-\frac{q}{p},2;3;1-\left(\frac{b}{a}\right)^p\right), & p<0 \\ \frac{1}{2b^{qp-q}} \cdot {}_2F_1\left(q-\frac{q}{p},1;3;1-\left(\frac{a}{b}\right)^p\right), & p>0 \end{cases}.$$

Substituting equations (2.3) and (2.4) into the above inequality results in the inequality (2.2), which completes the proof. □

Remark 6. *In Theorem 8,*

(i) If we take $p=1$, then we have inequality (1.4) in Theorem 2.

(ii) If we take $p=-1$, then we have the inequality (1.9) in Theorem 5.

Theorem 9. *Let $f: I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I$ with $a<b$, $p \in \mathbb{R}\setminus\{0\}$ and $f' \in L[a,b]$. If $|f'|^q$ is p-convex on $[a,b]$ for $q>1$, $\frac{1}{r}+\frac{1}{q}=1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx\right| \tag{2.5}$$
$$\le \frac{b^p-a^p}{2p} C_6^{\frac{1}{r}} \left(\frac{1}{q+1}\right)^{\frac{1}{q}} \left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}},$$

where

$$\begin{aligned} C_6 &= C_6(a,b;p;r) \\ &= \begin{cases} \frac{1}{a^{pr-r}} \cdot {}_2F_1\left(r-\frac{r}{p},1;2;1-\left(\frac{b}{a}\right)^p\right), & p<0 \\ \frac{1}{b^{pr-r}} \cdot {}_2F_1\left(r-\frac{r}{p},1;2;1-\left(\frac{a}{b}\right)^p\right), & p>0 \end{cases}, \end{aligned}$$

Proof. From Lemma 3, Hölder's inequality and the p-convexity of $|f'|^q$ on $[a,b]$,we have,

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx\right| \\ \le\ & \frac{b^p-a^p}{2p}\left(\int_0^1 \frac{1}{[ta^p+(1-t)b^p]^{r-r/p}}dt\right)^{\frac{1}{r}} \\ &\times \left(\int_0^1 |1-2t|^q \left|f'\left([ta^p+(1-t)b^p]^{1/p}\right)\right|^q dt\right)^{\frac{1}{q}} \\ \le\ & \frac{b^p-a^p}{2p} C_6^{\frac{1}{r}}(a,b;p;r) \left(\frac{1}{q+1}\right)^{\frac{1}{q}} \left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}, \end{aligned}$$

where an easy calculation gives

$$C_6(a,b;p;r) = \int_0^1 \frac{1}{[ta^p+(1-t)b^p]^{r-r/p}}dt \tag{2.6}$$
$$= \begin{cases} \frac{1}{a^{pr-r}} \cdot {}_2F_1\left(r-\frac{r}{p},1;2;1-\left(\frac{b}{a}\right)^p\right), & p<0 \\ \frac{1}{b^{pr-r}} \cdot {}_2F_1\left(r-\frac{r}{p},1;2;1-\left(\frac{a}{b}\right)^p\right), & p>0 \end{cases}$$

and

$$\int_0^1 |1-2t|^q \, tdt = \int_0^1 |1-2t|^q \, (1-t)\, dt = \frac{1}{2(q+1)}. \tag{2.7}$$

Substituting equations (2.6) and (2.7) into the above inequality results in the inequality (2.5), which completes the proof. □

3. Some applications for special means

Let us recall the following special means of two nonnegative number a, b with $b > a$:

(1) The arithmetic mean

$$A = A(a,b) := \frac{a+b}{2}.$$

(2) The geometric mean

$$G = G(a,b) := \sqrt{ab}.$$

(3) The harmonic mean

$$H = H(a,b) := \frac{2ab}{a+b}.$$

(4) The power mean

$$M_r = M_r(a,b) := \left(\frac{a^r+b^r}{2}\right)^{1/r}, \ r \neq 0.$$

(5) The Logarithmic mean

$$L = L(a,b) := \frac{b-a}{\ln b - \ln a}.$$

(6) The p-Logarithmic mean

$$L_p = L_p(a,b) := \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, \quad p \in \mathbb{R}\backslash\{-1,0\}.$$

(7) the Identric mean

$$I = I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}.$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature:

$$H \leq G \leq L \leq I \leq A.$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Proposition 1. *Let $0 < a < b$ and $p \in (-\infty, 1) \setminus \{-1\}$. Then we have the following inequality*

$$M_p.L_{p-1}^{p-1} \leq L_p^p \leq A.L_{p-1}^{p-1}$$

Proof. The assertion follows from the inequality (1.11) in Theorem 6, for $f : (0,\infty) \to \mathbb{R}, \ f(x) = x$. □

Proposition 2. *Let $0 < a < b$ and $p > 1$. Then we have the following inequality*

$$H(a^p, b^p) \leq L_{p-1}^{p-1}.L \leq A(a^p, b^p).$$

Proof. The assertion follows from the inequality (1.11) in Theorem 6, for $f : (0,\infty) \to \mathbb{R}, \ f(x) = x^{-p}$. □

Proposition 3. *Let $0 < a < b$. Then we have the following inequality*

$$L_p^p H \leq L_{p-1}^{p-1} G^{2+2p} \leq L_p^p M_p.$$

Proof. The assertion follows from the inequality (1.11) in Theorem 6, for $f : (0,\infty) \to \mathbb{R}, \ f(x) = 1/x$. □

4. Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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