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EXPONENTIAL DECAY AND NUMERICAL SOLUTION FOR A TIMOSHENKO SYSTEM WITH DELAY TERM IN THE INTERNAL FEEDBACK

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Abstract. In this work we study the asymptotic behavior as $t \to \infty$ of the solution for the Timoshenko system with delay term in the feedback. We use the semigroup theory for to prove the well-posedness of the system and for to establish the exponential stability. As far we know, there exist few results for problems with delay, where the asymptotic behavior is based on the Gearhart-Herbst-Pruss-Huang theorem to dissipative system. See [4],[5],[6]. Finally, we present numerical results of the solution of the system.

1. Introduction

In this paper we consider the following Timoshenko system

$$\rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) + \mu_1\varphi_t(x,t) + \mu_2\varphi_t(x,t-\tau) = 0, \tag{1}$$

$$\rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x+\psi)(x,t) + \mu_3\psi_t(x,t) + \mu_4\psi_t(x,t-\tau) = 0, \tag{2}$$

where φ is the transverse displacement of the beam, ψ is the rotation angle of the filament of the beam, $(x,t) \in (0,L)\times(0,\infty)$, $\tau > 0$ represents the time delay and $\rho_1, \rho_2, b, K, \mu_i$, $i = 1,2,3,4$, are positive constants. This beam, of length L is subjected to the following boundary conditions

$$\varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = 0, \quad t > 0, \tag{3}$$

and initial conditions $(\varphi_0, \varphi_1, \psi_0, \psi_1, f_0, g_0)$ belongs to a suitable functional space, defined for all $x \in (0,L)$ by

$$\begin{aligned} \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) &= \varphi_1(x), \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) &= \psi_1(x), \end{aligned} \tag{4}$$

and for $(x,t) \in (0,L)\times[0,\tau]$, that implies past history with $t-\tau \le 0$, by

$$\varphi_t(x,t-\tau) = f_0(x,t-\tau), \quad \psi_t(x,t-\tau) = g_0(x,t-\tau). \tag{5}$$

Note that $f_0(x,0) = \varphi_1(x)$ and $g_0(x,0) = \psi_1(x)$.
In the study of the asymptotic behavior, we use the result due to Gearhart. See [4, 5, 6].

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Theorem 1.1. *Let $S(t) = e^{\mathcal{A}\,t}$ be a C_0-Semigroup of contractions on a Hilbert space. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq i\beta, \beta \in \mathbb{R} \tag{6}$$

and

$$\lim_{|\beta|\to\infty} ||(i\beta I - \mathcal{A})^{-1}|| < \infty \tag{7}$$

hold.

Certainly, this approach is very different from other works in the literature, especially for problems with delay, where the exponential decay is made by the method of energy, see, for example [8, 9, 10], and references therein. The method of energy, in general, imposes a additional condition on the wave speeds, that is, $K\rho_2 = b\rho_1$, see [1, 2, 3]. Here we do not use any additional condition for the coefficients of the system. Our work improves the result obtained in [7] in the sense that delays has been introduced in the control (damping terms). The delays $\mu_2\varphi_t(x, t-\tau)$, $\mu_4\psi_t(x, t-\tau)$ makes the problem different from that considered in the literature. It is well known that small delays in the controls might turn such well-behaving system into a wild one. In recent years, the PDEs with time delay effects have become an active area of research.

The plan of this work is follows: in the next section, we introduce the Energy Space and prove that the full energy of the system decay. In the section 3, we introduce the semigroup representation for the system and prove that $\mathcal{A}$ the infinitesimal generator of the semigroup is dissipative, and more, that $\mathcal{A}$ generates a $e^{\mathcal{A}\,t}$, C_0-semigroup of contractions, that implies, prove the existence and regularity of solution. Finally in the section 4 by Theorem of Gearhart we prove that $e^{\mathcal{A}\,t}$ is exponentially stably.

2. Energy Space

For the Sobolev spaces we use the standard notation as in [11]. Let us proceed as [12]. We introduce the followings new dependents variables as in

$$z(x,\rho,t) = \varphi_t(x, t-\tau\rho), \quad w(x,\rho,t) = \psi_t(x, t-\tau\rho), \quad \rho \in (0,1), \tag{8}$$

that satisfies for $(x, \rho, t) \in (0, L) \times (0, 1) \times (0, \infty)$

$$\tau\, z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad \tau\, w_t(x,\rho,t) + w_\rho(x,\rho,t) = 0. \tag{9}$$

Therefore, problem (1)-(2) is equivalent to

$$\begin{aligned} \rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) + \mu_1\varphi_t(x,t) + \mu_2 z(x,1,t) &= 0,\\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x+\psi)(x,t) + \mu_3\psi_t(x,t) + \mu_4 w(x,1,t) &\\ &= 0,\\ \tau\, z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad \tau\, w_t(x,\rho,t) + w_\rho(x,\rho,t) &= 0, \end{aligned} \tag{10}$$

The above system subjected to the following initial and boundary conditions

$$
\begin{aligned}
\varphi(0,t)=\varphi(L,t)=\psi(0,t)=\psi(L,t)=0,\quad t>0,\\
z(x,0,t)=\varphi_t(x,t),\quad w(x,0,t)=\psi_t(x,t),\quad x\in(0,L),\qquad t>0,\\
\varphi(\cdot,0)=\varphi_0,\quad \varphi_t(\cdot,0)=\varphi_1,\quad x\in(0,L),\\
\psi(\cdot,0)=\psi_0,\quad \psi_t(\cdot,0)=\psi_1,\quad x\in(0,L),\\
z(x,1,0)=f_0(x,t-\tau),\quad w(x,1,0)=g_0(x,t-\tau),\quad \text{in}\quad (0,L)\times(0,\tau).
\end{aligned} \tag{11}
$$

Now, the energy space $\mathcal{H}$ is defined as

$$
\mathcal{H}=\{\mathrm{H}_0^1\times\mathrm{L}^2\times\mathrm{H}_0^1\times\mathrm{L}^2\times\mathrm{L}^2(0,1;\mathrm{L}^2)\times\mathrm{L}^2(0,1;\mathrm{L}^2)\}.
$$

For $\mu_1>\mu_2$, $\mu_3>\mu_4$ satisfying

$$
\tau\mu_2<\xi<\tau(2\mu_1-\mu_2),\quad \tau\mu_4<\eta<\tau(2\mu_3-\mu_4) \tag{12}
$$

respectively, we define the full energy of the system in the energy space as

$$
\begin{aligned}
E(t) &= \frac{1}{2}\int_0^L(\rho_1|\varphi_t|^2+\rho_2|\psi_t|^2+K|\varphi_x+\psi|^2+b|\psi_x|^2)dx\\
&+ \frac{\xi}{2}\int_0^L\int_0^1 z^2(x,\rho,t)\,d\rho\,dx+\frac{\eta}{2}\int_0^L\int_0^1 w^2(x,\rho,t)\,d\rho\,dx.
\end{aligned}
$$

Lemma 2.1. *There exists a positive constant C such that for any regular solution (φ,ψ,z,w) of the problem (10)-(11) and for any $t\geq 0$, we have*

$$
\frac{d}{dt}E(t)\leq -C\int_0^L(|\varphi_t|^2+|\psi_t|^2+z^2(x,1)+w^2(x,1))\,dx. \tag{13}
$$

Proof. 2.1. *We multiplying* (1) *by φ_t,* (2) *by ψ_t, and using integration by part to get*

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_0^L(\rho_1|\varphi_t|^2+\rho_2|\psi_t|^2+K|\varphi_x+\psi|^2+b|\psi_x|^2)dx = &-\mu_1\int_0^L|\varphi_t|^2\,dx\\
&-\mu_2\int_0^L\varphi_t\,z(1,t)\,dx\\
&-\mu_3\int_0^L|\psi_t|^2\,dx\\
&-\mu_4\int_0^L\psi_t\,w(1,t)\,dx
\end{aligned}
$$

and using the Energy $E(t)$ of the system, we obatin

$$
\begin{aligned}
\frac{d}{dt}E(t)=-\mu_1\int_0^L|\varphi_t|^2\,dx-\mu_2\int_0^L\varphi_t\,z(1,t)\,dx-\frac{d}{dt}\left\{\frac{\xi}{2}\int_0^L\int_0^1 z^2(x,\rho,t)\,d\rho\,dx\right\}\\
-\mu_3\int_0^L|\psi_t|^2\,dx-\mu_4\int_0^L\psi_t\,w(1,t)\,dx-\frac{d}{dt}\left\{\frac{\eta}{2}\int_0^L\int_0^1 w^2(x,\rho,t)\,d\rho\,dx\right\}
\end{aligned}
$$

using (14) and (15)

$$\begin{aligned}
\frac{d}{dt}\frac{\xi}{2}\int_0^L\int_0^1 z^2(x,\rho,t)\,d\rho\,dx &= -\frac{\xi}{\tau}\int_0^L\int_0^1 z(x,\rho,t)z_\rho(x,\rho,t)\,d\rho\,dx\\
&= -\frac{\xi}{2\tau}\int_0^L\int_0^1 \frac{\partial}{\partial\rho}z^2(x,\rho,t)\,d\rho\,dx\\
&= \frac{\xi}{2\tau}\int_0^L (z^2(x,0)-z^2(x,1))\,dx \qquad (14)
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\frac{\eta}{2}\int_0^L\int_0^1 w^2(x,\rho,t)\,d\rho\,dx &= -\frac{\eta}{\tau}\int_0^L\int_0^1 w(x,\rho,t)w_\rho(x,\rho,t)\,d\rho\,dx\\
&= -\frac{\eta}{2\tau}\int_0^L\int_0^1 \frac{\partial}{\partial\rho}w^2(x,\rho,t)\,d\rho\,dx\\
&= \frac{\eta}{2\tau}\int_0^L (w^2(x,0)-w^2(x,1))\,dx \qquad (15)
\end{aligned}$$

we obtain

$$\begin{aligned}
\frac{d}{dt}E(t) = -\mu_1\int_0^L|\varphi_t|^2\,dx-\mu_2\int_0^L\varphi_t\,z(1,t)\,dx \qquad (16)\\
+\frac{\xi}{2\tau}\int_0^L(z^2(x,0)-z^2(x,1))\,dx\\
-\mu_3\int_0^L|\psi_t|^2\,dx-\mu_4\int_0^L\psi_t\,w(1,t)\,dx\\
+\frac{\eta}{2\tau}\int_0^L(w^2(x,0)-w^2(x,1))\,dx \qquad (17)
\end{aligned}$$

Now, using Youngs's inequality we can rewritten the last equation as

$$\begin{aligned}
\frac{d}{dt}E(t)\le -\left(\mu_1-\frac{\xi}{2\tau}-\frac{\mu_2}{2}\right)\int_0^L|\varphi_t|^2\,dx-\left(\frac{\xi}{2\tau}-\frac{\mu_2}{2}\right)\int_0^L z^2(x,1)\,dx\\
-\left(\mu_3-\frac{\eta}{2\tau}-\frac{\mu_4}{2}\right)\int_0^L|\psi_t|^2\,dx-\left(\frac{\eta}{2\tau}-\frac{\mu_4}{2}\right)\int_0^L w^2(x,1)\,dx
\end{aligned}$$

from where our conclusion holds.

3. Existence of solution

Let us introduce the semigroup representation. To this end, let $U=(\varphi,\varphi_t,\psi,\psi_t,z,w)^T$ and rewrite the problem (10)-(11) as

$$\begin{aligned}
U_t &= \mathcal{A}\,U\\
U(0) &= U_0 \qquad (18)
\end{aligned}$$

where the operator $\mathcal{A}$ is defined for $U = (\varphi, u = \varphi_t, \psi, v = \psi_t, z, w)^T$ by

$$\mathcal{A}U = \begin{pmatrix} u \\ \frac{K}{\rho_1}(\varphi_x + \psi)_x - \frac{\mu_1}{\rho_1}u - \frac{\mu_2}{\rho_1}z(\cdot, 1) \\ v \\ \frac{b}{\rho_2}\psi_{xx} - \frac{K}{\rho_2}(\varphi_x + \psi) - \frac{\mu_3}{\rho_2}u - \frac{\mu_4}{\rho_2}w(\cdot, 1) \\ -\frac{1}{\tau}\, z_\rho \\ -\frac{1}{\tau}\, w_\rho \end{pmatrix},$$

with domain

$$D(\mathcal{A}) = \{(\varphi, \varphi_t, \psi, \psi_t, z, w)^T \in H \;:\; u = z(\cdot, 0, \cdot),\; v = w(\cdot, 0, \cdot),\; \text{in}(0, 1)\},$$

where for $x \in (0, L)$ we denote $\mathrm{H} = \mathrm{H}(0, L)$, $\mathrm{L} = \mathrm{L}(0, L)$ and

$$H = \{(\mathrm{H}^2 \cap \mathrm{H}^1_0) \times \mathrm{H}^1_0 \times (\mathrm{H}^2 \cap \mathrm{H}^1_0) \times \mathrm{H}^1_0 \times \mathrm{L}^2(0, 1; \mathrm{H}^1) \times \mathrm{L}^2(0, 1; \mathrm{H}^1)\}.$$

For $U = (\varphi^i, u^i, \psi^i, v^i, z^i, w^i)^T \in \mathcal{H},\; i = 1, 2$ and ξ, η as in (12) we defined the following inner product in the energy space as

$$\begin{aligned} \langle U^1, U^2\rangle &= \int_0^L [\rho_1 u^1 u^2 + \rho_2 v^1 v^2 + K(\varphi^1 + \psi^1)(\varphi^2 + \psi^2) + b\psi^1\psi^2) dx \\ &+ \xi \int_0^L \int_0^1 z^1(x, \rho) z^2(x, \rho)\, d\rho\, dx + \eta \int_0^L \int_0^1 w^1(x, \rho) w^2(x, \rho)\, d\rho\, dx. \end{aligned}$$

For to prove the existence of solution we begin with the proof that the operator $\mathcal{A}$ is dissipative.

Lemma 3.1. *For $U = (\varphi, u = \varphi_t, \psi, v = \psi_t, z, w)^T \in D(\mathcal{A})$, we have $\langle \mathcal{A}U, U\rangle \le 0$.*

Proof. 3.1.

$$\begin{aligned} \langle \mathcal{A}U, U\rangle = &-\mu_1 \int_0^L |\varphi_t|^2\, dx - \mu_2 \int_0^L \varphi_t\, z(1, t)\, dx - \frac{\xi}{\tau} \int_0^L \int_0^1 z(x, \rho, t) z_\rho(x, \rho, t)\, d\rho\, dx \\ &-\mu_3 \int_0^L |\psi_t|^2\, dx - \mu_4 \int_0^L \psi_t\, w(1, t)\, dx - \frac{\eta}{\tau} \int_0^L \int_0^1 w(x, \rho, t) w_\rho(x, \rho, t)\, d\rho\, dx. \end{aligned}$$

Using (14) and (15) in the equation above we obtain

$$\begin{aligned} \langle \mathcal{A}U, U\rangle = &-\mu_1 \int_0^L |\varphi_t|^2\, dx - \mu_2 \int_0^L \varphi_t\, z(1, t)\, dx + \frac{\xi}{2\tau} \int_0^L (z^2(x, 0) - z^2(x, 1))\, dx \\ &-\mu_3 \int_0^L |\psi_t|^2\, dx - \mu_4 \int_0^L \psi_t\, w(1, t)\, dx + \frac{\eta}{2\tau} \int_0^L (w^2(x, 0) - w^2(x, 1))\, dx. \end{aligned}$$

Now using (16) and Lemma 3.1 we concludes

$$\langle \mathcal{A}U, U\rangle = \frac{d}{dt}E(t) \le -C \int_0^L (|u|^2 + |v|^2 + z^2(x, 1) + w^2(x, 1))\, dx. \tag{19}$$

In the next lemma, we will prove an important property of resolvent of the operator $\mathcal{A}$.

Lemma 3.2. $0 \in \rho(\mathcal{A})$.

Proof. 3.2. *For any $F = (f_1, f_2, f_3, f_4, f_5, f_6)^T \in \mathcal{H}$ consider the equation $\mathcal{A}U = F$. This implies*

$$u = f_1, \tag{20}$$
$$K(\varphi_x + \psi)_x - \mu_1 u - \mu_2 z(\cdot, 1) = \rho_1 f_2, \tag{21}$$
$$v = f_3, \tag{22}$$
$$b\psi_{xx} - K(\varphi_x + \psi) - \mu_3 v - \mu_4 w(\cdot, 1) = \rho_2 f_4, \tag{23}$$
$$-z_\rho = \tau f_5, \tag{24}$$
$$-w_\rho = \tau f_6, \tag{25}$$

We plug $u = f_1$ obtained from (20) into (21) to get

$$K(\varphi_x + \psi)_x = \mu_1 u + \mu_2 z(\cdot, 1) + \rho_1 f_2 \quad \in L^2(0, L).$$

By Poincarè inequality we have

$$K(\varphi_x + \psi) \quad \in L^2(0, L). \tag{26}$$

Now we plug $v = f_3$ obtained from (26) and (22) into (23) to get

$$b\psi_{xx} = K(\varphi_x + \psi) + \mu_3 v + \mu_4 w(\cdot, 1) + \rho_2 f_4 \quad \in L^2(0, L). \tag{27}$$

By the standard theory in the linear elliptic equations , we have a unique $\psi \in H^2 \cap H_0^1$ satisfying (27). Then we plug ψ just obtained from solving (27) into (21) to get

$$K\varphi_{xx} = -K\psi_x + \mu_1 u + \mu_2 z(\cdot, 1) + \rho_1 f_2 \quad \in L^2(0, L). \tag{28}$$

Applying the standard theory in the linear elliptic equations again yields a unique solvability of $\varphi \in H^2 \cap H_0^1$ for (28).

From (24) we have using Poincarè inequality,

$$\frac{1}{C_p} \int_0^L \int_0^1 |z|^2 \, d\rho dx \leq \int_0^L \int_0^1 |z_\rho|^2 \, d\rho dx \in L^2(0, 1 : L^2(0, L)),$$

then $z \in L^2(0, 1 : H^1(0, L))$. The same idea we use for w. Thus the unique solvability of $\mathcal{A}U = F$ follows. It is clear from the theory of the linear elliptic equation, see Chapter 1 of [13]*, that $||U||_{\mathcal{H}} \leq C||F||_{\mathcal{H}}$ with C being a positive constant independent of U, and then $0 \in \rho(\mathcal{A})$.*

Now we will to prove that $\mathcal{A}$ generates a C_0-Semigroup of contractions.

Lemma 3.3. *The operator $\mathcal{A}$ generates a C_0-Semigroup of contractions on a Hilbert space $\mathcal{H}$.*

Proof. 3.3. *From Lemma 3.1 we have that $\mathcal{A}$ is dissipative operator, and from Lemma 3.2 follows that $0 \in \rho(A)$, them from Theorem 1.2.4, page 3 of* [13]*, we concludes that $\mathcal{A}$ generates a C_0-Semigroup of contractions on $\mathcal{H}$.*

In this step, we prove that the problem (10)-(11) is well-posedness, and in this direction, we have the following result

Theorem 3.4. *If $\mu_2 \leq \mu_1$ and $\mu_4 \leq \mu_3$, then there exists a unique solution $U \in C([0, \infty), \mathcal{H})$ of the (10)-(11). Moreover if $U_0 \in D(A)$, then $U \in C([0, \infty), D(\mathcal{A})) \cap C^1([0, \infty), \mathcal{H})$.*

Proof. 3.5. *From the classical semigroup theory, see for example* [14]*, follows by Lemma 3.3 that* $U(t) = e^{\mathcal{A}\,t}\,U_0$ *is the unique solution of the problem (10)-(11) in the conditions of theorem. The proof is complete.*

4. Asymptotic behavior

Now we are in position to present our principal result

Theorem 4.1. *The semigroup* $e^{\mathcal{A}\,t}$ *is exponentially stably.*

Proof. 4.2. *We now use Theorem 1.1 and we use a contradiction argument. We first prove (6). From Lemma 3.2 we have that* $0 \in \rho(\mathcal{A})$ *and follows from this fact and the contraction mapping theorem that for any real number* β *with* $|\beta| < ||\mathcal{A}||^{-1}$*, the operator* $i\beta I - \mathcal{A} = \mathcal{A}(i\beta\mathcal{A}^{-1} - I)$ *is invertible. Moreover,* $||(i\beta I - \mathcal{A})^{-1}||$ *is a continuous function of* β *in the interval* $(-||\mathcal{A}||^{-1}, ||\mathcal{A}||^{-1})$*.*

If $\sup\{||(i\beta I - \mathcal{A})^{-1}|| \,:\, |\beta| < ||\mathcal{A}||^{-1}\} = M < \infty$*, then by the contraction mapping theorem, the operator* $i\beta I - \mathcal{A} = (i\beta_0 I - \mathcal{A})(I + i(\beta - \beta_0)(i\beta_0 I - \mathcal{A})^{-1})$ *with* $|\beta_0| < ||\mathcal{A}||^{-1}$ *is invertible for* $|\beta - \beta_0| < 1/M$*. It turns out that by choosing* $|\beta_0|$ *as close to* $||\mathcal{A}||^{-1}$ *as we can, we conclude that*

$$\{\beta \,:\, |\beta| < ||\mathcal{A}||^{-1} + 1/M\} \subset \rho(\mathcal{A})$$

and $||(i\beta I - \mathcal{A})^{-1}||$ *is continuous function of* β *in the interval*

$$(-||\mathcal{A}||^{-1} - 1/M\,,\, ||\mathcal{A}||^{-1} + 1/M).$$

From argument above, it follows that if (6) is not true, then there is $w \in \mathbb{R}$ *with* $||\mathcal{A}||^{-1} \leq |w| < \infty$ *such that* $\{i\beta\,;\, |\beta| < |w|\} \subset \rho(\mathcal{A})$ *and*

$$\sup\{||(i\beta - \mathcal{A})^{-1}|| \,:\, |\beta| < |w|\} = \infty.$$

It turns out that there exists a sequence $\beta^n \in \mathbb{R}$ *with* $\beta^n \to w$*,* $|\beta^n| < |w|$ *and a sequence of complex vector functions* $U^n = (\varphi^n, u^n, \psi^n, v^n, z^n, w^n)^T$ *satisfying*

$$U^n \in D(\mathcal{A}) \quad \textit{with} \quad ||U^n||_{\mathcal{H}} = 1$$

such that

$$||(i\beta^n - \mathcal{A})U^n|| \to 0, \;\; \textit{as } n \to \infty,$$

and then

$$i\beta^n\varphi^n - u^n \to 0 \quad \textit{in} \quad \mathrm{H}_0^1 \tag{29}$$

$$i\beta^n\rho_1 u^n - K(\varphi_x^n + \psi^n)_x + \mu_1 u^n + \mu_2 z^n(\cdot,1) \to 0 \quad \textit{in} \quad \mathrm{L}^2$$

$$i\beta^n\psi^n - v^n \to 0 \quad \textit{in} \quad \mathrm{H}_0^1 \tag{30}$$

$$i\beta^n\rho_2 v^n - b\psi_{xx}^n + K(\varphi_x^n + \psi^n) + \mu_3 v^n + \mu_4 w^n(\cdot,1) \to 0 \quad \textit{in} \quad \mathrm{L}^2$$

$$i\beta^n\tau z^n - z_\rho^n \to 0 \quad \textit{in} \quad \mathrm{L}^2(0,1;\mathrm{L}^2)$$

$$i\beta^n\tau w^n - w_\rho^n \to 0 \quad \textit{in} \quad \mathrm{L}^2(0,1;\mathrm{L}^2)$$

Making the inner product of $(i\beta^n I - \mathcal{A})U^n$ *with* U^n *in* $\mathcal{H}$*, taking the real part, and using (19) we have*

$$\int_0^L (|u^n|^2 + |v^n|^2 + z^n(x,1)^2 + w^n(x,1)^2)\,dx \to 0,$$

from where follows that

$$u^n \to 0 \tag{31}$$

$$v^n \to 0 \tag{32}$$

$$z^n \to 0 \tag{33}$$

$$w^n \to 0 \tag{34}$$

Using (31) into (29) we obtain

$$\varphi^n \to 0, \tag{35}$$

and using (32) into (30) we obtain

$$\varphi^n \to 0. \tag{36}$$

Now using (31),(32),(33),(34),(35),(36) we concludes that $||U^n|| \to 0$ *which is a contradiction with* $||U^n|| = 1$ *and the proof of (6) is complete.*

Finally we prove (7) by a contradiction argument again. Suppose that (7) is not true. Then there exists a sequence β^n *with* $|\beta_n| \to \infty$ *and a sequence of complex vector functions* $U^n \in D(\mathcal{A})$ *with unit norm in* $\mathcal{H}$ *such that*

$$||(i\beta^n I - \mathcal{A})U^n|| \to 0, \ \ as\ n \to \infty.$$

Again we have

$$\int_0^L (|u^n|^2 + |v^n|^2 + z^n(x,1)^2 + w^n(x,1)^2)\,dx = -\langle \mathcal{A}U^n, U^n\rangle \to 0. \tag{37}$$

Making the inner product of $(i\beta^n I - \mathcal{A})U^n$ with U^n in $\mathcal{H}$ we obtain

$$i\beta^n ||U^n||^2 - \langle \mathcal{A}U^n, U^n\rangle \to 0.$$

From (37) we get

$$\beta^n ||U^n||^2 \to 0. \tag{38}$$

As $\beta^n \to \infty$ and $||U^n||$ is limited, we concludes that (38) is true only if $||U^n|| \to 0$ contradict $||U^n|| = 1$. The proof of theorem is complete.

5. Numerical Solution

We will solve numerically the system of Timoshenko (1)-(5) in the one-dimension domain Ω of the length L, using high-order schemes. We used the Implicit Compact Finite Difference Method of fourth-order for discretization of spacial variable and the classic Finite Difference for discretization of temporal variable.

5.1. **Discretization.** In order to get the discretization of the problem (1)-(5), we define the following sets:

$$\begin{aligned}
\Omega_h &= \{x_i:\ x_i = ih, \quad i = 0, 1, \ldots, I+1;\ \ h = L/(I+1)\},\\
\top^k &= \{t_n:\ t_n = nk,\ n = 0, 1, ..., N;\ \ k = Ch\},\\
\dagger^k &= \{t_n:\ t_n = nk,\ n = -M, -M+1, ..., 0;\ \ 0 < M < N\}
\end{aligned}$$

where $\mathcal{Q}_h^k = \Omega_h \times \top^k$ and $\mathcal{D}_h^k = \Omega_h \times \dagger^k$ are the computational mesh, and mesh of delay, respectively. The width of mesh $\mathcal{D}_h^k$ is $\tau = Mk$. In Figure 1 we show a mesh model for the full-domain $\mathcal{Q}_h^k \cup \mathcal{D}_h^k$. The points (x_i, t_n) are called nodes of

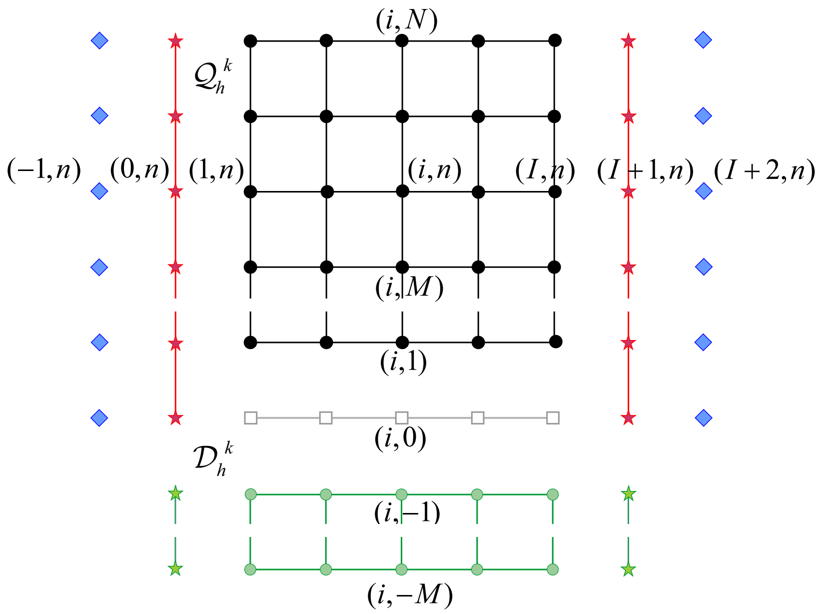

FIGURE 1. Model mesh for the full-domain: $\mathcal{Q}_h^k \cup \mathcal{D}_h^k$.

the mesh and, usually denote by (i,n). The classification of nodes is as follows: interiors (circles), boundaries (stars), initials (squares) and ghosts (diamonds).

Let $\chi = \chi(x,t)$ be a function with second order partial derivatives. Henceforth consider the following notation $\chi_i^n \equiv \chi(x_i,t_n)$. We define the following approximation of the derivatives of χ, according to Taylor,

$$(\chi_t)_i^n \approx \frac{1}{k}\delta_t^- \chi_i^n, \quad (\chi_t)_i^n \approx \frac{1}{2k}\delta_t^0 \chi_i^n, \qquad (\chi_t)_i^{(n-M)} \approx \frac{1}{k}\delta_t^- \chi_i^{(n-M)}$$

$$(\chi_x)_i^n \approx \frac{1}{2h}\delta_x^0 \chi_i^n, \quad (\chi_{tt})_i^n \approx \frac{1}{k^2}\delta_t^2 \chi_i^n, \quad (\chi_{xx})_i^n \approx \frac{1}{h^2}\left[\frac{\delta_x^2}{1+\frac{1}{12}\delta_x^2}\right]\chi_i^n \tag{39}$$

where the *finite difference operators* are given by

$$\delta_t^- \chi_i^n := \chi_i^n - \chi_i^{n-1}, \qquad \delta_t^0 \chi_i^n := \chi_i^{n+1} - \chi_i^{n-1},$$

$$\delta_t^- \chi_i^{(n-M)} := \chi_i^{(n-M)} - \chi_i^{(n-M)-1}, \qquad \delta_x^0 \chi_i^n := \chi_{i+1}^n - \chi_{i-1}^n,$$

$$\delta_t^2 \chi_i^n := \chi_i^{n+1} - 2\chi_i^n + \chi_i^{n-1}, \qquad \delta_x^2 \chi_i^n := \chi_{i+1}^n - 2\chi_i^n + \chi_{i-1}^n,$$

$$\left[1+\frac{1}{12}\delta_x^2\right]\chi_i^n := \frac{1}{12}\chi_{i+1}^n + \frac{5}{6}\chi_i^n + \frac{1}{12}\chi_{i-1}^n \tag{40}$$

The discrete formulation of equations (1)-(5) is obtained using (39),

$$\begin{aligned} \rho_1\left[1+\frac{1}{12}\delta_x^2\right]\delta_t^2\varphi_i^n - \alpha_1\delta_x^2\varphi_i^n - \alpha_2\left[1+\frac{1}{12}\delta_x^2\right]\delta_x^0\psi_i^n \quad &+ \\ \alpha_3\left[1+\frac{1}{12}\delta_x^2\right]\delta_t^0\varphi_i^n + \alpha_4\left[1+\frac{1}{12}\delta_x^2\right]\delta_t^-\varphi_i^{(n-M)} \quad &= \quad 0 \\ \text{in} \quad (x_i,t_n) \in \mathcal{Q}_h^k & \end{aligned} \tag{41}$$

$$\begin{aligned}\rho_2\Big[1+\frac{1}{12}\delta_x^2\Big]\delta_t^2\psi_i^n-\beta_1\delta_x^2\psi_i^n+\beta_2\Big[1+\frac{1}{12}\delta_x^2\Big]\delta_x^0\varphi_i^n+\beta_0\Big[1+\frac{1}{12}\delta_x^2\Big]\psi_i^n &\quad + \\ \beta_3\Big[1+\frac{1}{12}\delta_x^2\Big]\delta_t^0\psi_i^n+\beta_4\Big[1+\frac{1}{12}\delta_x^2\Big]\delta_t^-\psi_i^{(n-M)} &\quad = \quad 0\end{aligned}$$

$$\text{in}\quad (x_i,t_n)\in\mathcal{Q}_h^k \tag{42}$$

$$\varphi_i^0=(\varphi_0)_i,\quad \psi_i^0=(\varphi_0)_i,\quad (\varphi_t)_i^0=(\varphi_1)_i,\quad (\psi_t)_i^0=(\psi_1)_i\quad\text{in}\quad x_i\in\mathring{\Omega}_h \tag{43}$$

$$\varphi_0^n=\varphi_{I+1}^n=\psi_0^n=\psi_{I+1}^n=0\quad\text{on}\quad t_n\in\top^k \tag{44}$$

$$\varphi_i^{(n-M)}=(f_0)_i^{(n-M)},\quad \psi_i^{(n-M)}=(g_0)_i^{(n-M)},\quad\text{in}\quad (x_i,t_{(n-M)})\in\mathcal{D}_h^k \tag{45}$$

where, the parameters, are defined by

$$\begin{array}{llll}\alpha_1=Kk^2/h^2, & \alpha_2=Kk^2/2h, & & \alpha_3=\mu_1 k/2,\quad \alpha_4=\mu_2 k,\\ \beta_1=bk^2/h^2, & \beta_2=\alpha_2, & \beta_0=Kk^2, & \beta_3=\mu_3 k/2,\quad \beta_4=\mu_4 k\end{array}$$

Substituting (40) in (41)-(42), we have the following linear algebraic system:

$$A_1\Phi^{n+1} = B_1\Phi^n+C_1\Psi^n+D_1\Phi^{n-1}-E_1\delta_t^-\Phi^{(n-M)}+\Upsilon_1^n \tag{46}$$

$$A_2\Psi^{n+1} = B_2\Psi^n+C_2\Phi^n+D_2\Psi^{n-1}-E_2\delta_t^-\Psi^{(n-M)}+\Upsilon_2^n \tag{47}$$

where, $\Phi^{n+1}=(\varphi_1^{n+1},\varphi_2^{n+1},...,\varphi_I^{n+1})^{\mathrm{T}}$ and $\Psi^{n+1}=(\psi_1^{n+1},\psi_2^{n+1},...,\psi_I^{n+1})^{\mathrm{T}}$, $n=0,1,...,N-1$, are unknown vectors,

$$A_1=\text{tridiag}\Big(\frac{1}{12}(\rho_1+\alpha_3),\frac{5}{6}(\rho_1+\alpha_3),\frac{1}{12}(\rho_1+\alpha_3)\Big),$$

$$B_1=\text{tridiag}\Big(\frac{1}{6}(\rho_1+6\alpha_1),\frac{1}{3}(5\rho_1-6\alpha_1),\frac{1}{6}(\rho_1+6\alpha_1)\Big),$$

$$C_1=\text{pentadiag}\Big(-\frac{1}{12}\alpha_2,-\frac{5}{6}\alpha_2,0,\frac{5}{6}\alpha_2,\frac{1}{12}\alpha_2\Big),$$

$$D_1=\text{tridiag}\Big(\frac{1}{12}(-\rho_1+\alpha_3),\frac{5}{6}(-\rho_1+\alpha_3),\frac{1}{12}(-\rho_1+\alpha_3)\Big),$$

$$E_1=\text{tridiag}\Big(\frac{1}{12}\alpha_4,\frac{5}{6}\alpha_4,\frac{1}{12}\alpha_4\Big),$$

$$A_2=\text{tridiag}\Big(\frac{1}{12}(\rho_2+\beta_3),\frac{5}{6}(\rho_2+\beta_3),\frac{1}{12}(\rho_2+\beta_3)\Big),$$

$$B_2=\text{tridiag}\Big(\frac{1}{12}(2\rho_2+12\beta_1-\beta_0),\frac{1}{6}(10\rho_2-12\beta_1-5\beta_0),\frac{1}{12}(2\rho_2+12\beta_1-\beta_0)\Big),$$

$$C_2=-C_1,$$

$$D_2=\text{tridiag}\Big(\frac{1}{12}(-\rho_2+\beta_3),\frac{5}{6}(-\rho_2+\beta_3),\frac{1}{12}(-\rho_2+\beta_3)\Big),$$

$$E_2=\text{tridiag}\Big(\frac{1}{12}\beta_4,\frac{5}{6}\beta_4,\frac{1}{12}\beta_4\Big),$$

are matrices of order $I\times I$. Υ_1^n and Υ_2^n are vectors of order I that load the boundary data.

5.2. **Numerical test.** In order to verify the asymptotic behavior of the solution of the Timoshenko system, we consider the following data:

$$L = 2\pi, \quad \rho_1 = \rho_2 = K = b = 1.$$

Boundary condition:

$$\varphi(0,t) = \varphi(2\pi,t) = \psi(0,t) = \psi(2\pi,t) = 0$$

Initial condition:

$$\varphi_0(x) = 0, \quad \psi_0(x) = 0, \quad \varphi_1(x) = \sin(x), \quad \psi_1(x) = \cos(x)$$

Delay condition:

$$f_0(x,t-\tau) = \sin(x)\cos(t-\tau), \qquad g_0(x,t-\tau) = \cos(x)\cos(t-\tau)$$

Numerical data: $I = 18$, $C = 0.3$, $\tau = 10\%$ of the width of the mesh $\mathcal{Q}_h^k$, $TOL = 4 \times 10^{-5}$ (tolerance).

Table 1 shows seven cases where Timoshenko system may behave differently with the presence the terms of delay and damping. Each of these cases are plotted in Figure 2-8. Note that the asymptotic behavior of the solution was calculated by taking the maximum value of the function φ, in $x \in [0, 2\pi]$, throughout time. In Figure 2, it is observed that there is no asymptotic behavior of the solution, in contrast to Figures 3-6, where the asymptotic behavior of the solution is increasingly more acute. Figure 7 represents the case without delay, the presence of damping is very evident, obtaining the asymptotic behavior of the solution immediately. Figure 8 represents the case without delay and damping, and as was expected, there is no convergence of the solution. In Figure 9 we show the graph of function $\varphi(x,t)$, where $x \in [0, 2\pi]$, $t \in [-2.97, 29.76]$, $\mu_1 = \mu_3 = 1$, $\mu_2 = \mu_4 = 0.8$ and we choose only 300 iterations along time. With respect to rotation angle ψ we observe that it exhibits the same behavior that the function φ.

TABLE 1. Table for different cases.

Case	Damping	Delay	Iterations in time	Asymptotic behavior
1	$\mu_1 = \mu_3 = 1$	$\mu_2 = \mu_4 = 1$	3000	diverges
2	$\mu_1 = \mu_3 = 1$	$\mu_2 = \mu_4 = 0.9$	3000	converges
3	$\mu_1 = \mu_3 = 1$	$\mu_2 = \mu_4 = 0.8$	3000	converges
4	$\mu_1 = \mu_3 = 1$	$\mu_2 = \mu_4 = 0.7$	3000	converges
5	$\mu_1 = \mu_3 = 1$	$\mu_2 = \mu_4 = 0.6$	3000	converges
		$\vdots$		
6	$\mu_1 = \mu_3 = 1$	$\mu_2 = \mu_4 = 0$	159	converges
7	$\mu_1 = \mu_1 = 0$	$\mu_2 = \mu_4 = 0$	3000	diverges

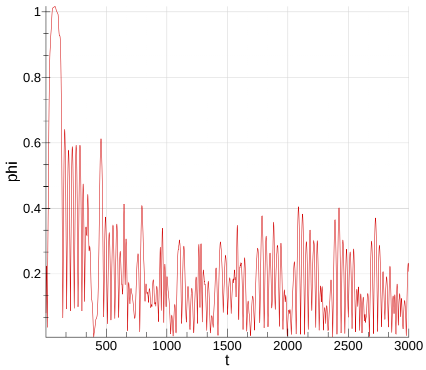

FIGURE 2. Case 1.

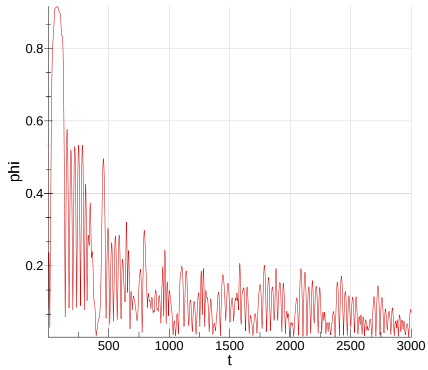

FIGURE 3. Case 2.

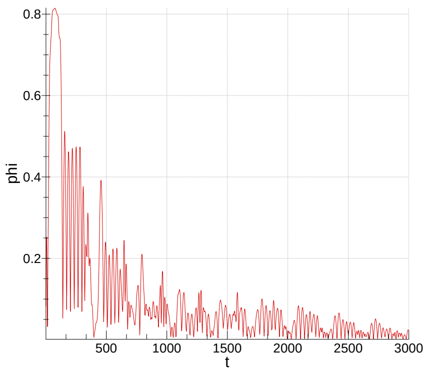

FIGURE 4. Case 3.

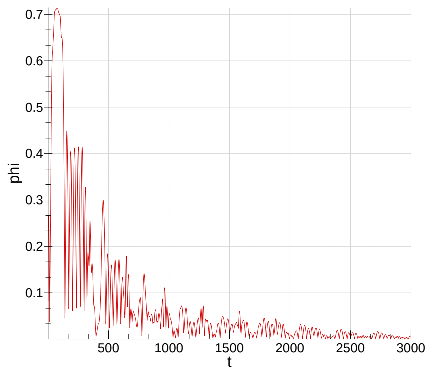

FIGURE 5. Case 4.

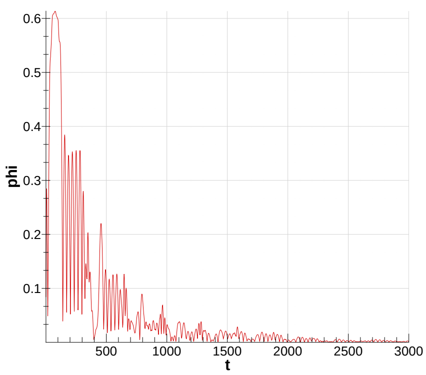

FIGURE 6. Case 5.

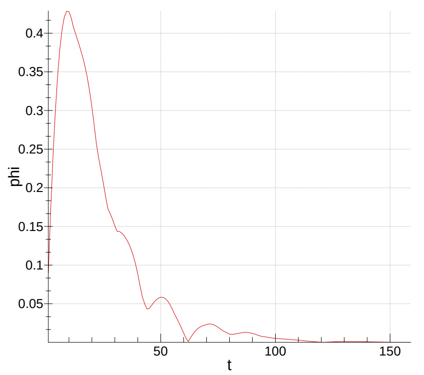

FIGURE 7. Case 6.

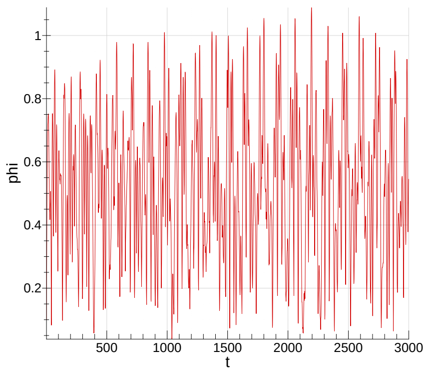

FIGURE 8. Case 7.

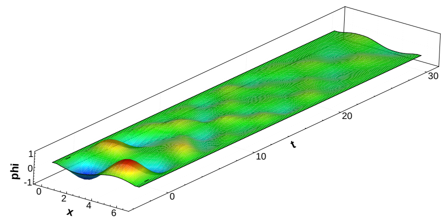

FIGURE 9. Graph of $\varphi(x,t)$.

6. Conclusion

We have demonstrated the well-posedness and asymptotic behavior solution of the Timoshenko system. Thus, it also was obtained numerically the asymptotic behavior of the solution confirming the theory developed.

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