International Journal of Analysis and Applications
ISSN 2291-8639
Volume 10, Number 1 (2016), 40-47
http://www.etamaths.com

FEJÉR–HADAMARD INEQULALITY FOR CONVEX FUNCTIONS ON THE COORDINATES IN A RECTANGLE FROM THE PLANE

G. FARID*, M. MARWAN, AND ATIQ UR REHMAN

Abstract. We give Fejér–Hadamard inequality for convex functions on coordinates in the rectangle from the plane. We define some mappings associated to it and discuss their properties.

1. Introduction

A real valued function $f : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, is called convex if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha) f(y),$$

where $\alpha \in [0, 1]$, for all $x, y \in I$.

Convex functions play a vital role in the theory of inequalities. A lot of inequalities are established using convex functions, e.g. see for convex functions in [1, 2, 6, 7]. The most classical and fundamental inequality is Hermite-Hadamard inequality, this is stated as follows:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

holds for every convex function $f : I \to \mathbb{R}$ and $a,\, b \in I$ with $a < b$.
This inequality is present in many textbooks and monographs devoted to convex functions and it is also extensively studied by many researchers. With the help of (1) researchers have produced many integral and differential inequalities (see [8, 9]), and operators. Very interesting historical remarks concerning the inequality (1) can be found in [13] (see also [14, pp. 62]).
In 1906, Fejér (see [16, page 138] and [11]) established the following weighted generalization of the Hermite–Hadamard inequality for symmetric functions.

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b g(x)dx \tag{2}$$

holds for every convex function $f : I \to \mathbb{R}\, a, b \in I$, and $g : [a, b] \to \mathbb{R}^+$ symmetric about $(a + b)/2$.

In [5] Dragomir gave the Hermite-Hadamard inequality on a rectangle in plane, by defining convex functions on coordinates.

Definition 1.1. Let us consider the two-dimensional interval $\Delta := [a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A function $f : \Delta \to \mathbb{R}$ will be called convex on the coordinates if the partial mappings $f_y : [a, b] \to \mathbb{R}$, $f_y(u) := f(u, y)$, and $f_x : [a, b] \to \mathbb{R}$, $f_x(v) := f(x, v)$, are convex where defined for all $y \in [c, d]$ and $x \in [a, b]$.

One can note that every convex mapping $f : \Delta \to \mathbb{R}$ is convex on the coordinates but the converse is not true. For example, $f(x, y) = xy$ is convex on coordinates in $\mathbb{R}^2$ but it is not convex.

2010 *Mathematics Subject Classification.* 26A51, 26D15, 65D30.
Key words and phrases. convex functions; Hadamard inequality; convex functions on coordinates.

Theorem 1.2. *Let $f : \Delta \to \mathbb{R}$ be convex on co-ordinate in Δ. Then we have:*

$$\begin{aligned} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) &\le \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(x,\frac{c+d}{2})dx + \frac{1}{d-c}\int_c^d f(\frac{a+b}{2},y)dy\right] \\ &\le \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx \\ &\le \frac{1}{4}\left[\frac{1}{(b-a)}\int_a^b f(x,c)dx + \frac{1}{(b-a)}\int_a^b f(x,d)dx\right. \\ &+ \left.\frac{1}{d-c}\int_c^d f(a,y)dy + \frac{1}{d-c}\int_c^d f(b,y)dy\right] \\ &\le \frac{1}{4}\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right]. \end{aligned}$$

There in [5] some mappings connected to above inequality are also considered and their properties are discussed. In this paper we are interested to give the Fejér–Hadamard inequality for a rectangle in plane via convex functions on coordinates. We also study some properties of mappings associated with the Fejér–Hadamard inequality for convex functions on coordinates.

2. Main results

Theorem 2.1. *Let $\Delta := [a,b]\times[c,d] \subset \mathbb{R}^2$ and $f : \Delta \to \mathbb{R}$ be a convex function on coordinates in Δ. Also let $g_1 : [a,b] \to \mathbb{R}^+$ and $g_2 : [c,d] \to \mathbb{R}^+$ be two integrable and symmetric functions about $(a+b)/2$ and $(c+d)/2$ respectively. Then one has the following inequalities*

$$\begin{aligned} & f\left(\frac{a+b}{2},\frac{c+d}{2}\right) \\ &\le \frac{1}{2}\left[\frac{1}{G_1}\int_a^b f\left(x,\frac{c+d}{2}\right)g_1(x)dx + \frac{1}{G_2}\int_c^d f\left(\frac{a+b}{2},y\right)g_2(y)dy\right] \\ &\le \frac{1}{G_1G_2}\int_a^b\int_c^d f(x,y)g_1(x)g_2(y)dydx \\ &\le \frac{1}{4}\left[\frac{1}{G_1}\int_a^b g_1(x)f(x,c)dx + \frac{1}{G_1}\int_a^b g_1(x)f(x,d)dx\right. \\ &+ \left.\frac{1}{G_2}\int_c^d g_2(y)f(a,y)dy + \frac{1}{G_2}\int_c^d g_2(y)f(b,y)dy\right] \\ &\le \frac{1}{4}\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right], \end{aligned} \tag{3}$$

where

$$G_1 = \int_a^b g_1(x)dx \text{ and } G_2 = \int_c^d g_2(y)dy.$$

These inequalities are sharp.

Proof. Since $f : \Delta \to \mathbb{R}$ is convex on coordinates, it follows that functions f_x and f_y are convex on $[c,d]$ and $[a,b]$ respectively. Thus from (2), we have

$$f\left(x,\frac{c+d}{2}\right) \le \frac{1}{G_2}\int_c^d f(x,y)g_2(y)dy \le \frac{f(x,c)+f(x,d)}{2} \tag{4}$$

and

$$f\left(\frac{a+b}{2},y\right) \le \frac{1}{G_1}\int_a^b f(x,y)g_1(x)dx \le \frac{f(a,y)+f(b,y)}{2}. \tag{5}$$

Multiplying (4) by $g_1(x)$

$$g_1(x)f\left(x,\frac{c+d}{2}\right)\leq\frac{1}{G_2}\int_c^d f(x,y)g_1(x)g_2(y)dy\leq g_1(x)\frac{f(x,c)+f(x,d)}{2}.$$

Now integrating on $[a,b]$, we get

$$\begin{aligned}\int_a^b g_1(x)f\left(x,\frac{c+d}{2}\right)dx&\leq\frac{1}{G_2}\int_a^b\int_c^d f(x,y)g_1(x)g_2(y)dydx\\&\leq\frac{1}{2}\left[\int_a^b g_1(x)f(x,c)dx+\int_a^b g_1(x)f(x,d)dx\right].\end{aligned}\tag{6}$$

Now multiplying (5) by $g_2(y)$ and integrating on $[c,d]$, we get

$$\begin{aligned}\int_c^d g_2(y)f\left(\frac{a+b}{2},y\right)dy&\leq\frac{1}{G_1}\int_a^b\int_c^d f(x,y)g_1(x)g_2(y)dydx\\&\leq\frac{1}{2}\left[\int_c^d g_2(y)f(a,y)dy+\int_c^d g_2(y)f(b,y)dy\right].\end{aligned}\tag{7}$$

Since $G_1,G_2>0$, dividing inequalities (6), (7) by G_1, G_2 respectively and adding we get second and third inequalities in (3).
From first part of (4) and (5), we have

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\leq\frac{1}{G_2}\int_c^d f\left(\frac{a+b}{2},y\right)g_2(y)dy$$

and

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\leq\frac{1}{G_1}\int_a^b f\left(x,\frac{c+d}{2}\right)g_1(x)dx.$$

Adding the above two inequalities we get the first inequality in (3).
Now from second part of (4) and (5), we can get

$$\frac{1}{G_1}\int_a^b f(x,c)g_1(x)dx\leq\frac{f(a,c)+f(b,c)}{2},$$

$$\frac{1}{G_1}\int_a^b f(x,d)g_1(x)dx\leq\frac{f(a,d)+f(b,d)}{2},$$

$$\frac{1}{G_2}\int_c^d f(a,y)g_2(y)dy\leq\frac{f(a,c)+f(a,d)}{2},$$

$$\frac{1}{G_2}\int_c^d f(b,y)g_2(y)dy\leq\frac{f(b,c)+f(b,d)}{2}.$$

By adding the above four inequalities, we get last inequality in (3).
If in (3) we choose $f(x)=xy$, then (3) becomes an equality, which shows that inequalities in (3) are sharp. □

Remark 2.2. If we put $g_1\equiv 1$ and $g_2\equiv 1$ in above theorem, then we get Theorem 1.2, which is the main theorem of [5].

For a mapping $f:\Delta\to\mathbb{R}$, we define the mapping $\widehat{H}:[0,1]^2\to\mathbb{R}$, as follows:

$$\widehat{H}(t,s)=\frac{1}{G_1G_2}\int_a^b\int_c^d f\left(tx+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)g_1(x)g_2(y)dydx.\tag{8}$$

The properties of this mapping are studied in the following theorem. We need a following Lemma to give desire results, which is due to Levin and Stečkin [16, pp. 200].

Lemma 2.3. *Let f be a convex function on $[a,b]$, g be a function symmetric about $(a+b)/2$ and nonincreasing on $[a,(a+b)/2]$. Then*

$$\int_a^b f(x)g(x)dx \geq \frac{1}{b-a}\int_a^b f(x)dx \int_a^b g(x)dx.$$

Theorem 2.4. *Suppose that $f:\Delta \to \mathbb{R}$ is convex on the coordinates in Δ. Then the mapping $\widehat{H}$, defined in (8), is convex on the coordinates on $[0,1]^2$. Further if g_1 is nonincreasing on $[a,(a+b)/2]$ and g_2 is nonincreasing on $[c,(c+d)/2]$, then*

$$\inf_{(t,s)\in[0,1]^2} \widehat{H}(t,s) = f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) = \widehat{H}(0,0)$$

and

$$\sup_{(t,s)\in[0,1]^2} \widehat{H}(t,s) = \frac{1}{G_1G_2}\int_a^b\int_c^d f(x,y)g_1(x)g_2(y)dydx = \widehat{H}(1,1).$$

Proof. For convexity, fix $s\in[0,1]$. Then for all $\alpha,\beta\geq 0$ with $\alpha+\beta=1$, and $t_1,t_2\in[0,1]$ we have

$$\widehat{H}(\alpha t_1+\beta t_2, s) = \frac{1}{G_1G_2}$$
$$\times \int_a^b\int_c^d f\left((\alpha t_1+\beta t_2)x + (1-(\alpha t_1+\beta t_2))\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right) g_1(x)g_2(y)dydx$$

which gives us

$$\begin{aligned}
&\widehat{H}(\alpha t_1+\beta t_2, s) = \frac{1}{G_1G_2}\int_a^b\int_c^d f\left(\alpha\left(t_1x+(1-t_1)\frac{a+b}{2}\right)\right.\\
&\left. + \beta\left(t_1x+(1-t_1)\frac{a+b}{2}\right), sy+(1-s)\frac{c+d}{2}\right) g_1(x)g_2(y)dydx\\
&\leq \frac{\alpha}{G_1G_2}\int_a^b\int_c^d f\left(t_1x+(1-t_1)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right) g_1(x)g_2(y)dydx\\
&+ \frac{\beta}{G_1G_2}\int_a^b\int_c^d f\left(t_2x+(1-t_2)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right) g_1(x)g_2(y)dydx\\
&= \alpha\widehat{H}(t_1,s)+\beta\widehat{H}(t_2,s).
\end{aligned}$$

If $t\in[0,1]$ is fixed, then for all $s_1,s_2\in[0,1]$ and $\alpha,\beta\geq 0$ with $\alpha+\beta=1$, we also have:

$$\widehat{H}(t,\alpha s_1+\beta s_2) \leq \alpha\widehat{H}(t,s_1)+\beta\widehat{H}(t,s_2)$$

and the statement is proved.
Now to prove the remaining part of the theorem, we take

$$\widehat{H}(t,s) = \frac{1}{G_1G_2}\int_a^b\int_c^d f\left(tx+(1-t)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right) g_2(y)g_1(x)dydx.$$

Since f is convex on the coordinates and $\frac{1}{G_2}\int_c^d g_2(y)dy = 1$, we apply Jensen's inequality for integrals on second coordinate to get

$$\widehat{H}(t,s) \geq \frac{1}{G_1}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}, \frac{1}{G_2}\int_c^d\left(sy+(1-s)\frac{c+d}{2}\right)g_2(y)dy\right) g_1(x)dx.$$

Now it follows from Lemma 2.3, that

$$\widehat{H}(t,s) \geq \frac{1}{G_1}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}, \frac{c+d}{2}\right) g_1(x)dx. \tag{9}$$

Since $\frac{1}{G_1}\int_a^b g_1(x)\,dx = 1$, Jensen's inequality for integrals leads to

$$\widehat{H}(t,s) \geq f\left(\frac{1}{G_1}\int_a^b \left(tx + (1-t)\frac{a+b}{2}\right) g_1(x)dx, \frac{c+d}{2}\right).$$

Now by Lemma 2.3, we have

$$\widehat{H}(t,s) \geq f\left(\frac{a+b}{2}, \frac{c+d}{2}\right).$$

This gives us the lower bound of $\widehat{H}$. To get upper bound, we use convexity on second coordinates of f to get

$$\widehat{H}(t,s) \leq \frac{1}{G_1G_2}\int_a^b \left[s\int_c^d f\left(tx + (1-t)\frac{a+b}{2}, y\right) g_2(y)dy \right.$$
$$\left. + (1-s)f\left(tx + (1-t)\frac{a+b}{2}, \frac{c+d}{2}\right) g_2(y)dy\right] g_1(x)dx.$$

This gives

$$\begin{aligned}\widehat{H}(t,s) &\leq \frac{s}{G_1G_2}\int_a^b\int_c^d f\left(tx + (1-t)\frac{a+b}{2}, y\right) g_1(x)g_2(y)\,dydx \\ &+ \frac{(1-s)}{G_1G_2}\int_a^b\int_c^d f\left(tx + (1-t)\frac{a+b}{2}, \frac{c+d}{2}\right) g_1(x)g_2(y)\,dydx \\ &\leq \frac{s}{G_1G_2}\int_a^b\int_c^d \left[tf(x,y) + (1-t)f\left(\frac{a+b}{2}, y\right)\right] g_1(x)g_2(y)\,dydx \\ &+ \frac{1-s}{G_1G_2}\int_a^b\int_c^d \left[tf\left(x, \frac{c+d}{2}\right) + (1-t)f\left(\frac{a+b}{2}, y\right)\right] g_1(x)g_2(y)dydx.\end{aligned}$$

On simplification, we have

$$\begin{aligned}\widehat{H}(t,s) &\leq \frac{st}{G_1G_2}\int_a^b\int_c^d f(x,y)g_1(x)g_2(y)\,dydx + \frac{s(1-t)}{G_2}\int_c^d f\left(\frac{a+b}{2}, y\right) g_2(y)dy \\ &+ \frac{(1-s)t}{G_1}\int_a^b f\left(x, \frac{c+d}{2}\right) g_1(x)dx + (1-s)(1-t)f\left(\frac{a+b}{2}, \frac{c+d}{2}\right).\end{aligned}$$

From inequalities (6) and (7), we have

$$\frac{1}{G_2}\int_c^d f\left(\frac{a+b}{2}, y\right) g_2(y)dy \leq \frac{1}{G_1G_2}\int_a^b\int_c^d f(x,y)g_1(x)g_2(y)\,dydx$$

and

$$\frac{1}{G_1}\int_a^b f\left(x, \frac{c+d}{2}\right) g_1(x) \leq \frac{1}{G_1G_2}\int_a^b\int_c^d f(x,y)g_1(x)g_2(y)dydx.$$

Using above inequalities, we deduce that

$$\begin{aligned}\widehat{H}(t,s) &\leq [st + s(1-t) + (1-s)t + (1-s)(1-t)]\,\frac{1}{G_1G_2}\int_a^b\int_c^d f(x,y)g_1(x)g_2(y)\,dydx \\ &= \frac{1}{G_1G_2}\int_a^b\int_c^d f(x,y)g_1(x)g_2(y)\,dydx, (t,s) \in [0,1]^2.\end{aligned}$$

Now we have to show the monotonicity of the mapping $\widehat{H}(t,s)$. For this firstly, we will show that $\widehat{H}(t,s) \geq \widehat{H}(t,0)$ for all $(t,s) \in [0,1]^2$. By (9), we have:

$$\widehat{H}(t,s) \geq \frac{1}{G_1}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}, \frac{c+d}{2}\right) g_1(x)dx = \widehat{H}(t,0)$$

for all $(t,s) \in [0,1]^2$.
Now let $0 \le s_1 \le s_2 \le 1$. By convexity of mapping $\widehat{H}(t,.)$ for all $t \in [0,1]$, we have

$$\frac{\widehat{H}(t,s_2) - \widehat{H}(t,s_1)}{s_2 - s_1} \ge \frac{\widehat{H}(t,s_1) - \widehat{H}(t,0)}{s_1} \ge 0.$$

This completes the proof. □

Remark 2.5. If we put $g_1 \equiv 1$ and $g_2 \equiv 1$, in Theorem 2.4 then we get Theorem 2 of [5].

If the function f is convex on Δ, instead of coordinated convex, then we have the following theorem.

Theorem 2.6. *Suppose that $f : \Delta \to \mathbb{R}$ is convex on Δ.*
(i) The mapping $\widehat{H}$ is convex on Δ.
(ii) Let $\widehat{h} : [0,1] \to \mathbb{R}$ be the mapping defined as $\widehat{h}(t) = \widehat{H}(t,t)$. Then $\widehat{h}$ is convex, monotonic nondecreasing on $[0,1]$ and one has the bounds:

$$\inf_{t\in[0,1]} \widehat{h}(t) = f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) = \widehat{H}(0,0)$$

and

$$\sup_{t\in[0,1]} \widehat{h}(t) = \frac{1}{G_1G_2}\int_a^b\int_c^d f(x,y)g_1(x)g_2(y)dydx = \widehat{H}(1,1).$$

Proof. (i) For convexity, let $(t_1,s_1),(t_2,s_2) \in [0,1]^2$ and $\alpha,\beta \ge 0$ with $\alpha+\beta = 1$. Then

$$\begin{aligned}
&\widehat{H}(\alpha t_1 + \beta t_2, \alpha s_1 + \beta s_2) = \frac{1}{G_1G_2}\\
&\times \int_a^b\int_c^d f\left[\alpha\left(t_1x + (1-t_1)\frac{a+b}{2}, s_1y + (1-s_1)\frac{c+d}{2}\right)\right.\\
&\left.+\beta\left(t_2x + (1-t_2)\frac{a+b}{2}, s_2y + (1-s_2)\frac{c+d}{2}\right)\right] g_1(x)g_2(y)dydx\\
&\le \frac{\alpha}{G_1G_2}\int_a^b\int_c^d f\left(t_1 + (1-t_1)\frac{a+b}{2}, s_1y + (1-s_1)\frac{c+d}{2}\right) g_1(x)g_2(y)dydx\\
&+ \frac{\beta}{G_1G_2}\int_a^b\int_c^d f\left(t_2x + (1-t_2)\frac{a+b}{2}, s_2y + (1-s_2)\frac{c+d}{2}\right) g_1(x)g_2(y)dydx\\
&= \alpha\widehat{H}(t_1,s_1) + \beta\widehat{H}(t_2,s_2).
\end{aligned}$$

Which shows that H is convex on $[0,1]^2$.
(ii) Now we prove the convexity of $\widehat{h}$ on $[0,1]$. For this, let $t_1,t_2 \in [0,1]$ and $\alpha,\beta \ge 0$ with $\alpha+\beta = 1$. Then

$$\begin{aligned}
\widehat{h}(\alpha t_1 + \beta t_2) &= \widehat{H}(\alpha t_1 + \beta t_2, \alpha t_1 + \beta t_2)\\
&= \widehat{H}(\alpha(t_1,t_1) + \beta(t_2,t_2))\\
&\le \alpha\widehat{H}(t_1,t_1) + \beta\widehat{H}(t_2,t_2)\\
&= \alpha\widehat{h}(t_1) + \beta\widehat{h}(t_2),
\end{aligned}$$

which shows the convexity of $\widehat{h}$ on $[0,1]$. Now to prove the remaining part of the theorem, we take

$$\widehat{h}(t) = \frac{1}{G_1G_2}\int_a^b\int_c^d f\left(tx + (1-t)\frac{a+b}{2}, ty + (1-t)\frac{c+d}{2}\right) g_2(y)g_1(x)dydx.$$

Since f is convex on the coordinates and $\frac{1}{G_2}\int_c^d g_2(y)dy = 1$, we apply Jensen's inequality for integrals on second coordinate to get

$$\widehat{h}(t) \ge \frac{1}{G_1}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}, \frac{1}{G_2}\int_c^d \left[ty + (1-t)\frac{c+d}{2}\right] g_2(y)dy\right) g_1(x)dx.$$

Now it follows from Lemma 2.3, that

$$\widehat{h}(t) \geq \frac{1}{G_1} \int_a^b f\left(tx + (1-t)\frac{a+b}{2}, \frac{c+d}{2}\right) g_1(x)dx. \tag{10}$$

Since $\frac{1}{G_1} \int_a^b g_1(x)\, dx = 1$, Jensen's inequality for integrals leads to

$$\widehat{h}(t) \geq f\left(\frac{1}{G_1} \int_a^b \left[tx + (1-t)\frac{a+b}{2}\right] g_1(x)dx, \frac{c+d}{2}\right).$$

Now by Lemma 2.3, we have

$$\widehat{h}(t) \geq f\left(\frac{a+b}{2}, \frac{c+d}{2}\right).$$

This gives us the lower bound of $\widehat{h}(.)$. To get upper bound, we use convexity on second coordinates of f to get

$$\begin{aligned}\widehat{h}(t) \leq \frac{1}{G_1G_2} \int_a^b \Bigg[t \int_c^d f\left(tx + (1-t)\frac{a+b}{2}, y\right) g_2(y)dy \\ + (1-t) f\left(tx + (1-t)\frac{a+b}{2}, \frac{c+d}{2}\right) g_2(y)dy \Bigg] g_1(x)dx.\end{aligned}$$

This gives

$$\begin{aligned}\widehat{h}(t) &\leq \frac{t}{G_1G_2} \int_a^b \int_c^d f\left(tx + (1-t)\frac{a+b}{2}, y\right) g_1(x)g_2(y)dydx \\ &+ \frac{(1-t)}{G_1G_2} \int_a^b \int_c^d f\left(tx + (1-t)\frac{a+b}{2}, \frac{c+d}{2}\right) g_1(x)g_2(y)dydx \\ &\leq \frac{t}{G_1G_2} \int_a^b \int_c^d \left[tf(x,y) + (1-t)f\left(\frac{a+b}{2}, y\right)\right] g_1(x)g_2(y)dydx \\ &+ \frac{1-t}{G_1G_2} \int_a^b \int_c^d \left[tf\left(x, \frac{c+d}{2}\right) + (1-t)f\left(\frac{a+b}{2}, y\right)\right] g_1(x)g_2(y)dydx.\end{aligned}$$

On simplification, we have

$$\begin{aligned}\widehat{h}(t) \leq \frac{t^2}{G_1G_2} \int_a^b \int_c^d f(x,y)g_1(x)g_2(y)dydx + \frac{t(1-t)}{G_2} \int_c^d f\left(\frac{a+b}{2}, y\right) g_2(y)dy \\ + \frac{(1-t)t}{G_1} \int_a^b f\left(x, \frac{c+d}{2}\right) g_1(x)dx + (1-t)^2 f\left(\frac{a+b}{2}, \frac{c+d}{2}\right).\end{aligned}$$

From inequalities (6) and (7), we have

$$\frac{1}{G_2} \int_c^d f\left(\frac{a+b}{2}, y\right) g_2(y)dy \leq \frac{1}{G_1G_2} \int_a^b \int_c^d f(x,y)g_1(x)g_2(y)dydx$$

and

$$\frac{1}{G_1} \int_a^b f\left(x, \frac{c+d}{2}\right) g_1(x) \leq \frac{1}{G_1G_2} \int_a^b \int_c^d f(x,y)g_1(x)g_2(y)dydx.$$

Using above inequalities, we deduce that

$$\begin{aligned}\widehat{h}(t) &\leq \left[t^2 + t(1-t) + (1-t)t + (1-t)^2\right] \frac{1}{G_1G_2} \int_a^b \int_c^d f(x,y)g_1(x)g_2(y)dydx \\ &= \frac{1}{G_1G_2} \int_a^b \int_c^d f(x,y)g_1(x)g_2(y)dydx,\ t \in [0,1].\end{aligned}$$

Now we have to show the monotonicity of the mapping $\widehat{h}$ for this firstly, we show that $\widehat{H}(t,t) \geq \widehat{H}(t,0)$ for all $t \in [0,1]$. By (10), we have:

$$\widehat{h}(t) \geq \frac{1}{G_1}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}, \frac{c+d}{2}\right) g_1(x)dx = \widehat{H}(t,0)$$

for all $t \in [0,1]$.

Now let $0 \leq t_1 \leq t_2 \leq 1$. By convexity of mapping $\widehat{H}(t,.)$ for all $t \in [0,1]$, we have

$$\frac{\widehat{h}(t_2)-\widehat{h}(t_1)}{t_2-t_1} \geq \frac{\widehat{h}(t_1)-\widehat{h}(0)}{t_1} \geq 0.$$

This completes the proof. □

Remark 2.7. If we put $g_1 \equiv 1$ and $g_2 \equiv 1$ in Theorem 2.6, then we get Theorem 3 of [5].

References

[1] S. Abramovich, G. Farid and J. E. Pečarić, *More about Jensens inequality and Cauchys means for superquadratic functions*, J. Math. Inequal. **7**(1) (2013), 11–24.

[2] S. I. Butt, J. E. Pečarić and Atiq Ur Rehman, *Non-symmetric Stolarsky means*, J. Math. Inequal. **7**(2) (2013), 227–237.

[3] S. S. Dragomir, J. E. Pečarić and J. Sáandor, *A note on the Jensen-Hadamard inequality*, L' Anal. Num. Theor. L'Approx. (Romania) **19** (1990), 21–28.

[4] S. S. Dragomir, D. Barbu and C. Buşe, *A probabilistic argument for the convergence of some sequences associated to Hadamard's inequality*, Studia Univ. Babeş-Bolgai, Mathematica **38** (1993), 29–34.

[5] S. S. Dragomir, *On Hadamards inequality for convex functions on the co-ordinates in a rec- tangle from the plane*, Taiwanese J Math. **4** (2001), 775–788.

[6] S. S. Dragomir, *On Hadamard's inequality for convex functions*, Mat. Balkanica **6** (1992), 215–222.

[7] S. S. Dragomir, *Refinements of the Hermite–Hadamard inequality for convex functions*, J. Inequal. Pure Appl. Math. **6**(5) (2005), Article 140.

[8] S. S. Dragomir and N. M. lonescu, *Some integral inequalities for differentiable convex functions.* Coll. Pap. of the Fac. of Sci. Kragujevac (Yugoslavia) **13** (1992), 11–16.

[9] S. S. Dragomir, *Some integral inequlities for differentiable convex functions*, Contributions, Macedonian Acad. of Sci. and Arts (Scopie) **16** (1992), 77–80.

[10] S. S. Dragomir, and N. M. Ionescu, *Some remarks in convex functions*, L'Anal. Num. Theor. L'Approx. (Romania), **21** (1992), 31–36.

[11] L. Fejér, Über die Fourierreihen, II. Math. Naturwiss Anz Ungar. Akad. Wiss. **24** (1906), 369–390

[12] D. S. Mitrinovišc, J. E. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, 1993.

[13] D. S. Mitrinović, I. B. Lacković, *Hermite and convexity*, Aequationes Math. **28** (1985).

[14] C. P. Niculescu, L. E. Persson, *Convex functions and their applications: A contemporary approach*, Springer, New York, 2006.

[15] J. E. Pečarić and S. S. Dragomir, *A generalisation of Hadamard's inequality for isotonic linear functionals*, Rodovi Math. (Sarajevo) **7** (1991), 103–107.

[16] J. E. Pečarić, F. Proschan, Y. L. Tong, Convex Functions, Partial Ordering, and Stasitcal Applications, Academic Press, Inc. 1992.

[17] J. E. Pečarić and S. S. Dragomir, *On some integral inequalities for convex functions*, Bull. Mat. Inst. Pol. Iasi **36** (1990), 19–23.

COMSATS Institute of Information Technology, Attock Campus, Pakistan

*Corresponding author: faridphdsms@hotmail.com