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ON WEAK AND STRONG CONVERGENCE THEOREMS OF MODIFIED SP-ITERATION SCHEME FOR TOTAL ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we study modified SP-iteration scheme for three total asymptotically nonexpansive mappings and also establish some weak and strong convergence theorems for mentioned mappings and scheme to converge to common fixed points in the framework of Banach spaces. Our results extend and generalize the previous works from the current existing literature.

1. Introduction

Let C be a nonempty subset of a Banach space E and $T\colon C \to C$ a nonlinear mapping. We denote the set of all fixed points of T by $F(T)$. The set of common fixed points of three mappings T_1, T_2 and T_3 will be denoted by $F = \cap_{i=1}^{3} F(T_i)$.

Definition 1.1. Let $T\colon C \to C$ be a mapping. Then
(1) T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \tag{1.1}$$

for all x, $y \in C$.

(2) T is said to be asymptotically nonexpansive if there exists a positive sequence $h_n \in [1, \infty)$ with $\lim_{n\to\infty} h_n = 1$ such that

$$\|T^n x - T^n y\| \leq h_n \|x - y\| \tag{1.2}$$

for all x, $y \in C$ and $n \geq 1$.

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [6] as a generalization of the class of nonexpansive mappings. They proved that if C is a nonempty closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive mapping on C, then has a fixed point.

T is said to be asymptotically noneexpansive in the intermediate sense if it is continuous and the following inequality holds:

$$\limsup_{n\to\infty} \sup_{x,y\in C} \big(\|T^n x - T^n y\| - \|x - y\|\big) \leq 0. \tag{1.3}$$

Observe that if we define

$$c_n = \limsup_{n\to\infty} \sup_{x,y\in C} \big(\|T^n x - T^n y\| - \|x - y\|\big) \text{ and } \nu_n = \max\{0, c_n\},$$

then $\nu_n \to 0$ as $n \to \infty$. It follows that (1.3) is reduced to

$$\|T^n x - T^n y\| \leq \|x - y\| + \nu_n, \tag{1.4}$$

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for all x, $y \in C$ and $n \geq 1$.

The class of mappings which are asymptotically nonexpansive in the intermediate sense was introduced by Bruck, Kuczumow and Reich [3]. It is known [10] that if C is a nonempty closed convex bounded subset of a uniformly convex Banach space E and T is asymptotically nonexpansive in the intermediate sense mapping, then T has a fixed point. It is worth mentioning that the class of mappings which are asymptotically nonexpansive in the intermediate contains properly the class of asymptotically nonexpansive mappings.

In 2006, Albert et al. [2] introduced the notion of total asymptotically nonexpansive mappings.

Definition 1.2. ([2]) The mapping T is said to be total asymptotically nonexpansive if

$$\|T^n x - T^n y\| \leq \|x - y\| + \mu_n \psi(\|x - y\|) + \nu_n, \tag{1.5}$$

for all x, $y \in C$ and $n \geq 1$, where $\{\mu_n\}$ and $\{\nu_n\}$ are nonnegative real sequences such that $\mu_n \to 0$ and $\nu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi\colon [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$. From the definition, we see that the class of total asymptotically nonexpansive mappings include the class of asymptotically nonexpansive mappings as a special case; see also [4] for more details.

Remark 1.3. From the above definition, it is clear that each asymptotically nonexpansive mapping is a total asymptotically nonexpansive mapping with $\nu_n = 0$, $\mu_n = k_n - 1$ for all $n \geq 1$, $\psi(t) = t$, $t \geq 0$.

(1) Mann iteration [12]: Chose $x_1 \in C$ and define

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T x_n, \; n \geq 1, \tag{1.6}$$

where $\{\alpha_n\}$ is a sequence in (0,1).

(2) Ishikawa iteration [9]: Chose $x_1 \in C$ and define

$$\begin{aligned} y_n &= (1 - \beta_n) x_n + \beta_n T x_n \\ x_{n+1} &= (1 - \alpha_n) x_n + \alpha_n T y_n, \; n \geq 1, \end{aligned} \tag{1.7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in (0,1).

(3) S-iteration [1]: Chose $x_1 \in C$ and define

$$\begin{aligned} y_n &= (1 - \beta_n) x_n + \beta_n T x_n \\ x_{n+1} &= (1 - \alpha_n) T x_n + \alpha_n T y_n, \; n \geq 1, \end{aligned} \tag{1.8}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in (0,1). Note that (1.8) is independent of (1.7) (and hence (1.6)). Agarwal, O'Regan and Sahu [1] showed that their process independent of those of Mann and Ishikawa and converges faster than both of these (see [[1], Proposition 3.1]).

(4) Modified S-iteration [1]: Chose $x_1 \in C$ and define

$$\begin{aligned} y_n &= (1 - \beta_n) x_n + \beta_n T^n x_n \\ x_{n+1} &= (1 - \alpha_n) T^n x_n + \alpha_n T^n y_n, \; n \geq 1, \end{aligned} \tag{1.9}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in (0,1).

(5) Noor iteration [13]: Chose $x_1 \in C$ and define

$$\begin{aligned} z_n &= (1 - \gamma_n) x_n + \gamma_n T x_n \\ y_n &= (1 - \beta_n) x_n + \beta_n T z_n \\ x_{n+1} &= (1 - \alpha_n) x_n + \alpha_n T y_n, \; n \geq 1, \end{aligned} \tag{1.10}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in [0,1].

(6) Modified Noor iteration [21]: Chose $x_1 \in C$ and define

$$\begin{aligned} z_n &= (1-\gamma_n)x_n + \gamma_n T^n x_n \\ y_n &= (1-\beta_n)x_n + \beta_n T^n z_n \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n,\ n \geq 1, \end{aligned} \tag{1.11}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in [0,1].

Recently, Phuengrattana and Suantai [16] introduced the following iteration scheme.

(7) SP-iteration [16]: Chose $x_1 \in C$ and define

$$\begin{aligned} z_n &= (1-\gamma_n)x_n + \gamma_n T x_n \\ y_n &= (1-\beta_n)z_n + \beta_n T z_n \\ x_{n+1} &= (1-\alpha_n)y_n + \alpha_n T y_n,\ n \geq 1, \end{aligned} \tag{1.12}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in [0,1].

Inspired and motivated by [16], we modify iteration scheme (1.12) for three total asymptotically nonexpansive self mappings of C as follows:

(8) Modified SP-iteration: Chose $x_1 \in C$ and define

$$\begin{aligned} z_n &= (1-\gamma_n)x_n + \gamma_n T_3^n x_n \\ y_n &= (1-\beta_n)z_n + \beta_n T_2^n z_n \\ x_{n+1} &= (1-\alpha_n)y_n + \alpha_n T_1^n y_n,\ n \geq 1, \end{aligned} \tag{1.13}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in [0,1].

Remark 1.4. If we take $T_1^n = T_2^n = T_3^n = T$ for all $n \geq 1$, then (1.13) reduces to the SP-iteration scheme (1.12).

The three-step iterative approximation problems were studied extensively by Noor [13, 14], Glowinsky and Le Tallec [7], and Haubruge et al [8]. It has been shown [7] that three-step iterative scheme gives better numerical results than the two step and one step approximate iterations. Thus we conclude that three step scheme plays an important and significant role in solving various problems, which arise in pure and applied sciences.

The purpose of this paper is to study modified SP-iteration scheme (1.13) and establish some strong and weak convergence theorems for total asymptotically nonexpansive mappings in the setting of Banach spaces. Our results extend and generalize the previous works from the current existing literature.

2. **Preliminaries**

For the sake of convenience, we restate the following definitions and lemmas.

Let E be a Banach space with its dimension greater than or equal to 2. The modulus of convexity of E is the function $\delta_E(\varepsilon)\colon (0,2] \to [0,1]$ defined by

$$\delta_E(\varepsilon) = \inf\Big\{1 - \|\frac{1}{2}(x+y)\| : \|x\| = 1,\ \|y\| = 1,\ \varepsilon = \|x-y\|\Big\}.$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$.

We recall the following:

Let $\mathcal{S} = \{x \in E : \|x\| = 1\}$ and let E^* be the dual of E, that is, the space of all continuous linear functionals f on E.

Definition 2.1. (i) *Opial condition*: The space E has Opial condition [15] if for any sequence $\{x_n\}$ in E, x_n converges to x weakly it follows that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$. Examples of Banach spaces satisfying Opial condition are Hilbert spaces and all spaces

$l^p(1 < p < \infty)$. On the other hand, $L^p[0, 2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial condition.

(ii) A mapping $T: C \to C$ is said to be demiclosed at zero, if for any sequence $\{x_n\}$ in K, the condition x_n converges weakly to $x \in C$ and Tx_n converges strongly to 0 imply $Tx = 0$.

(iii) A Banach space E has the Kadec-Klee property [19] if for every sequence $\{x_n\}$ in E, $x_n \to x$ weakly and $\|x_n\| \to \|x\|$ it follows that $\|x_n - x\| \to 0$.

Definition 2.2. *Condition* (A): The mapping $T: C \to E$ with $F(T) \neq \emptyset$ is said to satisfy *condition* (A) [18] if there is a nondecreasing function $f: [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0, \infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in C$, where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.

Now, we modify *Condition* (A) for three mappings.

Definition 2.3. *Condition* (B): Three mappings T_1, T_2, $T_3: C \to C$ are said to satisfy *condition* (B) if there is a nondecreasing function $f: [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(t) > 0$ for all $t \in (0, \infty)$ such that $a_1 \|x - T_1x\| + a_2 \|x - T_2x\| + a_3 T_3x \geq f(d(x, F))$ for all $x \in C$, where $d(x, F) = \inf\{\|x - p\| : p \in F = \cap_{i=1}^{3} F(T_i)\}$, where a_1, a_2 and a_3 are nonnegative real numbers such that $a_1 + a_2 + a_3 = 1$.

Note that *condition* (B) reduces to *condition* (A) when $T_1 = T_2 = T_3 = T$ and hence is more general than the demicompactness of T_1, T_2 and T_3 [18]. A mapping $T: C \to C$ is called: (1) *demicompact* if any bounded sequence $\{x_n\}$ in C such that $\{x_n - Tx_n\}$ converges has a convergent subsequence; (2) semicompact (or hemicompact) if any bounded sequence $\{x_n\}$ in C such that $\{x_n - Tx_n\} \to 0$ as $n \to \infty$ has a convergent subsequence. Every demicompact mapping is semicompact but the converse is not true in general.

Senter and Dotson [18] have approximated fixed points of a nonexpansive mapping T by Mann iterates whereas Maiti and Ghosh [11] and Tan and Xu [20] have approximated the fixed points using Ishikawa iterates under the *condition* (A) of [18]. Tan and Xu [20] pointed out that *condition* (A) is weaker than the compactness of C. We shall use *condition* (B) instead of compactness of C to study the strong convergence of $\{x_n\}$ defined by iteration scheme (1.13).

Lemma 2.4. *(See* [20]*) Let* $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ *and* $\{r_n\}_{n=1}^{\infty}$ *be sequences of nonnegative numbers satisfying the inequality*

$$\alpha_{n+1} \leq (1 + \beta_n)\alpha_n + r_n,\ \forall\, n \geq 1.$$

If $\sum_{n=1}^{\infty} \beta_n < \infty$ *and* $\sum_{n=1}^{\infty} r_n < \infty$*, then*

(i) $\lim_{n\to\infty} \alpha_n$ *exists;*

(ii) In particular, if $\{\alpha_n\}_{n=1}^{\infty}$ *has a subsequence which converges strongly to zero, then* $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.5. *(See* [17]*) Let* E *be a uniformly convex Banach space and* $0 < \alpha \leq t_n \leq \beta < 1$ *for all* $n \in \mathbb{N}$. *Suppose further that* $\{x_n\}$ *and* $\{y_n\}$ *are sequences of* E *such that* $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$ *and* $\lim_{n\to\infty} \|t_nx_n + (1-t_n)y_n\| = a$ *hold for some* $a \geq 0$. *Then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.6. *(See* [19]*) Let* E *be a real reflexive Banach space with its dual* E^* *has the Kadec-Klee property. Let* $\{x_n\}$ *be a bounded sequence in* E *and* p, $q \in w_w(x_n)$ *(where* $w_w(x_n)$ *denotes the set of all weak subsequential limits of* $\{x_n\}$*). Suppose* $\lim_{n\to\infty} \|tx_n + (1 - t)p - q\|$ *exists for all* $t \in [0, 1]$. *Then* $p = q$.

Lemma 2.7. *(See* [19]*) Let* K *be a nonempty convex subset of a uniformly convex Banach space* E. *Then there exists a strictly increasing continuous convex function* $\phi: [0, \infty) \to [0, \infty)$ *with* $\phi(0) = 0$ *such that for each Lipschitzian mapping* $T: C \to C$ *with the Lipschitz constant* L,

$$\|tTx + (1 - t)Ty - T(tx + (1 - t)y)\| \leq L\phi^{-1}\Big(\|x - y\| - \frac{1}{L}\|Tx - Ty\|\Big)$$

for all x, $y \in K$ *and all* $t \in [0, 1]$.

Proposition 2.8. *Let C be a nonempty subset of a Banach space E and $T_1, T_2, T_3\colon C \to C$ be three total asymptotically nonexpansive mappings. Then there exist nonnegative real sequences $\{\mu_n\}$ and $\{\nu_n\}$ in $[0,\infty)$ with $\mu_n \to 0$ and $\nu_n \to 0$ as $n \to \infty$ and a strictly increasing continuous function $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$ such that*

$$\|T_1^n x - T_1^n y\| \leq \|x-y\| + \mu_n \psi(\|x-y\|) + \nu_n, \tag{2.1}$$

$$\|T_2^n x - T_2^n y\| \leq \|x-y\| + \mu_n \psi(\|x-y\|) + \nu_n, \tag{2.2}$$

and

$$\|T_3^n x - T_3^n y\| \leq \|x-y\| + \mu_n \psi(\|x-y\|) + \nu_n, \tag{2.3}$$

for all $x, y \in C$ and $n \geq 1$.

Proof. Since $T_1, T_2, T_3\colon C \to C$ are three total asymptotically nonexpansive mappings, there exist nonnegative real sequences $\{\mu_{n_1}\}$, $\{\mu_{n_2}\}$, $\{\mu_{n_3}\}$, $\{\nu_{n_1}\}$, $\{\nu_{n_2}\}$ and $\{\nu_{n_3}\}$ in $[0,\infty)$ with $\mu_{n_1}, \mu_{n_2}, \mu_{n_3} \to 0$ and $\nu_{n_1}, \nu_{n_2}, \nu_{n_3} \to 0$ as $n \to \infty$ and strictly increasing continuous functions $\psi_1, \psi_2, \psi_3\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi_i(0) = 0$ for $i = 1,2,3$ such that

$$\|T_1^n x - T_1^n y\| \leq \|x-y\| + \mu_{n_1} \psi_1(\|x-y\|) + \nu_{n_1}, \tag{2.4}$$

$$\|T_2^n x - T_2^n y\| \leq \|x-y\| + \mu_{n_2} \psi_2(\|x-y\|) + \nu_{n_2}, \tag{2.5}$$

and

$$\|T_3^n x - T_3^n y\| \leq \|x-y\| + \mu_{n_3} \psi_3(\|x-y\|) + \nu_{n_3}, \tag{2.6}$$

for all $x, y \in C$ and $n \geq 1$.

Setting

$$\mu_n = \max\{\mu_{n_1}, \mu_{n_2}, \mu_{n_3}\}, \;\; \nu_n = \max\{\nu_{n_1}, \nu_{n_2}, \nu_{n_3}\}$$

and

$$\psi(r) = \max\{\psi_i(r), \text{for } i = 1,2,3 \;\text{ and }\; \text{for } r \geq 0\},$$

then we get that there exist nonnegative real sequences $\{\mu_n\}$ and $\{\nu_n\}$ with $\mu_n \to 0$ and $\nu_n \to 0$ as $n \to \infty$ and strictly increasing continuous function $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\psi(0) = 0$ such that

$$\begin{aligned}\|T_1^n x - T_1^n y\| &\leq \|x-y\| + \mu_{n_1} \psi_1(\|x-y\|) + \nu_{n_1} \\ &\leq \|x-y\| + \mu_n \psi(\|x-y\|) + \nu_n,\end{aligned}$$

$$\begin{aligned}\|T_2^n x - T_2^n y\| &\leq \|x-y\| + \mu_{n_2} \psi_2(\|x-y\|) + \nu_{n_2} \\ &\leq \|x-y\| + \mu_n \psi(\|x-y\|) + \nu_n,\end{aligned}$$

and

$$\begin{aligned}\|T_3^n x - T_3^n y\| &\leq \|x-y\| + \mu_{n_3} \psi_3(\|x-y\|) + \nu_{n_3} \\ &\leq \|x-y\| + \mu_n \psi(\|x-y\|) + \nu_n,\end{aligned}$$

for all $x, y \in C$ and $n \geq 1$. □

3. **Strong Convergence Theorems**

In this section, we prove some strong convergence theorems for three total asymptotically nonexpansive mappings in the framework of real Banach spaces. First, we shall need the following lemmas.

Lemma 3.1. *Let E be a real Banach space and C be a nonempty closed convex subset of E. Let $T_1, T_2, T_3\colon C \to C$ be three total asymptotically nonexpansive mappings with sequences $\{\mu_n\}$ and $\{\nu_n\}$ as defined in proposition 2.8 and $F = \cap_{i=1}^{3} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the iteration scheme defined by (1.13), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[\delta, 1-\delta]$ for all $n \in \mathbb{N}$ and for some $\delta \in (0,1)$ and the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$;

(ii) there exists a constant $M > 0$ such that $\psi(t) \leq M\,t$, $t \geq 0$.

Then $\lim_{n\to\infty}\|x_n - p\|$ *and* $\lim_{n\to\infty} d(x_n, F)$ *both exist for all* $p \in F$.

Proof. Let $p \in F$. Then from (1.13), we have

$$\begin{aligned}
\|z_n - p\| &= \|(1-\gamma_n)x_n + \gamma_n T_3^n x_n - p\| \\
&\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\|T_3^n x_n - p\| \\
&\leq (1-\gamma_n)\|x_n - p\| + \gamma_n[\|x_n - p\| \\
&\quad +\mu_n\psi(\|x_n - p\|) + \nu_n] \\
&\leq (1-\gamma_n)\|x_n - p\| + \gamma_n[\|x_n - p\| \\
&\quad +\mu_n M\|x_n - p\| + \nu_n] \\
&\leq \|x_n - p\| + \mu_n M\|x_n - p\| + \nu_n \\
&= (1+\mu_n M)\|x_n - p\| + \nu_n. \qquad (3.1)
\end{aligned}$$

Again from (1.13) and (3.1), we have

$$\begin{aligned}
\|y_n - p\| &= \|(1-\beta_n)z_n + \beta_n T_2^n z_n - p\| \\
&\leq (1-\beta_n)\|z_n - p\| + \beta_n\|T_2^n z_n - p\| \\
&\leq (1-\beta_n)\|z_n - p\| + \beta_n[\|z_n - p\| \\
&\quad +\mu_n\psi(\|z_n - p\|) + \nu_n] \\
&\leq (1-\beta_n)\|z_n - p\| + \beta_n[\|z_n - p\| \\
&\quad +\mu_n M\|z_n - p\| + \nu_n] \\
&\leq \|z_n - p\| + \mu_n M\|z_n - p\| + \nu_n \\
&= (1+\mu_n M)\|z_n - p\| + \nu_n \\
&\leq (1+\mu_n M)[(1+\mu_n M)\|x_n - p\| + \nu_n] + \nu_n \\
&\leq (1+\mu_n M)^2\|x_n - p\| + (2+\mu_n M)\nu_n. \qquad (3.2)
\end{aligned}$$

Finally, using (1.13) and (3.2), we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1-\alpha_n)y_n + \alpha_n T_1^n y_n - p\| \\
&\leq (1-\alpha_n)\|y_n - p\| + \alpha_n\|T_1^n y_n - p\| \\
&\leq (1-\alpha_n)\|y_n - p\| + \alpha_n[\|y_n - p\| \\
&\quad +\mu_n\psi(\|y_n - p\|) + \nu_n] \\
&\leq (1-\alpha_n)\|y_n - p\| + \alpha_n[\|y_n - p\| \\
&\quad +\mu_n M\|y_n - p\| + \nu_n] \\
&\leq \|y_n - p\| + \mu_n M\|y_n - p\| + \nu_n \\
&= (1+\mu_n M)\|y_n - p\| + \nu_n \\
&\leq (1+\mu_n M)[(1+\mu_n M)^2\|x_n - p\| \\
&\quad +(2+\mu_n M)\nu_n] + \nu_n \\
&\leq (1+\mu_n M)^3\|x_n - p\| + (1+\mu_n M)\times \\
&\quad (2+\mu_n M)\nu_n + \nu_n \\
&\leq (1+\mu_n Q_1)\|x_n - p\| + \nu_n Q_2 \qquad (3.3)
\end{aligned}$$

for some $Q_1, Q_2 > 0$.

For any $p \in F$, from (3.3), we obtain the following inequality

$$d(x_{n+1}, F) \leq (1+\mu_n Q_1)d(x_n, F) + \nu_n Q_2. \qquad (3.4)$$

Since $\sum_{n=1}^{\infty}\mu_n < \infty$ and $\sum_{n=1}^{\infty}\nu_n < \infty$, therefore applying Lemma 2.4(i) in (3.3) and (3.4), we have $\lim_{n\to\infty}\|x_n - p\|$ and $\lim_{n\to\infty} d(x_n, F)$ both exist. This completes the proof. □

Lemma 3.2. *Let E be a uniformly convex Banach space and C be a nonempty closed convex subset of E. Let $T_1, T_2, T_3\colon C \to C$ be three uniformly continuous and total asymptotically nonexpansive mappings with sequences $\{\mu_n\}$ and $\{\nu_n\}$ as defined in proposition 2.8 and $F = \cap_{i=1}^{3} F(T_i) \neq \emptyset$. Let*

$\{x_n\}$ be the iteration scheme defined by (1.13), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[\delta, 1-\delta]$ for all $n \in \mathbb{N}$ and for some $\delta \in (0,1)$ and the following conditions are satisfied:
(i) $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$;
(ii) there exists a constant $M > 0$ such that $\psi(t) \leq M\,t$, $t \geq 0$.
Then $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for $i = 1, 2, 3$.

Proof. By Lemma 3.1, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$, so we can assume that $\lim_{n\to\infty} \|x_n - p\| = c$. Then $c > 0$ otherwise there is nothing to prove.

Now (3.1) and (3.2) implies that

$$\limsup_{n\to\infty} \|z_n - p\| \leq c, \tag{3.5}$$

and

$$\limsup_{n\to\infty} \|y_n - p\| \leq c. \tag{3.6}$$

Also

$$\begin{aligned} \|T_1^n y_n - p\| &\leq \|y_n - p\| + \mu_n \psi(\|y_n - p\|) + \nu_n \\ &\leq \|y_n - p\| + \mu_n M \|y_n - p\| + \nu_n \\ &= (1 + \mu_n M)\|y_n - p\| + \nu_n, \end{aligned}$$

and so

$$\limsup_{n\to\infty} \|T_1^n y_n - p\| \leq c. \tag{3.7}$$

Since

$$c = \|x_{n+1} - p\| = \|(1-\alpha_n)(y_n - p) + \alpha_n (T_1^n y_n - p)\|.$$

It follows from Lemma 2.5 that

$$\lim_{n\to\infty} \|T_1^n y_n - y_n\| = 0. \tag{3.8}$$

Again note that

$$\begin{aligned} \|T_3^n x_n - p\| &\leq \|x_n - p\| + \mu_n \psi(\|x_n - p\|) + \nu_n \\ &\leq \|x_n - p\| + \mu_n M \|x_n - p\| + \nu_n \\ &= (1 + \mu_n M)\|x_n - p\| + \nu_n, \end{aligned}$$

$$\begin{aligned} \|T_2^n z_n - p\| &\leq \|z_n - p\| + \mu_n \psi(\|z_n - p\|) + \nu_n \\ &\leq \|z_n - p\| + \mu_n M \|z_n - p\| + \nu_n \\ &= (1 + \mu_n M)\|z_n - p\| + \nu_n. \end{aligned}$$

Hence, from above inequalities, we obtain

$$\limsup_{n\to\infty} \|T_3^n x_n - p\| \leq c, \tag{3.9}$$

and

$$\limsup_{n\to\infty} \|T_2^n z_n - p\| \leq c. \tag{3.10}$$

Further, note that

$$\begin{aligned} \|y_n - p\| &\leq \|y_n - T_1^n y_n\| + \|T_1^n y_n - p\| \\ &\leq \|y_n - T_1^n y_n\| + \|y_n - p\| + \mu_n \psi(\|y_n - p\|) + \nu_n \\ &\leq \|y_n - T_1^n y_n\| + \|y_n - p\| + \mu_n M \|y_n - p\| + \nu_n \\ &\leq \|y_n - T_1^n y_n\| + (1 + \mu_n M)\|y_n - p\| + \nu_n. \end{aligned}$$

It follows from (3.6) and (3.8) that

$$c \leq \liminf_{n\to\infty} \|y_n - p\|. \tag{3.11}$$

From (3.6) and (3.11), we get

$$\lim_{n\to\infty} \|y_n - p\| = c. \tag{3.12}$$

Now, we have

$$c = \lim_{n\to\infty} \|y_n - p\| = \|(1-\beta_n)(z_n - p) + \beta_n(T_2^n z_n - p)\|. \tag{3.13}$$

It follows from (3.5), (3.10) and Lemma 2.5 that

$$\lim_{n\to\infty} \|T_2^n z_n - z_n\| = 0. \tag{3.14}$$

Again note that

$$\begin{aligned} \|z_n - p\| &\leq \|z_n - T_2^n z_n\| + \|T_2^n z_n - p\| \\ &\leq \|z_n - T_2^n z_n\| + \|z_n - p\| + \mu_n \psi(\|z_n - p\|) + \nu_n \\ &\leq \|z_n - T_2^n z_n\| + \|z_n - p\| + \mu_n M \|z_n - p\| + \nu_n \\ &\leq \|z_n - T_2^n z_n\| + (1 + \mu_n M)\|z_n - p\| + \nu_n. \end{aligned}$$

It follows from (3.5) and (3.14) that

$$c \leq \liminf_{n\to\infty} \|z_n - p\|. \tag{3.15}$$

From (3.5) and (3.15), we get

$$\lim_{n\to\infty} \|z_n - p\| = c. \tag{3.16}$$

Now, we see that

$$c = \lim_{n\to\infty} \|z_n - p\| = \|(1-\gamma_n)(x_n - p) + \gamma_n(T_3^n x_n - p)\|. \tag{3.17}$$

It follows from Lemma 2.5 that

$$\lim_{n\to\infty} \|T_3^n x_n - x_n\| = 0. \tag{3.18}$$

Again note that

$$\begin{aligned} \|x_n - z_n\| &= \gamma_n \|x_n - T_3^n x_n\| \\ &\leq (1-\delta)\|x_n - T_3^n x_n\|. \end{aligned} \tag{3.19}$$

Using (3.18) in (3.19), we get

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.20}$$

Further, note that

$$\begin{aligned} \|x_n - y_n\| &= \beta_n \|z_n - T_2^n z_n\| \\ &\leq (1-\delta)\|z_n - T_2^n z_n\|. \end{aligned} \tag{3.21}$$

Using (3.14) in (3.21), we get

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.22}$$

Note that

$$\|x_n - T_2^n z_n\| \leq \|x_n - z_n\| + \|z_n - T_2^n z_n\|. \tag{3.23}$$

Using (3.14) and (3.20) in (3.23), we get

$$\lim_{n\to\infty} \|x_n - T_2^n z_n\| = 0. \tag{3.24}$$

Hence

$$\begin{aligned}\|x_n - T_2^n x_n\| &\leq \|x_n - T_2^n z_n\| + \|T_2^n z_n - T_2^n x_n\| \\ &\leq \|x_n - T_2^n z_n\| + \|z_n - x_n\| + \mu_n \psi(\|z_n - x_n\|) + \nu_n \\ &\leq \|x_n - T_2^n z_n\| + \|z_n - x_n\| + \mu_n M \|z_n - x_n\| + \nu_n \\ &= \|x_n - T_2^n z_n\| + (1 + \mu_n M)\|z_n - x_n\| + \nu_n. \qquad (3.25)\end{aligned}$$

Using (3.20) and (3.24) in (3.25), we get

$$\lim_{n\to\infty} \|x_n - T_2^n x_n\| = 0. \qquad (3.26)$$

Again notice that

$$\|x_n - T_1^n y_n\| \leq \|x_n - y_n\| + \|y_n - T_1^n y_n\|. \qquad (3.27)$$

Using (3.8) and (3.22) in (3.27), we get

$$\lim_{n\to\infty} \|x_n - T_1^n y_n\| = 0. \qquad (3.28)$$

Hence

$$\begin{aligned}\|x_n - T_1^n x_n\| &\leq \|x_n - T_1^n y_n\| + \|T_1^n x_n - T_1^n y_n\| \\ &\leq \|x_n - T_1^n y_n\| + \|x_n - y_n\| + \mu_n \psi(\|x_n - y_n\|) + \nu_n \\ &\leq \|x_n - T_1^n y_n\| + \|x_n - y_n\| + \mu_n M \|x_n - y_n\| + \nu_n \\ &= \|x_n - T_1^n y_n\| + (1 + \mu_n M)\|x_n - y_n\| + \nu_n. \qquad (3.29)\end{aligned}$$

Using (3.22) and (3.28) in (3.29), we get

$$\lim_{n\to\infty} \|x_n - T_1^n x_n\| = 0. \qquad (3.30)$$

By the definitions of x_{n+1}, we have

$$\|x_n - x_{n+1}\| \leq \|x_n - y_n\| + \|T_1^n y_n - y_n\|. \qquad (3.31)$$

Using (3.8) and (3.22) in (3.31), we get

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \qquad (3.32)$$

By (3.30), (3.31) and uniform continuity of T_1, we have

$$\begin{aligned}\|x_n - T_1 x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_1^{n+1} x_{n+1}\| \\ &\quad + \|T_1^{n+1} x_{n+1} - T_1^{n+1} x_n\| + \|T_1^{n+1} x_n - T_1 x_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_1^{n+1} x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\quad + \mu_{n+1} \psi(\|x_{n+1} - x_n\|) + \nu_{n+1} + \|T_1^{n+1} x_n - T_1 x_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T_1^{n+1} x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\quad + \mu_{n+1} M \|x_{n+1} - x_n\| + \nu_{n+1} + \|T_1^{n+1} x_n - T_1 x_n\| \\ &= (2 + \mu_{n+1} M)\|x_n - x_{n+1}\| + \|x_{n+1} - T_1^{n+1} x_{n+1}\| \\ &\quad + \|T_1^{n+1} x_n - T_1 x_n\| + \nu_{n+1} \\ &\to 0 \quad \text{as } n \to \infty. \qquad (3.33)\end{aligned}$$

Similarly, we can prove that

$$\|x_n - T_2 x_n\| = 0 \quad \text{and} \quad \|x_n - T_3 x_n\| = 0. \qquad (3.34)$$

This completes the proof. □

Theorem 3.3. *Let E be a real Banach space and C be a nonempty closed convex subset of E. Let $T_1, T_2, T_3 \colon C \to C$ be three total asymptotically nonexpansive mappings with sequences $\{\mu_n\}$ and $\{\nu_n\}$ as defined in proposition 2.8 and $F = \cap_{i=1}^3 F(T_i)$ is closed. Let $\{x_n\}$ be the iteration scheme defined by (1.13), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[\delta, 1-\delta]$ for all $n \in \mathbb{N}$ and for some $\delta \in (0,1)$ and the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$;

(ii) there exists a constant $M > 0$ such that $\psi(t) \leq M\, t$, $t \geq 0$.

Then $\{x_n\}$ converges strongly to a common fixed point of the mappings T_1, T_2 and T_3 if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = \inf\{\|x - p\| : p \in F\}$.

Proof. The necessity is obvious. Indeed, if $x_n \to q \in F$ as $n \to \infty$, then

$$d(x_n, F) = \inf_{q \in F} d(x_n, q) \leq \|x_n - q\| \to 0 \ (n \to \infty).$$

This shows that $\liminf_{n\to\infty} d(x_n, F) = 0$.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. By Lemma 3.1, we have that $\lim_{n\to\infty} d(x_n, F)$ exists. Further, by assumption $\liminf_{n\to\infty} d(x_n, F)$
$= 0$, from (3.4) and Lemma 2.4(ii), we conclude that $\lim_{n\to\infty} d(x_n, F) = 0$. Next, we show that $\{x_n\}$ is a Cauchy sequence.

From (3.3), we know that

$$\begin{aligned} \|x_{n+1} - p\| &\leq (1 + \mu_n Q_1)\|x_n - p\| + \nu_n Q_2 \\ &= (1 + d_n)\|x_n - p\| + Q_2\nu_n, \end{aligned} \tag{3.35}$$

where $d_n = Q_1\mu_n$ and for some $Q_1, Q_2 > 0$. Since $\sum_{n=1}^{\infty} \mu_n < \infty$, it follows that $\sum_{n=1}^{\infty} d_n < \infty$.

Since $1 + x \leq e^x$ for all $x \geq 0$, therefore from (3.35), we have

$$\begin{aligned} \|x_{n+m} - p\| &\leq (1 + d_{n+m-1})\|x_{n+m-1} - p\| + Q_2\nu_{n+m-1} \\ &\leq e^{d_{n+m-1}}\|x_{n+m-1} - p\| + Q_2\nu_{n+m-1} \\ &\leq e^{[d_{n+m-1}+d_{n+m-2}]}\|x_{n+m-2} - p\| + e^{d_{n+m-1}}Q_2\nu_{n+m-2} \\ &\quad +Q_2\nu_{n+m-1} \\ &\leq e^{[d_{n+m-1}+d_{n+m-2}]}\|x_{n+m-2} - p\| + e^{d_{n+m-1}}Q_2[\nu_{n+m-2} \\ &\quad +\nu_{n+m-1}] \\ &\ \ \vdots \\ &\leq \left(e^{\sum_{j=n}^{n+m-1} d_j}\right)\|x_n - p\| + \left(e^{\sum_{j=n}^{n+m-1} d_j}\right)Q_2 \sum_{j=n}^{n+m-1} \nu_j \\ &\leq \left(e^{\sum_{j=1}^{\infty} d_j}\right)\|x_n - p\| + \left(e^{\sum_{j=1}^{\infty} d_j}\right)Q_2 \sum_{j=n}^{n+m-1} \nu_j \\ &\leq Q_3\,\|x_n - p\| + Q_2Q_3 \sum_{j=n}^{n+m-1} \nu_j \end{aligned} \tag{3.36}$$

for all natural numbers m, n, where $Q_3 = e^{\sum_{j=1}^{\infty} d_j} < \infty$.

Now, given $\varepsilon > 0$, since $\lim_{n\to\infty} d(x_n, F) = 0$ and $\sum_{n=1}^{\infty} \nu_n < \infty$, there exists a natural number $n_1 > 0$ such that for all $n \geq n_1$, $d(x_n, F) < \frac{\varepsilon}{8Q_3}$ and $\sum_{j=1}^{\infty} \nu_j < \frac{\varepsilon}{4Q_2Q_3}$. So, we get $d(x_{n_1}, F) < \frac{\varepsilon}{4Q_3}$ and $\sum_{j=n_1}^{\infty} \nu_j < \frac{\varepsilon}{4Q_2Q_3}$. This means that there exists a $p_1 \in F$ such that $\|x_{n_1} - p_1\| \leq \frac{\varepsilon}{4Q_3}$. Hence, for all integers $n \geq n_1$ and $m \geq 1$, we obtain from (3.36) that

$$\begin{aligned} \|x_{n+m} - x_n\| &\leq \|x_{n+m} - p_1\| + \|x_n - p_1\| \\ &\leq Q_3 \|x_{n_1} - p_1\| + Q_2 Q_3 \sum_{j=n_1}^{n+m-1} \nu_j \\ &\quad + Q_3 \|x_{n_1} - p_1\| + Q_2 Q_3 \sum_{j=n_1}^{n+m-1} \nu_j \\ &= 2\Big(Q_3 \|x_{n_1} - p_1\| + Q_2 Q_3 \sum_{j=n_1}^{n+m-1} \nu_j\Big) \\ &\leq 2\Big(Q_3 \|x_{n_1} - p_1\| + Q_2 Q_3 \sum_{j=n_1}^{\infty} \nu_j\Big) \\ &< 2\Big(Q_3 . \frac{\varepsilon}{4Q_3} + Q_2 Q_3 . \frac{\varepsilon}{4Q_2 Q_3}\Big) = \varepsilon. \end{aligned}$$

This proves that $\{x_n\}$ is a Cauchy sequence in C. Thus, the completeness of E implies that $\{x_n\}$ must be convergent. Assume that $\lim_{n\to\infty} x_n = z$. We will prove that z is a common fixed point of T_1, T_2 and T_3, that is, we will show that $z \in F = \cap_{i=1}^3 F(T_i)$. Since C is closed, therefore $z \in C$. Next, we show that $z \in F$. Now $\lim_{n\to\infty} d(x_n, F) = 0$ gives that $d(z, F) = 0$. Since F is closed, $z \in F$. Thus, z is a common fixed point of the mappings T_1, T_2 and T_3. This completes the proof. □

We deduce the following result as corollary from Theorem 3.3 as follows.

Corollary 3.4. *Let E be a real Banach space and C be a nonempty closed convex subset of E. Let $T_1, T_2, T_3 \colon C \to C$ be three total asymptotically nonexpansive mappings with sequences $\{\mu_n\}$ and $\{\nu_n\}$ as defined in proposition 2.8 and $F = \cap_{i=1}^3 F(T_i)$ is closed. Let $\{x_n\}$ be the iteration scheme defined by (1.13), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[\delta, 1-\delta]$ for all $n \in \mathbb{N}$ and for some $\delta \in (0,1)$ and the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$;

(ii) there exists a constant $M > 0$ such that $\psi(t) \leq M\, t$, $t \geq 0$.

Then $\{x_n\}$ converges strongly to a point $p \in F$ if and only if there exists some subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to $p \in F$.

Theorem 3.5. *Let E be a real Banach space and C be a nonempty closed convex subset of E. Let $T_1, T_2, T_3 \colon C \to C$ be three total asymptotically nonexpansive mappings with sequences $\{\mu_n\}$ and $\{\nu_n\}$ as defined in proposition 2.8 and $F = \cap_{i=1}^3 F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the iteration scheme defined by (1.13), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[\delta, 1-\delta]$ for all $n \in \mathbb{N}$ and for some $\delta \in (0,1)$ and the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$;

(ii) there exists a constant $M > 0$ such that $\psi(t) \leq M\, t$, $t \geq 0$.

Then $\liminf_{n\to\infty} d(x_n, F) = \limsup_{n\to\infty} d(x_n, F) = 0$ if $\{x_n\}$ converges to a unique point in F.

Proof. Let $p \in F$. Since $\{x_n\}$ converges to p, $\lim_{n\to\infty} d(x_n, p) = 0$. So, for a given $\varepsilon > 0$, there exists $n_1 \in \mathbb{N}$ such that

$$d(x_n, p) < \varepsilon \text{ for all } n \geq n_1.$$

Taking the infimum over $p \in F(S, T)$, we obtain that

$$d(x_n, F) < \varepsilon \text{ for all } n \geq n_1.$$

This means that $\lim_{n\to\infty} d(x_n, F) = 0$. Thus we obtain that

$$\liminf_{n\to\infty} d(x_n, F) = \limsup_{n\to\infty} d(x_n, F) = 0.$$

This completes the proof. □

As an application of Theorem 3.3, we establish some strong convergence results as follows.

Theorem 3.6. *Let E be a real Banach space and C be a nonempty closed convex subset of E. Let $T_1, T_2, T_3 \colon C \to C$ be three total asymptotically nonexpansive mappings with sequences $\{\mu_n\}$ and $\{\nu_n\}$ as defined in proposition 2.8 and $F = \cap_{i=1}^{3} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the iteration scheme defined by (1.13), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[\delta, 1-\delta]$ for all $n \in \mathbb{N}$ and for some $\delta \in (0,1)$ and the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$;

(ii) there exists a constant $M > 0$ such that $\psi(t) \leq M\,t$, $t \geq 0$.

If one of the mappings in $\{T_i : i = 1,2,3\}$ is demicompact, then $\{x_n\}$ converges strongly to a common fixed point of the mappings T_1, T_2 and T_3.

Proof. Without loss of generality, we can assume that T_1 is demicompact. It follows from (3.33) in Lemma 3.2 that $\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0$ and $\{x_n\}$ is bounded, by demicompactness of T_1, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges strongly to some $q \in C$ as $k \to \infty$. From (3.33) in Lemma 3.2 we have

$$\lim_{k\to\infty} \|x_{n_k} - T_1 x_{n_k}\| = \|q - T_1 q\| = 0.$$

This implies that $q \in F(T_1)$. Similarly, we can prove that $q \in F(T_2)$ and $q \in F(T_3)$. Thus, we obtain that $q \in F = \cap_{i=1}^{3} F(T_i)$. It follows from Lemma 3.1 and Theorem 3.3 that $\{x_n\}$ must converges strongly to a common fixed point of the mappings T_1, T_2 and T_3. This completes the proof. □

Theorem 3.7. *Let E be a real Banach space and C be a nonempty closed convex subset of E. Let $T_1, T_2, T_3 \colon C \to C$ be three total asymptotically nonexpansive mappings with sequences $\{\mu_n\}$ and $\{\nu_n\}$ as defined in proposition 2.8 and $F = \cap_{i=1}^{3} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the iteration scheme defined by (1.13), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[\delta, 1-\delta]$ for all $n \in \mathbb{N}$ and for some $\delta \in (0,1)$ and the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$;

(ii) there exists a constant $M > 0$ such that $\psi(t) \leq M\,t$, $t \geq 0$.

If T_1, T_2 and T_3 satisfy condition (B), then $\{x_n\}$ converges strongly to a common fixed point of the mappings T_1, T_2 and T_3.

Proof. By Lemma 3.2, we know that

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0, \text{ for } i = 1,2,3. \tag{3.37}$$

From *condition* (B) and (3.37), we get

$$f(d(x_n, F) \quad \leq \quad a_1.\|x_n - T_1 x_n\| + a_2.\|x_n - T_2 x_n\| + a_3.\|x_n - T_3 x_n\| = 0,$$

that is, $f(d(x_n, F) = 0$. Since $f \colon [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(t) > 0$ for all $t \in (0,\infty)$, therefore we obtain

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

Now all the conditions of Theorem 3.3 are satisfied, therefore by its conclusion $\{x_n\}$ converges strongly to a common fixed point of the mappings T_1, T_2 and T_3. This completes the proof. □

4. **Weak Convergence Theorems**

In this section, we prove some weak convergence theorems of iteration scheme (1.13) for three total asymptotically nonexpansive mappings in a uniformly convex Banach space such that either it satisfies the Opial property or its dual space has the Kadec-Klee property (KK-property).

Theorem 4.1. *Let E be a uniformly convex Banach space satisfying Opial's condition and C be a nonempty closed convex subset of E. Let $T_1, T_2, T_3 \colon C \to C$ be three uniformly continuous and total asymptotically nonexpansive mappings with sequences $\{\mu_n\}$ and $\{\nu_n\}$ as defined in proposition 2.8 and $F = \cap_{i=1}^{3} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the iteration scheme defined by (1.13), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[\delta, 1-\delta]$ for all $n \in \mathbb{N}$ and for some $\delta \in (0,1)$ and the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$;

(ii) there exists a constant $M > 0$ such that $\psi(t) \leq M\,t$, $t \geq 0$.

If the mappings $I - T_i$ for all $i = 1,2,3$, where I denotes the identity mapping, are demiclosed at zero, then $\{x_n\}$ converges weakly to a common fixed point of the mappings T_1, T_2 and T_3.

Proof. Let $q \in F$, from Lemma 3.1 the sequence $\{\|x_n - q\|\}$ is convergent and hence bounded. Since E is uniformly convex, every bounded subset of E is weakly compact. Thus there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $q^* \in C$. From Lemma 3.2, we have

$$\lim_{k\to\infty} \|x_{n_k} - T_1 x_{n_k}\| = 0,\ \lim_{k\to\infty} \|x_{n_k} - T_2 x_{n_k}\| = 0,\ \lim_{k\to\infty} \|x_{n_k} - T_3 x_{n_k}\| = 0.$$

Since the mappings $I - T_i$ for all $i = 1, 2, 3$ are demiclosed at zero, therefore $T_i q^* = q^*$ for all $i = 1, 2, 3$, which means $q^* \in F$. Finally, let us prove that $\{x_n\}$ converges weakly to q^*. Suppose on contrary that there is a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to $p^* \in C$ and $q^* \neq p^*$. Then by the same method as given above, we can also prove that $p^* \in F$. From Lemma 3.1 the limits $\lim_{n\to\infty} \|x_n - q^*\|$ and $\lim_{n\to\infty} \|x_n - p^*\|$ exist. By virtue of the Opial condition of E, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - q^*\| &= \lim_{n_k\to\infty} \|x_{n_k} - q^*\| \\
&< \lim_{n_k\to\infty} \|x_{n_k} - p^*\| \\
&= \lim_{n\to\infty} \|x_n - p^*\| \\
&= \lim_{n_j\to\infty} \|x_{n_j} - p^*\| \\
&< \lim_{n_j\to\infty} \|x_{n_j} - q^*\| \\
&= \lim_{n\to\infty} \|x_n - q^*\|
\end{aligned}$$

which is a contradiction, so $q^* = p^*$. Thus $\{x_n\}$ converges weakly to a common fixed point of the mappings T_1, T_2 and T_3. This completes the proof. □

Lemma 4.2. *Under the conditions of Lemma 3.2 and for any* $p, q \in F$, $\lim_{n\to\infty} \|t x_n + (1-t)p - q\|$ *exists for all* $t \in [0, 1]$.

Proof. By Lemma 3.1, $\lim_{n\to\infty} \|x_n - z\|$ exists for all $z \in F$ and therefore $\{x_n\}$ is bounded. Letting

$$a_n(t) = \|t x_n + (1-t)p - q\|$$

for all $t \in [0, 1]$. Then $\lim_{n\to\infty} a_n(0) = \|p - q\|$ and $\lim_{n\to\infty} a_n(1) = \|x_n - q\|$ exists by Lemma 3.1. It, therefore, remains to prove the Lemma 4.2 for $t \in (0, 1)$. For all $x \in C$, we define the mapping $W_n \colon C \to C$ by:

$$U_n(x) = (1-\gamma_n)x + \gamma_n T_3^n x$$

$$V_n(x) = (1-\beta_n)U_n(x) + \beta_n T_2^n U_n(x)$$

and

$$W_n(x) = (1-\alpha_n)V_n(x) + \alpha_n T_1^n V_n(x)).$$

Then it follows that $z_n = U_n x_n$, $y_n = V_n x_n$, $x_{n+1} = W_n x_n$ and $W_n p = p$ for all $p \in F$. Now from (3.1), (3.2) and (3.3) of Lemma 3.1, we see that

$$\|U_n(x) - U_n(y)\| \leq (1 + \mu_n M)\|x - y\| + \nu_n$$

$$\|V_n(x) - V_n(y)\| \leq (1 + \mu_n M)^2\|x - y\| + (2 + \mu_n M)\nu_n$$

and

$$\begin{aligned}
\|W_n(x) - W_n(y)\| &\leq (1 + \mu_n Q_1)\|x - y\| + Q_2\nu_n \\
&= K_n\,\|x - y\| + Q_2\nu_n, \qquad (4.1)
\end{aligned}$$

for some $Q_1, Q_2 > 0$ and for all $x, y \in C$, where $K_n = 1 + \mu_n Q_1$ with $\sum_{n=1}^{\infty} \nu_n < \infty$ and $K_n \to 1$ as $n \to \infty$. Setting

$$H_{n,\,m} = W_{n+m-1} W_{n+m-2} \ldots W_n,\ m \geq 1 \qquad (4.2)$$

and

$$b_{n,m} = \|H_{n,m}(tx_n + (1-t)p) - (tH_{n,m}x_n + (1-t)H_{n,m}q)\|.$$

From (4.1) and (4.2), we have

$$\begin{aligned}
\|H_{n,m}(x) - H_{n,m}(y)\| &= \|W_{n+m-1}W_{n+m-2}\dots W_n(x) - W_{n+m-1}W_{n+m-2}\dots W_n(y)\| \\
&\leq K_{n+m-1}\|W_{n+m-2}\dots W_n(x) - W_{n+m-2}\dots W_n(y)\| \\
&\quad +Q_2\nu_{n+m-1} \\
&\leq K_{n+m-1}K_{n+m-2}\|W_{n+m-3}\dots W_n(x) - W_{n+m-3}\dots W_n(y)\| \\
&\quad +Q_2\nu_{n+m-1} + Q_2\nu_{n+m-2} \\
&\;\;\vdots \\
&\leq \Big(\prod_{j=n}^{n+m-1} K_j\Big)\|x-y\| + Q_2\sum_{j=n}^{n+m-1}\nu_j \\
&= M_n\|x-y\| + Q_2\sum_{j=n}^{n+m-1}\nu_j \qquad (4.3)
\end{aligned}$$

for all $x, y \in C$, where $M_n = \prod_{j=n}^{n+m-1} K_j$ and $H_{n,m}x_n = x_{n+m}$, $H_{n,m}p = p$ for all $p \in F$. Thus

$$\begin{aligned}
a_{n+m}(t) &= \|tx_{n+m} + (1-t)p - q\| \\
&\leq b_{n,m} + \|H_{n,m}(tx_n + (1-t)p) - q\| \\
&\leq b_{n,m} + M_n a_n(t) + Q_2\sum_{j=n}^{n+m-1}\nu_j \\
&\leq b_{n,m} + M_n a_n(t) + Q_2\sum_{j=1}^{\infty}\nu_j. \qquad (4.4)
\end{aligned}$$

By using [[5], Theorem 2.3], we have

$$\begin{aligned}
b_{n,m} &\leq \varphi^{-1}(\|x_n - u\| - \|H_{n,m}x_n - H_{n,m}u\|) \\
&\leq \varphi^{-1}(\|x_n - u\| - \|x_{n+m} - u + u - H_{n,m}u\|) \\
&\leq \varphi^{-1}(\|x_n - u\| - (\|x_{n+m} - u\| - \|H_{n,m}u - u\|))
\end{aligned}$$

and so the sequence $\{b_{n,m}\}$ converges uniformly to 0, i.e., $b_{n,m} \to 0$ as $n \to \infty$. Since $\lim_{n\to\infty} M_n = 1$, $Q_2 > 0$ and $\nu_j \to 0$ as $j \to \infty$, therefore from (4.4), we have

$$\limsup_{n\to\infty} a_n(t) \leq \lim_{n,m\to\infty} b_{n,m} + \liminf_{n\to\infty} a_n(t) + 0 = \liminf_{n\to\infty} a_n(t).$$

This shows that $\lim_{n\to\infty} a_n(t)$ exists, that is, $\lim_{n\to\infty}\|tx_n + (1-t)p - q\|$ exists for all $t \in [0,1]$. This completes the proof. □

Theorem 4.3. *Let E be a real uniformly convex Banach space such that its dual E^* has the Kadec-Klee property and C be a nonempty closed convex subset of E. Let $T_1, T_2, T_3\colon C \to C$ be three uniformly continuous and total asymptotically nonexpansive mappings with sequences $\{\mu_n\}$ and $\{\nu_n\}$ as defined in proposition 2.8 and $F = \cap_{i=1}^3 F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the iteration scheme defined by (1.13), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[\delta, 1-\delta]$ for all $n \in \mathbb{N}$ and for some $\delta \in (0,1)$ and the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty}\mu_n < \infty$, $\sum_{n=1}^{\infty}\nu_n < \infty$;

(ii) there exists a constant $M > 0$ such that $\psi(t) \leq M\,t$, $t \geq 0$.

If the mappings $I - T_i$ for all $i = 1, 2, 3$, where I denotes the identity mapping, are demiclosed at zero, then $\{x_n\}$ converges weakly to a common fixed point of the mappings T_1, T_2 and T_3.

Proof. By Lemma 3.1, $\{x_n\}$ is bounded and since E is reflexive, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to some $p \in C$. By Lemma 3.2, we have

$$\lim_{j\to\infty}\|x_{n_j} - T_i x_{n_j}\| = 0 \text{ for all } i = 1, 2, 3.$$

Since by hypothesis the mappings $I - T_i$ for all $i = 1, 2, 3$ are demiclosed at zero, therefore $T_i p = p$ for all $i = 1, 2, 3$, which means $p \in F$. Now, we show that $\{x_n\}$ converges weakly to p. Suppose $\{x_{n_i}\}$ is another subsequence of $\{x_n\}$ converges weakly to some $q \in C$. By the same method as above, we have $q \in F$ and $p, q \in w_w(x_n)$. By Lemma 4.2, the limit

$$\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$$

exists for all $t \in [0, 1]$ and so $p = q$ by Lemma 2.6. Thus, the sequence $\{x_n\}$ converges weakly to $p \in F$. This completes the proof. □

Example 4.4. Let E be the real line with the usual norm $|.|$, $C = [0, \infty)$. Assume that $T_1(x) = x$, $T_2(x) = \frac{x}{3}$ and $T_3(x) = \sin x$ for all $x \in C$. Let ϕ be the strictly increasing continuous function such that $\phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$. Let $\{\mu_n\}_{n\geq 1}$ and $\{\nu_n\}_{n\geq 1}$ be two nonnegative real sequences defined by $\mu_n = \frac{1}{n^2}$ and $\nu_n = \frac{1}{n^3}$ for all $n \geq 1$ with $\mu_n \to 0$ and $\nu_n \to 0$ as $n \to \infty$. Then T_1, T_2 and T_3 are total asymptotically nonexpansive mappings with common fixed point 0, that is, $F = F(T_1) \cap F(T_2) \cap T_3 = \{0\}$.

5. Conclusion

In this paper, we establish some weak and strong convergence theorems for modified SP iteration scheme for three total asymptotically nonexpansive mappings in the framework of real Banach spaces. The results presented in this paper extend and generalize several results from the current existing literature to the case of more general class of mappings, spaces and iteration schemes considered in this paper.

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