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EXISTENCE OF POSITIVE SOLUTIONS FOR A COUPLED SYSTEM OF (p,q)-LAPLACIAN FRACTIONAL HIGHER ORDER BOUNDARY VALUE PROBLEMS

K. R. PRASAD[1], B. M. B. KRUSHNA[2,*] AND L. T. WESEN[1,3]

Abstract. In this paper, we establish the existence of at least three positive solutions for a system of (p,q)-Laplacian fractional order two-point boundary value problems by applying five functionals fixed point theorem under suitable conditions on a cone in a Banach space.

1. Introduction

In the universe, many real world problems can be formulated as mathematical models to analyze the situations and to predict future. Most of these models involve the rate of change of the dependent variable which leads to formation of the differential equations. One goal of differential equations is to understand the phenomena of nature by developing mathematical models. Fractional calculus is an extension of classical calculus and deals with the generalization of integration and differentiation to an arbitrary real order.

A class of differential equations governed by nonlinear differential operators appears frequently and generated by great deal of interest in studying special types of problems. In this theory, the most applicable operator is the classical p-Laplacian operator. These types of problems arise in mathematical modeling of viscoelastic flows, turbulent filtration in porous media, biophysics, plasma physics and chemical reaction design. For a detailed description on applications of p-Laplacian operator, we refer [10].

The positivity of boundary value problems associated with ordinary differential equations were studied by many authors [14, 1, 2] and extended to p-Laplacian boundary value problems [4, 22, 8]. Later these results are further extended to fractional order boundary value problems [6, 5, 9, 21, 12, 19] by utilizing various fixed point theorems on cones.

Recently researchers are concentrating on the theory of fractional order boundary value problems associated with p-Laplacian operator. Yang and Yan [22] studied the existence of positive solutions for third order Sturm–Liouville boundary value problems with p-Laplacian operator by applying the fixed point index method. Chai [7] obtained the existence and multiplicity of positive solutions for a class of p-Laplacian fractional order boundary value problems by means of the fixed point theorem. Prasad and Krushna [18, 20] derived sufficient conditions for the existence of positive solutions to p-Laplacian fractional order boundary value problems.

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Motivated by the papers mentioned above, in this paper, we are concerned with establishing the existence of positive solutions for a coupled system of (p,q)-Laplacian fractional order differential equations

$$D_{0^+}^{\beta_1}\Big(\phi_p\Big(D_{0^+}^{\alpha_1}x(t)\Big)\Big)=f_1\big(t,x(t),y(t)\big),\ t\in(0,1), \tag{1.1}$$

$$D_{0^+}^{\beta_2}\Big(\phi_q\Big(D_{0^+}^{\alpha_2}y(t)\Big)\Big)=f_2\big(t,x(t),y(t)\big),\ t\in(0,1), \tag{1.2}$$

satisfying the boundary conditions

$$\left.\begin{aligned} x^{(j)}(0)=0,\ j=0,1,\cdots,n-2,\ x^{(n-2)}(1)=0,\\ \phi_p\Big(D_{0^+}^{\alpha_1}x(0)\Big)=0,\ \phi_p\Big(D_{0^+}^{\alpha_1}x(1)\Big)=0,\end{aligned}\right\} \tag{1.3}$$

$$\left.\begin{aligned} y^{(j)}(0)=0,\ j=0,1,\cdots,n-2,\ y^{(n-2)}(1)=0,\\ \phi_q\Big(D_{0^+}^{\alpha_2}y(0)\Big)=0,\ \phi_q\Big(D_{0^+}^{\alpha_2}y(1)\Big)=0,\end{aligned}\right\} \tag{1.4}$$

where $\alpha_i\in(n-1,n], n\geq 3$, $\beta_i\in(1,2]$, $\phi_p(s)=|s|^{p-2}s$, $\phi_q(s)=|s|^{q-2}s$, $p,q>1$, $\phi_p^{-1}=\phi_q$, $\phi_q^{-1}=\phi_p$, $\dfrac{1}{p}+\dfrac{1}{q}=1$, $f_i:[0,1]\times\mathbb{R}^2\to\mathbb{R}^+$ are continuous and $D_{0^+}^{\alpha_i}, D_{0^+}^{\beta_i}$, for $i=1,2$ are the standard Riemann–Liouville fractional order derivatives.

By a positive solution of the coupled system of fractional order boundary value problem (1.1)-(1.4), we mean $\big(x(t),y(t)\big)\in\Big(C^{\alpha_1+\beta_1}[0,1]\times C^{\alpha_2+\beta_2}[0,1]\Big)$ satisfying the boundary value problem (1.1)-(1.4) with $x(t)\geq 0$, $y(t)\geq 0$, for all $t\in[0,1]$ and $(x,y)\neq(0,0)$.

The rest of the paper is organized as follows. In Section 2, the solution of the boundary value problems (1.1), (1.3) and (1.2), (1.4) are expressed in terms of Green functions and the bounds for these Green functions are estimated. In Section 3, the existence of at least three positive solutions for a coupled system of (p,q)-Laplacian fractional order boundary value problem (1.1)-(1.4) are established, by using five functionals fixed point theorem. In Section 4, as an application, the results are demonstrated with an example.

2. Green Functions and Bounds

In this section, the solution of the boundary value problems (1.1), (1.3) and (1.2), (1.4) are expressed in terms of the equivalent integral equations involving Green functions and the bounds for the Green functions are estimated, which are essential to establish the main results.

Lemma 2.1. *Let $h_1(t)\in C[0,1]$. Then the fractional order differential equation,*

$$D_{0^+}^{\alpha_1}x(t)+h_1(t)=0,\ t\in(0,1), \tag{2.1}$$

satisfying

$$x^{(j)}(0)=0,\ j=0,1,\cdots,n-2,\ x^{(n-2)}(1)=0, \tag{2.2}$$

has a unique solution,

$$x(t)=\int_0^1 G_1(t,s)h_1(s)ds,$$

where $G_1(t,s)$ is the Green's function for the problem (2.1)-(2.2) and is given by

$$G_1(t,s)=\begin{cases} G_{11}(t,s),\ 0\leq t\leq s\leq 1,\\ G_{12}(t,s),\ 0\leq s\leq t\leq 1,\end{cases} \tag{2.3}$$

$$G_{11}(t,s) = \frac{t^{\alpha_1-1}(1-s)^{\alpha_1-n+1}}{\Gamma(\alpha_1)},$$
$$G_{12}(t,s) = \frac{t^{\alpha_1-1}(1-s)^{\alpha_1-n+1}-(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}.$$

For details refer to [19].

Lemma 2.2. *Let $h_2(t) \in C[0,1]$. Then the fractional order differential equation,*

$$D_{0^+}^{\beta_1}\Big(\phi_p\Big(D_{0^+}^{\alpha_1}x(t)\Big)\Big) = h_2(t),\ t \in (0,1), \tag{2.4}$$

satisfying

$$\phi_p\Big(D_{0^+}^{\alpha_1}x(0)\Big) = 0,\ \phi_p\Big(D_{0^+}^{\alpha_1}x(1)\Big) = 0, \tag{2.5}$$

has a unique solution,

$$x(t) = \int_0^1 G_1(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau)h_2(\tau)d\tau\Big)ds, \tag{2.6}$$

where

$$H_1(t,s) = \begin{cases} \dfrac{\big[t(1-s)\big]^{\beta_1-1}}{\Gamma(\beta_1)}, & 0 \le t \le s \le 1, \\ \dfrac{\big[t(1-s)\big]^{\beta_1-1}-(t-s)^{\beta_1-1}}{\Gamma(\beta_1)}, & 0 \le s \le t \le 1. \end{cases} \tag{2.7}$$

Proof. An equivalent integral equation for (2.4) is given by

$$\phi_p\Big(D_{0^+}^{\alpha_1}x(t)\Big) = \frac{1}{\Gamma(\beta_1)}\int_0^t (t-\tau)^{\beta_1-1}h_2(\tau)d\tau + k_1t^{\beta_1-1} + k_2t^{\beta_1-2}.$$

From (2.5), one gets that $k_2 = 0$ and $k_1 = \dfrac{-1}{\Gamma(\beta_1)}\displaystyle\int_0^1 (1-\tau)^{\beta_1-1}h_2(\tau)d\tau$. Then,

$$\begin{aligned}\phi_p\Big(D_{0^+}^{\alpha_1}x(t)\Big) &= \frac{1}{\Gamma(\beta_1)}\int_0^t (t-\tau)^{\beta_1-1}h_2(\tau)d\tau - \frac{t^{\beta_1-1}}{\Gamma(\beta_1)}\int_0^1 (1-\tau)^{\beta_1-1}h_2(\tau)d\tau \\ &= -\int_0^1 H_1(t,\tau)h_2(\tau)d\tau.\end{aligned}$$

Therefore, $D_{0^+}^{\alpha_1}x(t) + \phi_q\Big(\displaystyle\int_0^1 H_1(t,\tau)h_2(\tau)d\tau\Big) = 0$. Hence $x(t)$ in (2.6) is the solution to the fractional order boundary value problem (2.4), (1.3). □

Lemma 2.3. *The Green's function $G_1(t,s)$ given in (2.3) is nonnegative, for all $(t,s) \in [0,1]\times[0,1]$.*

Proof. Consider the Green's function $G_1(t,s)$ given by (2.3). Let $0 \le t \le s \le 1$. Then, we have

$$G_{11}(t,s) = \frac{1}{\Gamma(\alpha_1)}\big[t^{\alpha_1-1}(1-s)^{\alpha_1-n+1}\big] \ge 0.$$

Let $0 \le s \le t \le 1$. Then, we have

$$\begin{aligned}
G_{12}(t,s) =& \frac{1}{\Gamma(\alpha_1)}\left[t^{\alpha_1-1}(1-s)^{\alpha_1-n+1} - (t-s)^{\alpha_1-1}\right] \\
\ge& \frac{1}{\Gamma(\alpha_1)}\left[t^{\alpha_1-1}(1-s)^{\alpha_1-n+1} - (t-ts)^{\alpha_1-1}\right] \\
=& \frac{t^{\alpha_1-1}}{\Gamma(\alpha_1)}\left[\left(1+(n-2)s+\frac{1}{2}(n^2-3n+2)s^2+\cdots\right)-1\right](1-s)^{\alpha_1-1} \\
\ge& 0.
\end{aligned}$$

□

Lemma 2.4. *For $t \in I = \left[\frac{1}{4}, \frac{3}{4}\right]$, the Green's function $G_1(t,s)$ given in* (2.3) *satisfies the following inequalities*

$$\begin{aligned}
&(P1)\ G_1(t,s) \le G_1(1,s),\ \ for\ all\ (t,s) \in [0,1]\times[0,1], \\
&(P2)\ G_1(t,s) \ge \left(\frac{1}{4}\right)^{\alpha_1-1} G_1(1,s),\ \ for\ all\ (t,s) \in I\times[0,1].
\end{aligned}$$

Proof. Consider the Green's function $G_1(t,s)$ given by (2.3). Let $0 \le t \le s \le 1$. Then, we have

$$\frac{\partial G_{11}(t,s)}{\partial t} = \frac{1}{\Gamma(\alpha_1)}\left[(\alpha_1-1)t^{\alpha_1-2}(1-s)^{\alpha_1-n+1}\right] \ge 0.$$

Therefore, $G_{11}(t,s)$ is increasing in t, which implies $G_{11}(t,s) \le G_{11}(1,s)$. Let $0 \le s \le t \le 1$. Then, we have

$$\begin{aligned}
&\frac{\partial G_{12}(t,s)}{\partial t} \\
&= \frac{1}{\Gamma(\alpha_1)}\left[(\alpha_1-1)t^{\alpha_1-2}(1-s)^{\alpha_1-n+1} - (\alpha_1-1)(t-s)^{\alpha_1-2}\right] \\
&\ge \frac{1}{\Gamma(\alpha_1)}\left[(\alpha_1-1)t^{\alpha_1-2}(1-s)^{\alpha_1-n+1} - (\alpha_1-1)(t-ts)^{\alpha_1-2}\right] \\
&\ge \frac{t^{\alpha_1-2}}{\Gamma(\alpha_1-1)}\left[1-\left(1-(n-3)s+\frac{1}{2}(n^2-7n+12)s^2+\cdots\right)\right](1-s)^{\alpha_1-n+1} \\
&\ge 0.
\end{aligned}$$

Therefore, $G_{12}(t,s)$ is increasing in t, which implies $G_{12}(t,s) \le G_{12}(1,s)$. Let $0 \le t \le s \le 1$ and $t \in I$. Then

$$\begin{aligned}
G_{11}(t,s) =& \frac{1}{\Gamma(\alpha_1)}\left[t^{\alpha_1-1}(1-s)^{\alpha_1-n+1}\right] \\
\ge& t^{\alpha_1-1}\frac{1}{\Gamma(\alpha_1)}\left[(1-s)^{\alpha_1-n+1} - (1-s)^{\alpha_1-1}\right] \\
=& t^{\alpha_1-1}G_{11}(1,s) \ge \left(\frac{1}{4}\right)^{\alpha_1-1} G_{11}(1,s).
\end{aligned}$$

Let $0 \le s \le t \le 1$ and $t \in I$. Then

$$\begin{aligned}
G_{12}(t,s) =& \frac{1}{\Gamma(\alpha_1)}\Big[t^{\alpha_1-1}(1-s)^{\alpha_1-n+1} - (t-s)^{\alpha_1-1}\Big]\\
\ge& \frac{1}{\Gamma(\alpha_1)}\Big[t^{\alpha_1-1}(1-s)^{\alpha_1-n+1} - (t-ts)^{\alpha_1-1}\Big]\\
=& t^{\alpha_1-1}G_{12}(1,s) \ge \Big(\frac{1}{4}\Big)^{\alpha_1-1}G_{12}(1,s).
\end{aligned}$$

□

Lemma 2.5. *For $t, s \in [0,1]$, the Green's function $H_1(t,s)$ given in* (2.7) *satisfies the following inequalities*

$$(Q1)\ H_1(t,s) \ge 0,$$
$$(Q2)\ H_1(t,s) \le H_1(s,s).$$

For details refer to [20].

Lemma 2.6. *Let $\xi_1 \in (\frac{1}{4}, \frac{3}{4})$. Then the Green's function $H_1(t,s)$ holds the inequality,*

$$\min_{t\in I} H_1(t,s) \ge \vartheta_1^*(s)H_1(s,s),\ for\ 0 < s < 1, \tag{2.8}$$

where

$$\vartheta_1^*(s) = \begin{cases} \dfrac{[\frac{3}{4}(1-s)]^{\beta_1-1} - (\frac{3}{4}-s)^{\beta_1-1}}{[s(1-s)]^{\beta_1-1}}, & s \in (0,\xi_1],\\[2ex] \dfrac{1}{(4s)^{\beta_1-1}}, & s \in [\xi_1, 1). \end{cases}$$

For details refer to [20].

Lemma 2.7. *Let $g_1(t) \in C[0,1]$, then the fractional order differential equation,*

$$D_{0^+}^{\alpha_2} y(t) + g_1(t) = 0,\ t \in (0,1), \tag{2.9}$$

satisfying

$$y^{(j)}(0) = 0,\ j = 0, 1, \cdots, n-2,\ y^{(n-2)}(1) = 0, \tag{2.10}$$

has a unique solution,

$$y(t) = \int_0^1 G_2(t,s)g_1(s)ds,$$

where $G_2(t,s)$ is the Green's function for the problem (2.9)-(2.10) *and is given by*

$$G_2(t,s) = \begin{cases} G_{21}(t,s),\ 0 \le t \le s \le 1,\\ G_{22}(t,s),\ 0 \le s \le t \le 1, \end{cases} \tag{2.11}$$

$$\begin{aligned}
G_{21}(t,s) =& \frac{t^{\alpha_2-1}(1-s)^{\alpha_2-n+1}}{\Gamma(\alpha_2)},\\
G_{22}(t,s) =& \frac{t^{\alpha_2-1}(1-s)^{\alpha_2-n+1} - (t-s)^{\alpha_2-1}}{\Gamma(\alpha_2)}.
\end{aligned}$$

For details refer to [19].

Lemma 2.8. *Let $g_2(t) \in C[0,1]$. Then the fractional order differential equation,*

$$D_{0^+}^{\beta_2}\Big(\phi_q\Big(D_{0^+}^{\alpha_2}y(t)\Big)\Big) = g_2(t), \ t \in (0,1), \tag{2.12}$$

satisfying

$$\phi_q\Big(D_{0^+}^{\alpha_2}y(0)\Big) = 0, \ \phi_q\Big(D_{0^+}^{\alpha_2}y(1)\Big) = 0, \tag{2.13}$$

has a unique solution,

$$y(t) = \int_0^1 G_2(t,s)\phi_p\Big(\int_0^1 H_2(s,\tau)g_2(\tau)d\tau\Big)ds,$$

where

$$H_2(t,s) = \begin{cases} \dfrac{\big[t(1-s)\big]^{\beta_2-1}}{\Gamma(\beta_2)}, & 0 \le t \le s \le 1, \\ \dfrac{\big[t(1-s)\big]^{\beta_2-1} - (t-s)^{\beta_2-1}}{\Gamma(\beta_2)}, & 0 \le s \le t \le 1. \end{cases} \tag{2.14}$$

Proof. Proof is similar to Lemma 2.2. □

Lemma 2.9. *The Green's function $G_2(t,s)$ given in* (2.11) *is nonnegative, for all $(t,s) \in [0,1]\times[0,1]$.*

Proof. Proof is similar to Lemma 2.3. □

Lemma 2.10. *For $I = \left[\frac{1}{4}, \frac{3}{4}\right]$, the Green's function $G_2(t,s)$ given in* (2.11) *satisfies the following inequalities*

$$(C1)\ G_2(t,s) \le G_2(1,s), \ for\ all\ (t,s) \in [0,1]\times[0,1],$$
$$(C2)\ G_2(t,s) \ge \Big(\frac{1}{4}\Big)^{\alpha_2-1} G_2(1,s), \ for\ all\ (t,s) \in I\times[0,1].$$

Proof. Proof is similar to Lemma 2.4. □

Lemma 2.11. *For $t,s \in [0,1]$, the Green's function $H_2(t,s)$ given in* (2.14) *satisfies the following inequalities*

$$(D1)\ H_2(t,s) \ge 0,$$
$$(D2)\ H_2(t,s) \le H_2(s,s).$$

For details refer to [20].

Lemma 2.12. *Let $\xi_2 \in (\frac{1}{4}, \frac{3}{4})$. Then the Green's function $H_2(t,s)$ holds the inequality,*

$$\min_{t\in I} H_2(t,s) \ge \vartheta_2^*(s)H_2(s,s), \ for\ 0 < s < 1, \tag{2.15}$$

where

$$\vartheta_2^*(s) = \begin{cases} \dfrac{[\frac{3}{4}(1-s)]^{\beta_2-1} - (\frac{3}{4}-s)^{\beta_2-1}}{[s(1-s)]^{\beta_2-1}}, & s \in (0,\xi_2], \\ \dfrac{1}{(4s)^{\beta_2-1}}, & s \in [\xi_2, 1). \end{cases} \tag{2.16}$$

For details refer to [20].

3. Existence of Positive Solutions

In this section, we establish sufficient conditions for the existence of at least three positive solutions for a system of (p,q)-Laplacian fractional order boundary value problem (1.1)-(1.4), by using five functionals fixed point theorem.

Let γ, β, θ be nonnegative continuous convex functionals on P and α, ψ be nonnegative continuous concave functionals on P, then for nonnegative numbers h', a', b', d' and c', convex sets are defined.

$$\begin{aligned}
P(\gamma, c') &= \{y \in P : \gamma(y) < c'\},\\
P(\gamma, \alpha, a', c') &= \{y \in P : a' \le \alpha(y); \gamma(y) \le c'\},\\
Q(\gamma, \beta, d', c') &= \{y \in P : \beta(y) \le d'; \gamma(y) \le c'\},\\
P(\gamma, \theta, \alpha, a', b', c') &= \{y \in P : a' \le \alpha(y); \theta(y) \le b'; \gamma(y) \le c'\},\\
Q(\gamma, \beta, \psi, h', d', c') &= \{y \in P : h' \le \psi(y); \beta(y) \le d'; \gamma(y) \le c'\}.
\end{aligned}$$

In establishing the positive solutions for a coupled system of (p,q)-Laplacian fractional order boundary value problem (1.1)-(1.4), the following so called Five Functionals Fixed Point Theorem is fundamental.

Theorem 3.1. [3] *Let P be a cone in the real Banach space B. Suppose α and ψ are nonnegative continuous concave functionals on P and γ, β, θ are nonnegative continuous convex functionals on P, such that for some positive numbers c' and e', $\alpha(y) \le \beta(y)$ and $\| y \| \le e'\gamma(y)$, for all $y \in \overline{P(\gamma, c')}$. Suppose further that $T : \overline{P(\gamma, c')} \to \overline{P(\gamma, c')}$ is completely continuous and there exist constants h', d', a' and $b' \ge 0$ with $0 < d' < a'$ such that each of the following is satisfied.*

(B1) $\{y \in P(\gamma, \theta, \alpha, a', b', c') : \alpha(y) > a'\} \ne \emptyset$ *and*
$\alpha(Ty) > a'$ *for* $y \in P(\gamma, \theta, \alpha, a', b', c')$,

(B2) $\{y \in Q(\gamma, \beta, \psi, h', d', c') : \beta(y) > d'\} \ne \emptyset$ *and*
$\beta(Ty) > d'$ *for* $y \in Q(\gamma, \beta, \psi, h', d', c')$,

(B3) $\alpha(Ty) > a'$ *provided* $y \in P(\gamma, \alpha, a', c')$ *with* $\theta(Ty) > b'$,

(B4) $\beta(Ty) < d'$ *provided* $y \in Q(\gamma, \beta, \psi, h', d', c')$ *with* $\psi(Ty) < h'$.

Then T has at least three fixed points $y_1, y_2, y_3 \in \overline{P(\gamma, c')}$ such that $\beta(y_1) < d', a' < \alpha(y_2)$ and $d' < \beta(y_3)$ with $\alpha(y_3) < a'$.

Consider the Banach space $B = E \times E$, where $E = \{x : x \in C[0,1]\}$ equipped with the norm $\|(x,y)\| = \|x\|_0 + \|y\|_0$, for $(x,y) \in B$ and the norm is defined as

$$\|x\|_0 = \max_{t \in [0,1]} |x(t)|.$$

Define a cone $P \subset B$ by

$$P = \Big\{(x,y) \in B : x(t) \ge 0,\ y(t) \ge 0,\ t \in [0,1] \text{ and } \min_{t \in I} \big[x(t) + y(t)\big] \ge \eta \|(x,y)\|\Big\},$$

where

$$\eta = \min\left\{\left(\frac{1}{4}\right)^{\alpha_1 - 1}, \left(\frac{1}{4}\right)^{\alpha_2 - 1}\right\}. \tag{3.1}$$

Define the nonnegative continuous concave functionals α, ψ and the nonnegative continuous convex functionals β, θ, γ on P by

$$\alpha(x,y) = \min_{t\in I}\big\{|x|+|y|\big\}, \psi(x,y) = \min_{t\in I_1}\big\{|x|+|y|\big\},$$
$$\gamma(x,y) = \max_{t\in[0,1]}\big\{|x|+|y|\big\}, \beta(x,y) = \max_{t\in I_1}\big\{|x|+|y|\big\}, \theta(x,y) = \max_{t\in I}\big\{|x|+|y|\big\},$$

where $I_1 = \left[\frac{1}{3}, \frac{2}{3}\right]$. For any $(x,y)\in P$,

$$\alpha(x,y) = \min_{t\in I}\big\{|x|+|y|\big\} \le \max_{t\in I_1}\big\{|x|+|y|\big\} = \beta(x,y), \tag{3.2}$$

$$\|(x,y)\| \le \frac{1}{\eta}\min_{t\in I}\big\{|x|+|y|\big\} \le \frac{1}{\eta}\max_{t\in[0,1]}\big\{|x|+|y|\big\} = \frac{1}{\eta}\gamma(x,y). \tag{3.3}$$

Let

$$\vartheta^*(s) = \min\Big\{\vartheta_1^*(s), \vartheta_2^*(s)\Big\}. \tag{3.4}$$

Define

$$\mathcal{L} = \min\left\{\frac{1}{\int_0^1 G_1(1,s)\phi_q\Big(\int_0^1 H_1(\tau,\tau)d\tau\Big)ds}, \frac{1}{\int_0^1 G_2(1,s)\phi_p\Big(\int_0^1 H_2(\tau,\tau)d\tau\Big)ds}\right\}, \text{ and}$$

$$\mathcal{M} = \max\left\{\frac{1}{\int_{s\in I}\eta G_1(1,s)\phi_q\Big(\int_{\tau\in I}\vartheta^*(\tau)H_1(\tau,\tau)d\tau\Big)ds}, \frac{1}{\int_{s\in I}\eta G_2(1,s)\phi_p\Big(\int_{\tau\in I}\vartheta^*(\tau)H_2(\tau,\tau)d\tau\Big)ds}\right\}.$$

Theorem 3.2. *Suppose there exist $0 < a' < b' < \dfrac{b'}{\eta} < c'$ such that f_1, f_2 satisfies the following conditions:*

$$(A1)\begin{cases} f_1\big(t,x(t),y(t)\big) < \phi_p\Big(\dfrac{a'\mathcal{L}}{2}\Big) \text{ and } f_2\big(t,x(t),y(t)\big) < \phi_q\Big(\dfrac{a'\mathcal{L}}{2}\Big), \\ t\in[0,1] \text{ and } x,y\in\big[\eta a', a'\big], \end{cases}$$

$$(A2)\begin{cases} f_1\big(t,x(t),y(t)\big) > \phi_p\Big(\dfrac{b'\mathcal{M}}{2}\Big) \text{ and } f_2\big(t,x(t),y(t)\big) > \phi_q\Big(\dfrac{b'\mathcal{M}}{2}\Big), \\ t\in I \text{ and } x,y\in\Big[b', \dfrac{b'}{\eta}\Big], \end{cases}$$

$$(A3)\begin{cases} f_1\big(t,x(t),y(t)\big) < \phi_p\Big(\dfrac{c'\mathcal{L}}{2}\Big) \text{ and } f_2\big(t,x(t),y(t)\big) < \phi_q\Big(\dfrac{c'\mathcal{L}}{2}\Big), \\ t\in[0,1] \text{ and } x,y\in\Big[0, c'\Big]. \end{cases}$$

Then the (p,q)-Laplacian fractional order boundary value problem (1.1)-(1.4) *has at least three positive solutions, (x_1,x_2), (y_1,y_2) and (z_1,z_2) such that $\beta(x_1,x_2) < a'$, $b' < \alpha(y_1,y_2)$ and $a' < \beta(z_1,z_2)$ with $\alpha(z_1,z_2) < b'$.*

Proof. Let $T_1, T_2 : P \to E$ and $T : P \to B$ be the operators defined by

$$\begin{cases} T_1(x,y)(t) = \displaystyle\int_0^1 G_1(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f_1\big(\tau, x(\tau), y(\tau)\big)d\tau\Big)ds, \\ T_2(x,y)(t) = \displaystyle\int_0^1 G_2(t,s)\phi_p\Big(\int_0^1 H_2(s,\tau)f_2\big(\tau, x(\tau), y(\tau)\big)d\tau\Big)ds, \end{cases}$$

and

$$T(x,y)(t) = \Big(T_1(x,y)(t), T_2(x,y)(t)\Big), \text{ for } (x,y) \in B.$$

It is obvious that a fixed point of T is the solution of the fractional order boundary value problem (1.1)-(1.4). Three fixed points of T are sought. First, it is shown that $T : P \to P$. Let $(x,y) \in P$. Clearly, $T_1\big(x,y\big)(t) \geq 0$ and $T_2\big(x,y\big)(t) \geq 0$, for $t \in [0,1]$. Also for $(x,y) \in P$,

$$\begin{cases} \|T_1(x,y)\|_0 \leq \displaystyle\int_0^1 G_1(1,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f_1\big(\tau, x(\tau), y(\tau)\big)d\tau\Big)ds, \\ \|T_2(x,y)\|_0 \leq \displaystyle\int_0^1 G_2(1,s)\phi_p\Big(\int_0^1 H_2(s,\tau)f_2\big(\tau, x(\tau), y(\tau)\big)d\tau\Big)ds, \end{cases}$$

and

$$\begin{aligned} \min_{t\in I} T_1(x,y)(t) &= \min_{t\in I} \int_0^1 G_1(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f_1\big(\tau, x(\tau), y(\tau)\big)d\tau\Big)ds \\ &\geq \eta \int_0^1 G_1(1,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f_1\big(\tau, x(\tau), y(\tau)\big)d\tau\Big)ds \\ &\geq \eta\|T_1(x,y)\|_0. \end{aligned}$$

Similarly, $\min\limits_{t\in I} T_2\big(x,y\big)(t) \geq \eta\|T_2\big(x,y\big)\|_0$. Therefore,

$$\begin{aligned} \min_{t\in I} \Big\{T_1(x,y)(t) + T_2(x,y)(t)\Big\} &\geq \eta\|T_1(x,y)\|_0 + \eta\|T_2(x,y)\|_0 \\ &= \eta\Big(\|T_1(x,y)\|_0 + \|T_2(x,y)\|_0\Big) \\ &= \eta\Big\|\Big(T_1(x,y), T_2(x,y)\Big)\Big\| \\ &= \eta\|T(x,y)\|. \end{aligned}$$

Hence $T(x,y) \in P$ and so $T : P \to P$. Moreover the operator T is completely continuous. From (3.2) and (3.3), for each $(x,y) \in P$, $\alpha(x,y) \leq \beta(x,y)$ and $\|(x,y)\| \leq \dfrac{1}{\eta}\gamma(x,y)$. It is shown that $T : \overline{P(\gamma, c')} \to \overline{P(\gamma, c')}$. Let $(x,y) \in \overline{P(\gamma, c')}$. Then $0 \leq |x| + |y| \leq c'$. The condition $(A3)$ is used to obtain

$$\begin{aligned} \gamma\big(T(x,y)(t)\big) &= \max_{t\in[0,1]}\Big[\int_0^1 G_1(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f_1\big(\tau, x(\tau), y(\tau)\big)d\tau\Big)ds + \\ &\quad \int_0^1 G_2(t,s)\phi_p\Big(\int_0^1 H_2(s,\tau)f_2\big(\tau, x(\tau), y(\tau)\big)d\tau\Big)ds\Big] \\ &\leq \int_0^1 G_1(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau)\phi_p\Big(\frac{c'\mathcal{L}}{2}\Big)d\tau\Big)ds + \\ &\quad \int_0^1 G_2(t,s)\phi_p\Big(\int_0^1 H_2(s,\tau)\phi_q\Big(\frac{c'\mathcal{L}}{2}\Big)d\tau\Big)ds \end{aligned}$$

$$
\begin{aligned}
&< \frac{c'\mathcal{L}}{2}\int_0^1 G_1(1,s)\phi_q\Big(\int_0^1 H_1(\tau,\tau)d\tau\Big)ds+\\
&\frac{c'\mathcal{L}}{2}\int_0^1 G_2(1,s)\phi_p\Big(\int_0^1 H_2(\tau,\tau)d\tau\Big)ds\\
&< \frac{c'}{2}+\frac{c'}{2}=c'.
\end{aligned}
$$

Therefore $T:\overline{P(\gamma,c')}\to\overline{P(\gamma,c')}$. Now the conditions $(B1)$ and $(B2)$ of Theorem 3.1 are to be verified. It is obvious that

$$
\begin{aligned}
\frac{b'(\eta+1)}{2\eta} &\in \Big\{(x,y)\in P\Big(\gamma,\theta,\alpha,b',\frac{b'}{\eta},c'\Big):\alpha(x,y)>b'\Big\}\neq\emptyset \text{ and}\\
\frac{\eta a'+a'}{2} &\in \Big\{(x,y)\in Q\Big(\gamma,\beta,\psi,\eta a',a',c'\Big):\beta(x,y)<a'\Big\}\neq\emptyset.
\end{aligned}
$$

Next, let $(x,y)\in P\Big(\gamma,\theta,\alpha,b',\dfrac{b'}{\eta},c'\Big)$ or $(x,y)\in Q\big(\gamma,\beta,\psi,\eta a',a',c'\big)$. Then, $b'\leq\Big\{|x(t)|+|y(t)|\Big\}\leq\dfrac{b'}{\eta}$ and $\eta a'\leq\Big\{|x(t)|+|y(t)|\Big\}\leq a'$. Now the condition $(A2)$ is applied to get

$$
\begin{aligned}
\alpha\big(T(x,y)(t)\big) &= \min_{t\in I}\Big[\int_0^1 G_1(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f_1\big(\tau,x(\tau),y(\tau)\big)d\tau\Big)ds+\\
&\int_0^1 G_2(t,s)\phi_p\Big(\int_0^1 H_2(s,\tau)f_2\big(\tau,x(\tau),y(\tau)\big)d\tau\Big)ds\Big]\\
&\geq \eta\Big[\int_0^1 G_1(1,s)\phi_q\Big(\int_0^1 \vartheta^*(\tau)H_1(\tau,\tau)\phi_p\Big(\frac{b'\mathcal{M}}{2}\Big)d\tau\Big)ds+\\
&\int_0^1 G_2(1,s)\phi_p\Big(\int_0^1 \vartheta^*(\tau)H_2(\tau,\tau)\phi_q\Big(\frac{b'\mathcal{M}}{2}\Big)d\tau\Big)ds\Big]\\
&> \frac{b'\mathcal{M}}{2}\int_{s\in I}\eta G_1(1,s)\phi_q\Big(\int_{\tau\in I}\vartheta^*(\tau)H_1(\tau,\tau)d\tau\Big)ds+\\
&\frac{b'\mathcal{M}}{2}\int_{s\in I}\eta G_2(1,s)\phi_p\Big(\int_{\tau\in I}\vartheta^*(\tau)H_2(\tau,\tau)d\tau\Big)ds\\
&\geq \frac{b'}{2}+\frac{b'}{2}=b'.
\end{aligned}
$$

Clearly the condition $(A1)$ leads to

$$
\begin{aligned}
\beta\big(T(x,y)(t)\big) &= \max_{t\in I_1}\Big[\int_0^1 G_1(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f_1\big(\tau,x(\tau),y(\tau)\big)d\tau\Big)ds+\\
&\int_0^1 G_2(t,s)\phi_p\Big(\int_0^1 H_2(s,\tau)f_2\big(\tau,x(\tau),y(\tau)\big)d\tau\Big)ds\Big]\\
&\leq \int_0^1 G_1(1,s)\phi_q\Big(\int_0^1 H_1(s,\tau)\phi_p\Big(\frac{a'\mathcal{L}}{2}\Big)d\tau\Big)ds+\\
&\int_0^1 G_2(1,s)\phi_p\Big(\int_0^1 H_2(s,\tau)\phi_q\Big(\frac{a'\mathcal{L}}{2}\Big)d\tau\Big)ds
\end{aligned}
$$

$$\begin{aligned}
&< \frac{a'\mathcal{L}}{2}\int_0^1 G_1(1,s)\phi_q\Big(\int_0^1 H_1(\tau,\tau)d\tau\Big)ds+ \\
&\frac{a'\mathcal{L}}{2}\int_0^1 G_2(1,s)\phi_p\Big(\int_0^1 H_2(\tau,\tau)d\tau\Big)ds \\
&\leq \frac{a'}{2}+\frac{a'}{2}=a'.
\end{aligned}$$

To see that $(B3)$ is satisfied, let $(x,y)\in P\big(\gamma,\alpha,b',c'\big)$ with $\theta\big(T(x,y)(t)\big) > \dfrac{b'}{\eta}$. Then

$$\begin{aligned}
\alpha\big(T(x,y)(t)\big) &= \min_{t\in I}\Big[\int_0^1 G_1(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f_1\big(\tau,x(\tau),y(\tau)\big)d\tau\Big)ds+ \\
&\int_0^1 G_2(t,s)\phi_p\Big(\int_0^1 H_2(s,\tau)f_2\big(\tau,x(\tau),y(\tau)\big)d\tau\Big)ds\Big] \\
&\geq \eta\Big[\int_0^1 G_1(1,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f_1\big(\tau,x(\tau),y(\tau)\big)d\tau\Big)ds+ \\
&\int_0^1 G_2(1,s)\phi_p\Big(\int_0^1 H_2(s,\tau)f_2\big(\tau,x(\tau),y(\tau)\big)d\tau\Big)ds\Big] \\
&\geq \eta\max_{t\in[0,1]}\Big[\int_0^1 G_1(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f_1\big(\tau,x(\tau),y(\tau)\big)d\tau\Big)ds+ \\
&\int_0^1 G_2(t,s)\phi_p\Big(\int_0^1 H_2(s,\tau)f_2\big(\tau,x(\tau),y(\tau)\big)d\tau\Big)ds\Big] \\
&\geq \eta\max_{t\in I}\Big[\int_0^1 G_1(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f_1\big(\tau,x(\tau),y(\tau)\big)d\tau\Big)ds+ \\
&\int_0^1 G_2(t,s)\phi_q\Big(\int_0^1 H_2(s,\tau)f_2\big(\tau,x(\tau),y(\tau)\big)d\tau\Big)ds\Big] \\
&= \eta\theta\big(T(x,y)(t)\big) > b'.
\end{aligned}$$

Finally it is shown that $(B4)$ holds. Let $(x,y)\in Q\big(\gamma,\beta,a',c'\big)$ with $\psi\big(T(x,y)\big) < \eta a'$. Then we have

$$\begin{aligned}
\beta\big(T(x,y)(t)\big) &= \max_{t\in I_1}\Big[\int_0^1 G_1(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f_1\big(\tau,x(\tau),y(\tau)\big)d\tau\Big)ds+ \\
&\int_0^1 G_2(t,s)\phi_p\Big(\int_0^1 H_2(s,\tau)f_2\big(\tau,x(\tau),y(\tau)\big)d\tau\Big)ds\Big]
\end{aligned}$$

$$\leq \max_{t\in[0,1]}\Big[\int_0^1 G_1(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f_1\big(\tau,x(\tau),y(\tau)\big)d\tau\Big)ds+$$
$$\int_0^1 G_2(t,s)\phi_p\Big(\int_0^1 H_2(s,\tau)f_2\big(\tau,x(\tau),y(\tau)\big)d\tau\Big)ds\Big]$$
$$\leq \frac{1}{\eta}\min_{t\in I}\Big[\int_0^1 G_1(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f_1\big(\tau,x(\tau),y(\tau)\big)d\tau\Big)ds+$$
$$\int_0^1 G_2(t,s)\phi_p\Big(\int_0^1 H_2(s,\tau)f_2\big(\tau,x(\tau),y(\tau)\big)d\tau\Big)ds\Big]$$
$$\leq \frac{1}{\eta}\min_{t\in I_1}\Big[\int_0^1 G_1(t,s)\phi_q\Big(\int_0^1 H_1(s,\tau)f_1\big(\tau,x(\tau),y(\tau)\big)d\tau\Big)ds+$$
$$\int_0^1 G_2(t,s)\phi_p\Big(\int_0^1 H_2(s,\tau)f_2\big(\tau,x(\tau),y(\tau)\big)d\tau\Big)ds\Big]$$
$$= \frac{1}{\eta}\psi\big(T(x,y)(t)\big) < a'.$$

It is been proved that all the conditions of Theorem 3.1 are satisfied. Therefore the system of (p,q)-Laplacian fractional order boundary value problem (1.1)-(1.4) has at least three positive solutions (x_1,x_2), (y_1,y_2) and (z_1,z_2) such that $\beta(x_1,x_2) < a', b' < \alpha(y_1,y_2)$ and $a' < \beta(z_1,z_2)$ with $\alpha(z_1,z_2) < b'$. This completes the proof. □

4. Example

In this section, as an application, the results are demonstrated with an example.

Consider the system of (p,q)-Laplacian fractional order differential equations

$$D_{0^+}^{1.8}\Big(\phi_p\Big(D_{0^+}^{3.8}x(t)\Big)\Big) = f_1(t,x,y),\ t\in(0,1), \tag{4.1}$$

$$D_{0^+}^{1.7}\Big(\phi_q\Big(D_{0^+}^{3.9}y(t)\Big)\Big) = f_2(t,x,y),\ t\in(0,1), \tag{4.2}$$

satisfying the boundary conditions

$$\left.\begin{array}{l} x(0)=x'(0)=x''(0)=0 \text{ and } x''(1)=0,\\ \phi_p\Big(D_{0^+}^{3.8}x(0)\Big)=\phi_p\Big(D_{0^+}^{3.8}x(1)\Big)=0, \end{array}\right\} \tag{4.3}$$

$$\left.\begin{array}{l} y(0)=y'(0)=y''(0)=0 \text{ and } y''(1)=0,\\ \phi_q\Big(D_{0^+}^{3.9}y(0)\Big)=\phi_q\Big(D_{0^+}^{3.9}y(1)\Big)=0, \end{array}\right\} \tag{4.4}$$

where

$$f_1(t,x,y) = \begin{cases} \dfrac{e^{2t}}{57}+\dfrac{cos(x+y)}{9}+\dfrac{11(x+y)^3}{9}, & 0\leq x+y\leq 4,\\[2mm] \dfrac{cos(x+y)}{9}+\dfrac{e^{2t}}{57}+\dfrac{5632}{9}, & x+y>4, \end{cases}$$

$$f_2(t,x,y) = \begin{cases} \dfrac{sin(x+y)}{9}+\dfrac{11(x+y)^3}{9}+\dfrac{e^{2t}}{56}, & 0\leq x+y\leq 4,\\[2mm] \dfrac{e^{2t}}{56}+\dfrac{sin(x+y)}{9}+\dfrac{5632}{9}, & x+y>4. \end{cases}$$

Clearly f_i, for $i = 1, 2$ are continuous and increasing on $[0, \infty)$. Let $p = 2$. By direct calculations, one can determine $\eta = 0.0205$, $\mathcal{L} = 27.9632$ and $\mathcal{M} = 314.5214$. Choosing $a' = 0.5, b' = 4$ and $c' = 200$, then $0 < a' < b' < \dfrac{b'}{\eta} < c'$ and f_1, f_2 satisfies

$$(a)\begin{cases} f_1\big(t,x,y\big) < 0.5091 = \phi_p\Big(\dfrac{a'\mathcal{L}}{2}\Big) \text{ and } f_2\big(t,x,y\big) < 0.5091 = \phi_q\Big(\dfrac{a'\mathcal{L}}{2}\Big), \\ t \in [0,1] \text{ and } x, y \in \big[\eta a', a'\big] = [0.0103, 0.05], \end{cases}$$

$$(b)\begin{cases} f_1\big(t,x,y\big) > 1356 = \phi_p\Big(\dfrac{b'\mathcal{M}}{2}\Big) \text{ and } f_2\big(t,x,y\big) > 1356 = \phi_q\Big(\dfrac{b'\mathcal{M}}{2}\Big), \\ t \in I = [0.25, 0.75] \text{ and } x, y \in \Big[b', \dfrac{b'}{\eta}\Big] = [4, 195.12], \end{cases}$$

$$(c)\begin{cases} f_1\big(t,x,y\big) < 2796.32 = \phi_p\Big(\dfrac{c'\mathcal{L}}{2}\Big) \text{ and } f_2\big(t,x,y\big) < 2796.32 = \phi_q\Big(\dfrac{c'\mathcal{L}}{2}\Big), \\ t \in [0,1] \text{ and } x, y \in \Big[0, c'\Big] = [0, 200]. \end{cases}$$

Then all the conditions of Theorem 3.2 are satisfied. Thus by Theorem 3.2, the (p, q)-Laplacian fractional order boundary value problem (4.1)-(4.4) has at least three positive solutions.

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[1]Department of Applied Mathematics, Andhra University, Visakhapatnam, 530 003, India

[2]Department of Mathematics, MVGR College of Engineering, Vizianagaram, 535 005, India

[3]Department of Mathematics, Jimma University, Jimma, Oromia, 378, Ethiopia

*Corresponding author