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ON THE WALLIS FORMULA

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Abstract. By virtue of complex methods and tools, the authors express the famous Wallis formula as a sum involving binomial coefficients, establish the expansions for $\sin^k x$ and $\cos^k x$ in terms of $\cos(mx)$, find the general formulas for the derivatives of $\sin^k x$ and $\cos^k x$, and recover the general multiple-angle formulas for $\sin(kx)$ and $\cos(kx)$, where $k \in \mathbb{N}$ and $m \in \mathbb{Z}$.

1. Introduction

It is well known [8, 9, 16, 18, 23] that

$$I_n = \int_0^{\pi/2} \cos^n x \,\mathrm{d}\, x = \int_0^{\pi/2} \sin^n x \,\mathrm{d}\, x = \frac{(n-1)!!}{n!!} \times \begin{cases} \frac{\pi}{2} & \text{for } n \text{ even} \\ 1 & \text{for } n \text{ odd} \end{cases} \tag{1.1}$$

for $n \in \mathbb{N}$, where $n!!$ denotes a double factorial. Usually we call (1.1) the Wallis cosine or sine formula, or simply say, the Wallis formula, in the literature. In mathematical analysis, the Wallis formula (1.1) is derived generally by integrating by parts and mathematical induction.

The formula (1.1) may also be represented by

$$I_n = \frac{\sqrt{\pi}\,\Gamma((n+1)/2)}{n\Gamma(n/2)} = \frac{\sqrt{\pi}}{2} \frac{\Gamma((n+1)/2)}{\Gamma((n+2)/2)},$$

where $\Gamma(x)$ stands for the classical Euler gamma function which may defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \,\mathrm{d}\, t, \quad \Re(z) > 0.$$

The Wallis ratio is defined [42] as

$$W_n = \frac{(2n-1)!!}{(2n)!!} = \frac{(2n)!}{2^{2n}(n!)^2} = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}, \quad n \in \mathbb{N}.$$

It is clear that for $n \in \mathbb{N}$

$$W_n = \frac{2}{\pi} I_{2n} = \frac{1}{2^{2n}} \binom{2n}{n} \tag{1.2}$$

and

$$I_{2n-1} I_{2n} = \frac{\pi}{4n}.$$

There have existed plenty of literature about bounding the Wallis ratio. See, for example, [4, 5, 6, 7, 9, 16, 17, 19, 20, 22, 42, 43, 47].

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In [18], the Wallis formula (1.1) was generalized as

$$I(t)=\int_0^{\pi/2}\cos^t x\,\mathrm{d}\,x=\int_0^{\pi/2}\sin^t x\,\mathrm{d}\,x=\frac{\sqrt{\pi}}{2}\frac{\Gamma((t+1)/2)}{\Gamma((t+2)/2)},\quad t\ge 0.$$

See also [27, Section 2.3] and [48, 49].

In [2, p. 123], it was claimed that if $I_{m,n}$ is a primitive of $\sin^m x\cos^n x$ for $m,n\in\mathbb{R}$, then

$$I_{m+2,n}=-\frac{\sin^{m+1}x\cos^{n+1}x}{m+n+2}+\frac{m+1}{m+n+2}I_{m,n}$$

is a primitive of $\sin^{m+2}x\cos^n x$ if $m+n+2\ne 0$. With the aid of this formula the formula (1.1) may be recovered.

In [3, 10], by establishing double inequalities for I_{2n-1} and I_{2n}, the double inequality

$$\frac{\sqrt{\pi}}{\sqrt{1+(9\pi/16-1)/n}}\le\int_{-\sqrt{n}}^{\sqrt{n}}e^{-x^2}\,\mathrm{d}\,x<\frac{\sqrt{\pi}}{\sqrt{1-3/(4n)}}$$

was obtained for $n\in\mathbb{N}$. As a result, the probability integral

$$\int_0^\infty e^{-x^2}\,\mathrm{d}\,x=\frac{\sqrt{\pi}}{2}$$

was recovered. For more information, please refer to [2, p. 123], [22, 34] and related references therein.

In [13, 44], among other things, the sequence nI_n^2 for $n\in\mathbb{N}$, which originates from computation of the probability of intersecting between a plane couple and a convex body, was proved to be increasing.

For recent developments on the gamma function and the ratios of two gamma functions, please refer to the papers [11, 12, 14, 15, 21, 24, 25, 26, 29, 30, 32, 33, 35, 36, 37, 40, 41, 45, 46], the expository and survey articles [27, 28, 38, 39] and closely related references therein.

The aims of this paper are, by virtue of complex methods and tools, to express the sequence I_{2n-1} as a sum involving binomial coefficients and to recover the identity (1.2). As by-products, the expansions for $\sin^k x$ and $\cos^k x$ in terms of $\cos(mx)$ for $m\in\mathbb{Z}$, the derivatives for $\sin^k x$ and $\cos^k x$, and the general multiple-angle formulas for $\sin(kx)$ and $\cos(kx)$ are established and recovered.

2. Main results

Now we are in a position to establish and recover our main results and by-products.

Theorem 2.1. *For $n\in\mathbb{N}$, we have*

$$I_{2n-1}=\frac{(-1)^{n+1}}{2^{2n-1}}\sum_{k=0}^{2n-1}\frac{(-1)^k}{2n-2k-1}\binom{2n-1}{k}. \tag{2.1}$$

First proof. Let $i=\sqrt{-1}$ be the imaginary unit. Then for $n\in\mathbb{N}$ we have

$$I_{2n-1}=\int_0^{\pi/2}\left(\frac{e^{ix}+e^{-ix}}{2}\right)^{2n-1}\mathrm{d}\,x$$

$$
\begin{aligned}
&= \frac{1}{2^{2n-1}} \int_0^{\pi/2} \sum_{\ell=0}^{2n-1} \binom{2n-1}{\ell} e^{i\ell x} e^{-i(2n-1-\ell)x} \,\mathrm{d}\, x \\
&= \frac{1}{2^{2n-1}} \sum_{\ell=0}^{2n-1} \binom{2n-1}{\ell} \int_0^{\pi/2} e^{i(2\ell-2n+1)x} \,\mathrm{d}\, x \\
&= \frac{1}{2^{2n-1}} \sum_{\ell=0}^{2n-1} \binom{2n-1}{\ell} \frac{1}{i(2\ell-2n+1)} \big[e^{i(2\ell-2n+1)\pi/2} - 1\big] \\
&= \frac{1}{2^{2n-1}} \sum_{\ell=0}^{2n-1} \binom{2n-1}{\ell} \frac{1}{2\ell-2n+1} i \big[1 - e^{i(2\ell-2n+1)\pi/2}\big] \\
&= \frac{1}{2^{2n-1}} \sum_{\ell=0}^{2n-1} \binom{2n-1}{\ell} \frac{1}{2\ell-2n+1} \sin\frac{(2\ell-2n+1)\pi}{2} \\
&= \frac{1}{2^{2n-1}} \sum_{\ell=0}^{2n-1} \binom{2n-1}{\ell} \frac{1}{2\ell-2n+1} \cos[(\ell-n)\pi] \\
&= \frac{1}{2^{2n-1}} \sum_{\ell=0}^{2n-1} \binom{2n-1}{\ell} \frac{(-1)^{\ell-n}}{2\ell-2n+1}.
\end{aligned}
$$

The formula (2.1) follows. □

Second proof. For $n \in \mathbb{N}$, we have

$$
\begin{aligned}
I_n &= \int_0^{\pi/2} \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^n \mathrm{d}\, x \\
&= \frac{1}{2^n} \int_0^{\pi/2} \big[e^{i(x-\pi/2)} - e^{-i(x+\pi/2)}\big]^n \,\mathrm{d}\, x \\
&= \frac{1}{2^n} \int_0^{\pi/2} \sum_{\ell=0}^{n} (-1)^{n-\ell} \binom{n}{\ell} e^{i\ell(x-\pi/2)} e^{-i(n-\ell)(x+\pi/2)} \,\mathrm{d}\, x \\
&= \frac{1}{2^n} \sum_{\ell=0}^{n} (-1)^{n-\ell} \binom{n}{\ell} \int_0^{\pi/2} e^{i[(2\ell-n)x-n\pi/2]} \,\mathrm{d}\, x \\
&= \frac{1}{2^n} \sum_{\ell=0}^{n} (-1)^{n-\ell} \binom{n}{\ell} \int_0^{\pi/2} \cos\left[(2\ell-n)x - n\frac{\pi}{2}\right] \mathrm{d}\, x.
\end{aligned}
$$

Therefore, it follows that

$$
\begin{aligned}
I_{2n-1} &= \frac{-1}{2^{2n-1}} \sum_{\ell=0}^{2n-1} (-1)^{\ell} \binom{2n-1}{\ell} \int_0^{\pi/2} \cos\left[(2\ell-2n+1)x - (2n-1)\frac{\pi}{2}\right] \mathrm{d}\, x \\
&= \frac{(-1)^n}{2^{2n-1}} \sum_{\ell=0}^{2n-1} (-1)^{\ell} \binom{2n-1}{\ell} \int_0^{\pi/2} \sin[(2\ell-2n+1)x] \,\mathrm{d}\, x \\
&= \frac{(-1)^{n+1}}{2^{2n-1}} \sum_{\ell=0}^{2n-1} (-1)^{\ell} \binom{2n-1}{\ell} \frac{1}{2\ell-2n+1} \left[\cos\frac{(2\ell-2n+1)\pi}{2} - 1\right]
\end{aligned}
$$

$$= \frac{(-1)^n}{2^{2n-1}} \sum_{\ell=0}^{2n-1} (-1)^\ell \binom{2n-1}{\ell} \frac{1}{2\ell - 2n + 1}.$$

The proof is completed. □

Corollary 2.1. *For $\ell \in \mathbb{N}$, we have*

$$\cos^\ell x = \frac{1}{2^\ell} \sum_{q=0}^{\ell} \binom{\ell}{q} \cos[(2q-\ell)x], \tag{2.2}$$

$$\sin^\ell x = \frac{(-1)^\ell}{2^\ell} \sum_{q=0}^{\ell} (-1)^q \binom{\ell}{q} \cos\left[(2q-\ell)x - \frac{\ell}{2}\pi\right], \tag{2.3}$$

and

$$\sum_{q=0}^{\ell} \binom{\ell}{q} \sin[(2q-\ell)x] = 0, \tag{2.4}$$

$$\sum_{q=0}^{\ell} (-1)^q \binom{\ell}{q} \sin\left[(2q-\ell)x - \frac{\ell}{2}\pi\right] = 0. \tag{2.5}$$

Proof. From the second proof of Theorem 2.1, we conclude that

$$\cos^\ell x = \frac{1}{2^\ell}(e^{ix} + e^{-ix})^\ell = \frac{1}{2^\ell} \sum_{q=0}^{\ell} \binom{\ell}{q} e^{qix} e^{-(\ell-q)ix} = \frac{1}{2^\ell} \sum_{q=0}^{\ell} \binom{\ell}{q} e^{(2q-\ell)ix}$$

$$= \frac{1}{2^\ell} \sum_{q=0}^{\ell} \binom{\ell}{q} \{\cos[(2q-\ell)x] + i\sin[(2q-\ell)x]\}.$$

Equating the real and imaginary parts in the above equality gives equalities (2.2) and (2.4).

Similarly, we have

$$\sin^\ell x = \frac{1}{(2i)^\ell} \sum_{q=0}^{\ell} (-1)^{\ell-q} \binom{\ell}{q} e^{qix} e^{-(\ell-q)ix} = \frac{(-1)^\ell}{(2i)^\ell} \sum_{q=0}^{\ell} (-1)^q \binom{\ell}{q} e^{(2q-\ell)ix}$$

$$= \frac{(-1)^\ell}{2^\ell} e^{-\pi i\ell/2} \sum_{q=0}^{\ell} (-1)^q \binom{\ell}{q} e^{(2q-\ell)ix} = \frac{(-1)^\ell}{2^\ell} \sum_{q=0}^{\ell} (-1)^q \binom{\ell}{q} e^{[(2q-\ell)x - \pi\ell/2]i}$$

$$= \frac{(-1)^\ell}{2^\ell} \sum_{q=0}^{\ell} (-1)^q \binom{\ell}{q} \left\{\cos\left[(2q-\ell)x - \frac{\ell}{2}\pi\right] + i\sin\left[(2q-\ell)x - \frac{\ell}{2}\pi\right]\right\}.$$

Hence, we obtain equalities (2.3) and (2.5). □

Corollary 2.2. *For $m, k \in \mathbb{N}$, we have*

$$\frac{\mathrm{d}^m \cos^k x}{\mathrm{d}\, x^m} = \frac{1}{2^k} \sum_{q=0}^{k} \binom{k}{q} (2q-k)^m \cos\left[\frac{\pi}{2}m + (2q-k)x\right], \tag{2.6}$$

$$\frac{\mathrm{d}^m \sin^k x}{\mathrm{d}\, x^m} = \frac{(-1)^k}{2^k} \sum_{q=0}^{k} (-1)^q \binom{k}{q} (2q-k)^m \cos\left[(m-k)\frac{\pi}{2} + (2q-k)x\right], \tag{2.7}$$

and

$$\sum_{q=0}^{k}\binom{k}{q}(2q-k)^m\sin\left[\frac{\pi}{2}m+(2q-k)x\right]=0,$$

$$\sum_{q=0}^{k}(-1)^q\binom{k}{q}(2q-k)^m\sin\left[(m-k)\frac{\pi}{2}+(2q-k)x\right]=0.$$

Proof. These identities follow from directly differentiating on all the sides of the identities in Corollary 2.1. □

Remark 2.1. The formulas (2.6) and (2.7) were established and applied in the paper [31].

Theorem 2.2. *For $n\in\mathbb{N}$, we have*

$$I_{2n}=\frac{\pi}{2^{2n+1}}\binom{2n}{n}. \tag{2.8}$$

First proof. A direct calculation reveals that

$$\begin{aligned}
I_{2n}&=\int_0^{\pi/2}\left(\frac{e^{ix}+e^{-ix}}{2}\right)^{2n}\,\mathrm{d}\,x\\
&=\frac{1}{2^{2n}}\int_0^{\pi/2}\sum_{\ell=0}^{2n}\binom{2n}{\ell}e^{i\ell x}e^{-i(2n-\ell)x}\,\mathrm{d}\,x\\
&=\frac{1}{2^{2n}}\sum_{\ell=0}^{2n}\binom{2n}{\ell}\int_0^{\pi/2}e^{i(2\ell-2n)x}\,\mathrm{d}\,x\\
&=\frac{1}{2^{2n}}\left[\left(\sum_{\ell=0}^{n-1}+\sum_{\ell=n+1}^{2n}\right)\binom{2n}{\ell}\int_0^{\pi/2}e^{i(2\ell-2n)x}\,\mathrm{d}\,x+\frac{\pi}{2}\binom{2n}{n}\right]\\
&=\frac{\pi}{2^{2n+1}}\binom{2n}{n}+\frac{1}{2^{2n}}\left(\sum_{\ell=0}^{n-1}+\sum_{\ell=n+1}^{2n}\right)\binom{2n}{\ell}\frac{1}{i(2\ell-2n)}\left[e^{i(2\ell-2n)\pi/2}-1\right]\\
&=\frac{\pi}{2^{2n+1}}\binom{2n}{n}+\frac{1}{2^{2n}}\left(\sum_{\ell=0}^{n-1}+\sum_{\ell=n+1}^{2n}\right)\binom{2n}{\ell}\frac{i}{2\ell-2n}\left[1-e^{i(2\ell-2n)\pi/2}\right]\\
&=\frac{\pi}{2^{2n+1}}\binom{2n}{n}+\frac{1}{2^{2n}}\left(\sum_{\ell=0}^{n-1}+\sum_{\ell=n+1}^{2n}\right)\binom{2n}{\ell}\frac{1}{2(\ell-n)}\sin\frac{2(\ell-n)\pi}{2}\\
&=\frac{\pi}{2^{2n+1}}\binom{2n}{n}.
\end{aligned}$$

Consequently, the formula (2.8) is proved. □

Second proof. By virtue of (2.3), it follows that

$$\begin{aligned}
I_{2n}&=\frac{1}{2^{2n}}\sum_{\ell=0}^{2n}(-1)^\ell\binom{2n}{\ell}\int_0^{\pi/2}\cos[(2\ell-2n)x-n\pi]\,\mathrm{d}\,x\\
&=\frac{(-1)^n}{2^{2n}}\sum_{\ell=0}^{2n}(-1)^\ell\binom{2n}{\ell}\int_0^{\pi/2}\cos[(2\ell-2n)x]\,\mathrm{d}\,x
\end{aligned}$$

$$= \frac{(-1)^n}{2^{2n}}\left[(-1)^n\binom{2n}{n}\frac{\pi}{2} + \left(\sum_{\ell=0}^{n-1} + \sum_{\ell=n+1}^{2n}\right)(-1)^\ell\binom{2n}{\ell}\frac{1}{2\ell-2n}\sin\frac{(2\ell-2n)\pi}{2}\right]$$

$$= \frac{\pi}{2^{2n+1}}\binom{2n}{n}.$$

As a result, the formula (2.8) is proved. □

Third proof. Letting $\ell = 2n$ and integrating from 0 to $\frac{\pi}{2}$ on both sides of (2.2) arrive at the formula (2.8). □

Remark 2.2. In [2, p. 100], the formula (2.8) was proved alternatively.

3. General multiple-angle formulas for sine and cosine

Let $i = \sqrt{-1}$ be the imaginary unit. Then

$$i^k = \begin{cases} i, & k = 1 + 4\ell, \\ -1, & k = 2 + 4\ell, \\ -i, & k = 3 + 4\ell, \\ 1, & k = 4 + 4\ell, \end{cases}$$

where $k \in \mathbb{N}$ and $\ell \ge 0$. The quantity i^k may also be computed by

$$i^k = (-1)^{\frac{1}{2}\left[k - \frac{1-(-1)^k}{2}\right]} i^{\frac{1-(-1)^k}{2}}$$

and

$$i^k = e^{k\pi i/2} = \cos\frac{k\pi}{2} + i\sin\frac{k\pi}{2}.$$

It is well known [1, p. 72] that the first few multiple-angle formulas are

$$\sin(2x) = 2\sin x\cos x,$$

$$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x,$$

$$\sin(3x) = 3\sin x - 4\sin^3 x = 4\sin x\sin\left(\frac{\pi}{3} + x\right)\sin\left(\frac{\pi}{3} - x\right),$$

$$\cos(3x) = 4\cos^3 x - 3\cos x = 4\cos x\cos\left(\frac{\pi}{3} + x\right)\cos\left(\frac{\pi}{3} - x\right),$$

$$\sin(4x) = 8\cos^3 x\sin x - 4\cos x\sin x, \quad \cos(4x) = 8\cos^4 x - 8\cos^2 x + 1.$$

Theorem 3.1. *For $k \ge 2$, the general multiple-angle formulas for the sine and cosine functions are*

$$\sin(kx) = \sum_{\ell=0}^{k}\binom{k}{\ell}\sin\frac{\ell\pi}{2}\sin^\ell x\cos^{k-\ell} x$$

and

$$\cos(kx) = \sum_{\ell=0}^{k}\binom{k}{\ell}\cos\frac{\ell\pi}{2}\sin^\ell x\cos^{k-\ell} x.$$

Proof. By the formula

$$e^{kxi} = \cos(kx) + i\sin(kx),$$

we have

$$e^{kxi} = \left(e^{xi}\right)^k = (\cos x + i\sin x)^k$$

$$
\begin{aligned}
&= \sum_{\ell=0}^{k} \binom{k}{\ell} i^\ell \sin^\ell x \cos^{k-\ell} x \\
&= \sum_{\ell=0}^{k} \binom{k}{\ell} \left[\cos \frac{\ell\pi}{2} + i \sin \frac{\ell\pi}{2}\right] \sin^\ell x \cos^{k-\ell} x \\
&= \sum_{\ell=0}^{k} \binom{k}{\ell} \cos \frac{\ell\pi}{2} \sin^\ell x \cos^{k-\ell} x + i \sum_{\ell=0}^{k} \binom{k}{\ell} \sin \frac{\ell\pi}{2} \sin^\ell x \cos^{k-\ell} x.
\end{aligned}
$$

Further equating the real and imaginary parts yields the required general multiple-angle formulas for the sine and cosine functions. The proof of Theorem 3.1 is complete. □

Corollary 3.1. *For $k \ge 2$, we have*

$$
\begin{aligned}
\sin(kx) &= \sum_{\ell=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{2\ell+1} \sin \frac{(2\ell+1)\pi}{2} \sin^{2\ell+1} x \cos^{k-2\ell-1} x \\
&= \sum_{\ell=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{2\ell+1} (-1)^\ell \sin^{2\ell+1} x \cos^{k-2\ell-1} x
\end{aligned}
$$

and

$$
\begin{aligned}
\cos(kx) &= \sum_{\ell=0}^{\lfloor k/2 \rfloor} \binom{k}{2\ell} \cos(\ell\pi) \sin^{2\ell} x \cos^{k-2\ell} x \\
&= \sum_{\ell=0}^{\lfloor k/2 \rfloor} \binom{k}{2\ell} (-1)^\ell \sin^{2\ell} x \cos^{k-2\ell} x,
\end{aligned}
$$

where $\lfloor x \rfloor$ is called as the floor function which expresses the biggest integer not more than x.

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