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NEW INTEGRAL INEQUALITIES IN QUANTUM CALCULUS

KAMEL BRAHIM[1,*], SABRINA TAF[2] AND BOCHRA NEFZI[1]

Abstract. In this paper, we study the q-analogue of Klamkin-McLenaghan's and Grueb-Reinboldt's inequalities then we use the Riemann-Liouville fractional q-integral to get some new integral results.

1. INTRODUCTION

Let us consider

(1.1)
$$T(f,g;a,b)=\frac{1}{b-a}\int_a^b f(x)g(x)dx-\left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right)$$

where f and g are two integrable functions on $[a,b]$,
and

$$T_q(f,g;a,b)=\frac{1}{b-a}\int_a^b f(x)g(x)d_qx-\left(\frac{1}{b-a}\int_a^b f(x)d_qx\right)\left(\frac{1}{b-a}\int_a^b g(x)d_qx\right)$$

where f and g are two functions defined on $[a,b]_q$.
The well-known Grüss integral inequality can be stated as follows (see [10, 15]):

$$|T(f,g;a,b)|\le\frac{1}{4}(M-m)(N-n), \tag{1.2}$$

provided that f and g are two integrable functions on $[a,b]$ such that

$$0<m\le f(x)\le M<\infty,\ 0<n\le g(x)\le N<\infty,\ ,x\in[a,b]. \tag{1.3}$$

The constant $\frac{1}{4}$ is best possible.
Gauchman gave the q-integral Grüss inequality as follows (see [8]):

$$|T_q(f,g;a,b)|\le\frac{1}{4}(M-m)(N-n),$$

In [12], the authors proved the following Klamkin-McLenaghan inequality

(1.4)
$$\sum_{k=1}^n w_ka_k^2\sum_{k=1}^n w_kb_k^2-\left(\sum_{k=1}^n w_ka_kb_k\right)^2\le\left(\sqrt{\frac{M}{n}}-\sqrt{\frac{m}{N}}\right)^2\sum_{k=1}^n w_ka_kb_k\sum_{k=1}^n w_kb_k^2,$$

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where $0 < \frac{m}{N} \leq \frac{a_k}{b_k} \leq \frac{M}{n} < \infty$, $w_k > 0$, $\quad k = 1, \ldots, n$.
In [9], the authors proved the following Grueb-Reinboldt inequality

$$\sum_{k=1}^{n} w_k a_k^2 \sum_{k=1}^{n} w_k b_k^2 \leq \frac{(MN + mn)^2}{4nmNM} \left(\sum_{k=1}^{n} w_k a_k b_k \right)^2 \tag{1.5}$$

In [6], Dragomir and Diamond proved that

$$|T(f, g; a, b)| \leq \frac{1}{4} \cdot \frac{(M - m)(N - n)}{\sqrt{mMnN}} \cdot \frac{1}{b - a} \int_a^b f(x)dx \cdot \frac{1}{b - a} \int_a^b g(x)dx, \tag{1.6}$$

and

$$|T(f, g; a, b)| \leq \left(\sqrt{M} - \sqrt{m} \right) \left(\sqrt{N} - \sqrt{n} \right) \sqrt{\frac{1}{b - a} \int_a^b f(x)dx \cdot \frac{1}{b - a} \int_a^b g(x)dx}. \tag{1.7}$$

In recent years, many researches have studies (1.1) and number of inequalities appeared in literature (see [1, 2, 3, 4, 5, 14, 18]).

The main objective of this paper is to establish some new q-fractional integral inequalities of Klamkin-McLenaghan and Grueb-Reinboldt type.

This paper is organized as follows: in section 2, we present some preliminary results and notation. In section 3, we state the q-analogue of Klamkin-McLenaghan and Grueb-Reinboldt inequalities, then we establish some new q-fractional integral inequalities.

2. Basic Definitions

For the convenience of the reader, we provide in this section a summary of the mathematical notations and definitions used in this paper (see [7, 13, 16]). We write for $a, b \in \mathbb{C}$ and $q \in (0, 1)$,

$$(a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k), \qquad (a - b)^{(\alpha)} = a^\alpha \frac{(\frac{b}{a}; q)_\infty}{(q^\alpha \frac{b}{a}; q)_\infty}.$$

The q-Jackson integral from 0 to a is defined by (see [11])

$$\int_0^a f(x) d_q x = (1 - q) a \sum_{n=0}^{\infty} f(aq^n) q^n, \tag{2.1}$$

provided the sum converges absolutely.
The q-Jackson integral in a generic interval $[a, b]$ is given by (see [11])

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x. \tag{2.2}$$

In the case $a = bq^n$, we can write

$$\int_a^b f(x) d_q x = (1 - q) b \sum_{k=0}^{n-1} f(bq^k) q^k. \tag{2.3}$$

The fractional q-integral of the Riemann-Liouville type is (see [16])

$$\left(J_q^\alpha f\right)(x) = \frac{1}{\Gamma_q(\alpha)}\int_0^x (x-qt)^{(\alpha-1)} f(t)d_qt; \quad \alpha > 0$$

$$(2.4) \qquad = \frac{x^\alpha}{\Gamma_q(\alpha)}(1-q)\sum_{n=0}^{\infty}(1-q^{n+1})^{(\alpha-1)} f(xq^n)q^n.$$

where

$$\Gamma_q(\alpha) = \frac{1}{1-q}\int_0^1 \left(\frac{u}{1-q}\right)^{\alpha-1} e_q(qu)d_qu, \qquad \textit{and} \quad e_q(t) = \prod_{k=0}^{\infty}(1-q^kt).$$

The q-fractional integration has the following semi-group property for $\alpha, \beta \in \mathbb{R}^+$

$$(J_q^\beta J_q^\alpha f)(x) = (J_q^{\alpha+\beta} f)(x).$$

For the expression (2.4), when $f(x) = x^\lambda$, we get another expression that will be used later:

$$J_q^\alpha(x^\lambda) = \frac{\Gamma_q(\lambda+1)}{\Gamma(\alpha+\lambda+1)}x^{\alpha+\lambda}.$$

Finally, for $b > 0$ and $a = bq^n, n = 1, 2, ..., \infty$, we write

$$(2.5) \qquad [a,b]_q = \{bq^k : 0 \leq k \leq n\}, \qquad [0,b]_q = \{bq^k : k \in \mathbb{N}\}.$$

3. Main results

Theorem 1. *Let f and g be two functions defined on $[a,b]_q$ satisfying the condition*

$$(3.1) \qquad 0 < m \leq f(t) \leq M < \infty, \ 0 < n \leq g(t) \leq N < \infty, \ t \in [a,b]_q$$

Then one has the inequality

(3.2)

$$|T_q(f,g;a,b)| \leq \left(\sqrt{M}-\sqrt{m}\right)\left(\sqrt{N}-\sqrt{n}\right)\sqrt{\frac{1}{b-a}\int_a^b f(x)d_qx \cdot \frac{1}{b-a}\int_a^b g(x)d_qx}.$$

The following Lemma is used to prove Theorem 1:

Lemma 1. *Let h and l are two functions defined on $[a,b]_q$ such that*

$$(3.3) \quad 0 < m_1 \leq h(t) \leq M_1 < \infty, \ 0 < n_1 \leq l(t) \leq N_1 < \infty, \ t \in [a,b]_q.$$

Then, we have

(3.4)

$$\int_a^b h^2(x)d_qx\int_a^b l^2(x)d_qx - \left(\int_a^b h(x)l(x)d_qx\right)^2 \leq \left(\sqrt{\frac{M_1}{n_1}}-\sqrt{\frac{m_1}{N_1}}\right)^2\int_a^b h(x)l(x)d_qx\int_a^b l^2(x)d_qx$$

Proof. From the condition (3.8), we have

$$m_1\sqrt{q^k} \leq h(bq^k)\sqrt{q^k} \leq M_1\sqrt{q^k},$$

and

$$n_1\sqrt{q^k} \leq l(bq^k)\sqrt{q^k} \leq N_1\sqrt{q^k}.$$

Using the Klamkin-McLenaghan inequality (1.4), we obtain

$$\begin{aligned}\sum_{k=0}^{n-1} h^2(bq^k)q^k \sum_{k=0}^{n-1} l^2(bq^k)q^k &- \left(\sum_{k=0}^{n-1} h(bq^k)l(bq^k)q^k\right)^2 \\ &\le \left(\sqrt{\frac{M_1}{n_1}} - \sqrt{\frac{m_1}{N_1}}\right)^2 \sum_{k=0}^{n-1} h(bq^k)l(bq^k)q^k \sum_{k=0}^{n-1} l^2(bq^k)q^k.\end{aligned}$$

From (2.3), we get

$$\begin{aligned}\int_a^b h^2(x)d_qx \int_a^b l^2(x)d_qx &- \left(\int_a^b h(x)l(x)d_qx\right)^2 \\ &\le \left(\sqrt{\frac{M_1}{n_1}} - \sqrt{\frac{m_1}{N_1}}\right)^2 \int_a^b h(x)l(x)d_qx \int_a^b l^2(x)d_qx\end{aligned}$$

Lemma 1 is thus proved. □

Proof of Theorem 1:
Using the Cauchy-Schwartz inequality for double integrals, we have

$$|T_q(f,g;a,b)| = \left|\frac{1}{2(b-a)^2}\int_a^b\int_a^b (f(x)-f(y))(g(x)-g(y))d_qxd_qy\right|$$

$$\begin{aligned}&\le \frac{1}{2(b-a)^2}\left[\int_a^b\int_a^b (f(x)-f(y))^2 d_qxd_qy \cdot \int_a^b\int_a^b (g(x)-g(y))^2 d_qxd_qy\right]^{\frac{1}{2}} \\ &= \frac{1}{2(b-a)^2}\left[4\left[(b-a)\int_a^b f^2(x)d_qx - \left(\int_a^b f(x)d_qx\right)^2\right]\right. \\ &\times \left.\left[(b-a)\int_a^b g^2(x)d_qx - \left(\int_a^b g(x)d_qx\right)^2\right]\right]^{\frac{1}{2}} \\ &= \left[\frac{1}{b-a}\int_a^b f^2(x)d_qx - \left(\frac{1}{b-a}\int_a^b f(x)d_qx\right)^2\right]^{\frac{1}{2}} \\ &\times \left[\frac{1}{b-a}\int_a^b g^2(x)d_qx - \left(\frac{1}{b-a}\int_a^b g(x)d_qx\right)^2\right]^{\frac{1}{2}}. \qquad (3.5)\end{aligned}$$

By Lemma 1, we get

$$\begin{aligned}\frac{1}{b-a}\int_a^b f^2(x)d_qx &- \left(\frac{1}{b-a}\int_a^b f(x)d_qx\right)^2 \\ &\le \left(\sqrt{M}-\sqrt{m}\right)^2 \frac{1}{b-a}\int_a^b f(x)d_qx \qquad (3.6)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{b-a}\int_a^b g^2(x)d_qx &- \left(\frac{1}{b-a}\int_a^b g(x)d_qx\right)^2 \\ &\le \left(\sqrt{N}-\sqrt{n}\right)^2 \frac{1}{b-a}\int_a^b g(x)d_qx \qquad (3.7)\end{aligned}$$

From (3.5), (3.6) and (3.7), we deduce the desired inequality (3.9).
Theorem 1 is thus proved.

Corollary 1. *Let f and g be two functions defined on $[0,b]_q$ satisfying the condition*

$$0 < m \leq f(t) \leq M < \infty,\ 0 < n \leq g(t) \leq N < \infty,\ t \in [0,b]_q \tag{3.8}$$

Then one has the inequality

$$|T_q(f,g;0,b)| \leq \frac{1}{b}\left(\sqrt{M}-\sqrt{m}\right)\left(\sqrt{N}-\sqrt{n}\right)\sqrt{\int_0^b f(x)d_qx \cdot \int_0^b g(x)d_qx}. \tag{3.9}$$

Proof. By taking $a = bq^n$ in the previous theorem and by tending n to ∞ we obtain the result. □

Theorem 2. *Let f and g be two functions defined on $[a,b]_q$ satisfying the condition (3.1). Then one has the inequality*

$$|T_q(f,g;a,b)| \leq \frac{(M-m)(N-n)}{4\sqrt{nmNM}} \cdot \frac{1}{b-a}\int_a^b f(x)d_qx \cdot \frac{1}{b-a}\int_a^b g(x)d_qx \tag{3.10}$$

Lemma 2. [17] *Let f and g are two functions defined on $[a,b]_q$ satisfying the condition (3.8). Then we have*

$$\int_a^b f^2(x)d_qx \int_a^b g^2(x)d_qx \leq \frac{(MN+mn)^2}{4nmNM}\left(\int_a^b f(x)g(x)d_qx\right)^2 \tag{3.11}$$

Proof of Theorem 2:
Using Lemma 2, we get

$$(b-a)\int_a^b f^2(x)d_qx \leq \frac{(M+m)^2}{4mM}\left(\int_a^b f(x)d_qx\right)^2,$$

Hence

$$\frac{1}{b-a}\int_a^b f^2(x)d_qx - \left(\frac{1}{b-a}\int_a^b f(x)d_qx\right)^2 \leq \frac{(M-m)^2}{4mM}\left(\frac{1}{b-a}\int_a^b f(x)d_qx\right)^2. \tag{3.12}$$

Similarly, we have

$$\frac{1}{b-a}\int_a^b g^2(x)d_qx - \left(\frac{1}{b-a}\int_a^b g(x)d_qx\right)^2 \leq \frac{(N-n)^2}{4nN}\left(\frac{1}{b-a}\int_a^b g(x)d_qx\right)^2. \tag{3.13}$$

From (3.5), (3.12) and (3.13), we deduce the desired inequality (3.14).
Theorem 2 is thus proved.

Corollary 2. *Let f and g be two functions defined on $[0,b]_q$ satisfying the condition (3.8). Then one has the inequality*

$$|T_q(f,g;0,b)| \leq \frac{(M-m)(N-n)}{4b^2\sqrt{nmNM}} \cdot \int_0^b f(x)d_qx \cdot \int_0^b g(x)d_qx \tag{3.14}$$

Theorem 3. *Let f and g be two positive functions defined on $[0,\infty)$ satisfying the condition*

$$0 < m \leq f(\tau) \leq M < \infty,\ 0 < n \leq g(\tau) \leq N < \infty,\ \tau \in [0,t],\ t > 0. \tag{3.15}$$

Then for all $\alpha > 0,\ b > 0$ *we have*

$$\left|\frac{b^{\alpha}}{\Gamma_q(\alpha+1)} J_q^{\alpha}(f(b)g(b)) - J_q^{\alpha} f(b) J_q^{\alpha} g(b)\right| \tag{3.16}$$

$$\leq \left(\sqrt{M}-\sqrt{m}\right)\left(\sqrt{N}-\sqrt{n}\right)\frac{b^{\alpha}}{\Gamma_q(\alpha+1)}\sqrt{J_q^{\alpha} f(b) J_q^{\alpha} g(b)}$$

The following Lemma is used to prove Theorem 3:

Lemma 3. *Let* h *and* l *be two positive functions on* $[0,\infty)$ *such that*

$$0 < m_1 \leq h(\tau) \leq M_1 < \infty,\ 0 < n_1 \leq l(\tau) \leq N_1 < \infty,\ \tau \in [0,t],\ t > 0. \tag{3.17}$$

Then for all $\alpha > 0,\ b > 0$*, we have*

$$J_q^{\alpha} h^2(b) J_q^{\alpha} l^2(b) - \left(J_q^{\alpha}(h(b)l(b))\right)^2 \leq \left(\sqrt{\frac{M_1}{n_1}} - \sqrt{\frac{m_1}{N_1}}\right)^2 J_q^{\alpha}(h(b)l(b)) J_q^{\alpha} l^2(b). \tag{3.18}$$

Proof. Using the Klamkin-McLenaghan inequality (1.4), we obtain

$$\begin{aligned} & \sum_{k=0}^{\infty} h^2(bq^k)(1-q^{k+1})^{(\alpha-1)} q^k \sum_{k=0}^{\infty} l^2(bq^k)(1-q^{k+1})^{(\alpha-1)} q^k \\ - & \left(\sum_{k=0}^{\infty} h(bq^k) l(bq^k)(1-q^{k+1})^{(\alpha-1)} q^k\right)^2 \\ \leq & \left(\sqrt{\frac{M_1}{n_1}} - \sqrt{\frac{m_1}{N_1}}\right)^2 \sum_{k=0}^{\infty} h(bq^k) l(bq^k)(1-q^{k+1})^{(\alpha-1)} q^k \sum_{k=0}^{\infty} l^2(bq^k)(1-q^{k+1})^{(\alpha-1)} q^k. \end{aligned}$$

From (2.4), we obtain

$$J_q^{\alpha} h^2(b) J_q^{\alpha} l^2(b) - \left(J_q^{\alpha}(h(b)l(b))\right)^2 \leq \left(\sqrt{\frac{M_1}{n_1}} - \sqrt{\frac{m_1}{N_1}}\right)^2 J_q^{\alpha}(h(b)l(b)) J_q^{\alpha} l^2(b).$$

Lemma 3 is thus proved. □

Proof of Theorem 3:
Define

$$Q(\tau,\rho) = (f(\tau) - f(\rho))(g(\tau) - g(\rho)) \tag{3.19}$$

Multiplying (3.19) by $\frac{(b-q\tau)^{(\alpha-1)}(b-q\rho)^{(\alpha-1)}}{\Gamma_q^2(\alpha)}$ and double integrating with respect to τ and ρ from 0 to b, we get

$$\begin{aligned} & \frac{1}{\Gamma_q^2(\alpha)} \int_0^b \int_0^b (b-q\tau)^{(\alpha-1)}(b-q\rho)^{(\alpha-1)} Q(\tau,\rho) d_q\tau d_q\rho \\ = & \ 2\frac{b^{\alpha}}{\Gamma_q(\alpha+1)} J_q^{\alpha}(f(b)g(b)) - 2J_q^{\alpha} f(b) J_q^{\alpha} g(b). \end{aligned} \tag{3.20}$$

On the other hand, using the Cauchy-Schwartz inequality we get

$$\begin{aligned}
&\left|\frac{1}{\Gamma_q^2(\alpha)}\int_0^b\int_0^b (b-q\tau)^{(\alpha-1)}(b-q\rho)^{(\alpha-1)}Q(\tau,\rho)d_q\tau d_q\rho\right| \\
\leq\ & \left[\frac{1}{\Gamma_q^2(\alpha)}\int_0^b\int_0^b (b-q\tau)^{(\alpha-1)}(b-q\rho)^{(\alpha-1)}(f(\tau)-f(\rho))^2 d_q\tau d_q\rho\right]^{\frac{1}{2}} \\
\times\ & \left[\frac{1}{\Gamma_q^2(\alpha)}\int_0^b\int_0^b (b-q\tau)^{(\alpha-1)}(b-q\rho)^{(\alpha-1)}(g(\tau)-g(\rho))^2 d_q\tau d_q\rho\right]^{\frac{1}{2}}.
\end{aligned}$$

Then,

$$\begin{aligned}
&\left|\frac{1}{\Gamma_q^2(\alpha)}\int_0^b\int_0^b (b-q\tau)^{(\alpha-1)}(b-q\rho)^{(\alpha-1)}Q(\tau,\rho)d_q\tau d_q\rho\right| \\
\leq\ & 2\left[\frac{b^\alpha}{\Gamma_q(\alpha+1)}J_q^\alpha f^2(b)-\left(J_q^\alpha f(b)\right)^2\right]^{\frac{1}{2}}\times\left[\frac{b^\alpha}{\Gamma_q(\alpha+1)}J_q^\alpha g^2(b)-\left(J_q^\alpha g(b)\right)^2\right]^{\frac{1}{2}}
\end{aligned}\tag{3.21}$$

Using Lemma 3, we get

$$\frac{b^\alpha}{\Gamma_q(\alpha+1)}J_q^\alpha f^2(b)-\left(J_q^\alpha f(b)\right)^2\leq\left(\sqrt{M}-\sqrt{m}\right)^2\frac{b^\alpha}{\Gamma_q(\alpha+1)}\left(J_q^\alpha f(b)\right)\tag{3.22}$$

and

$$\frac{b^\alpha}{\Gamma_q(\alpha+1)}J_q^\alpha g^2(b)-\left(J_q^\alpha g(b)\right)^2\leq\left(\sqrt{N}-\sqrt{n}\right)^2\frac{b^\alpha}{\Gamma_q(\alpha+1)}\left(J_q^\alpha g(b)\right).\tag{3.23}$$

Using (3.20), (3.22) and (3.23), we deduce the desired inequality (3.16).
Theorem 3 is thus proved.

Theorem 4. *Let f and g be two positive functions defined on $[0,\infty)$ satisfying the condition (3.15). Then for all $\alpha>0,\ b>0$, we have*

$$\left|\frac{b^\alpha}{\Gamma_q(\alpha+1)}J_q^\alpha(f(b)g(b))-J_q^\alpha f(b)J_q^\alpha g(b)\right|\leq\frac{(M-m)(N-n)}{4\sqrt{mMnN}}J_q^\alpha f(b)J_q^\alpha g(b).\tag{3.24}$$

To prove Theorem 4 we need the following result:

Lemma 4. *Let h and l be two positive functions on $[0,\infty)$ such that the condition (3.17). Then for all $\alpha>0,\ b>0$, we have*

$$J_q^\alpha h^2(b)J_q^\alpha l^2(b)\leq\frac{(MN+mn)^2}{4nmNM}(J_q^\alpha h(b)l(b))^2\tag{3.25}$$

Proof. Using Grueb-Reinboldt inequality (1.5), we obtain

$$\begin{aligned}
&\left(\sum_{k=0}^\infty h^2(bq^k)(1-q^{k+1})^{\alpha-1}q^k\right)\left(\sum_{k=0}^\infty l^2(bq^k)(1-q^{k+1})^{\alpha-1}q^k\right) \\
\leq\ & \frac{(MN+mn)^2}{4nmNM}\left(\sum_{k=0}^\infty h(bq^k)l(bq^k)(1-q^{k+1})^{\alpha-1}q^k\right)^2
\end{aligned}\tag{3.26}$$

Which implies that

$$\frac{(J_q^\alpha h^2(b))(J_q^\alpha l^2(b))}{(J_q^\alpha h(b)l(b))^2} \leq \frac{(MN+mn)^2}{4nmNM} \tag{3.27}$$

Lemma 4 is thus proved. □

Proof of Theorem 4:
Using Lemma 4, we get

$$\frac{b^\alpha}{\Gamma_q(\alpha+1)} J_q^\alpha f^2(b) \leq \frac{(M+m)^2}{4mM} (J_q^\alpha f(b))^2$$

thus,

$$\frac{b^\alpha}{\Gamma_q(\alpha+1)} J_q^\alpha f^2(b) - (J_q^\alpha f(b))^2 \leq \frac{(M-m)^2}{4mM} (J_q^\alpha f(b))^2 \tag{3.28}$$

By the same, we obtain

$$\frac{b^\alpha}{\Gamma_q(\alpha+1)} J_q^\alpha g^2(b) - (J_q^\alpha g(b))^2 \leq \frac{(N-n)^2}{4nN} (J_q^\alpha g(b))^2 \tag{3.29}$$

Using (3.20), (3.28) and (3.29), we deduce the desired inequality (3.24).
Theorem 4 is thus proved.

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[1]Departement of Mathematics, Faculty of Science of Tunis, Tunisia

[2]Department of Mathematics, Faculty SEI, UMAB University of Mostaganem, Algeria

*Corresponding author