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ABSOLUTE MONOTONICITY OF FUNCTIONS RELATED TO ESTIMATES OF FIRST EIGENVALUE OF LAPLACE OPERATOR ON RIEMANNIAN MANIFOLDS

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Abstract. The authors find the absolute monotonicity and complete monotonicity of some functions involving trigonometric functions and related to estimates the lower bounds of the first eigenvalue of Laplace operator on Riemannian manifolds.

1. Background and mail results

In [38, 39], J. Q. Zhong and H. C. Yang obtained that the first eigenvalue λ_1 of Laplace operator on a compact Riemannian monifold M with non-negative Ricci curvature satisfies

$$\lambda_1 \ge \frac{\pi^2}{d^2}, \tag{1.1}$$

where d denotes the diameter of M. The inequality (1.1) improves corresponding results in [11, 12]. For proving the inequality (1.1), the authors introduced in [38, Lemma 4] and [39, Lemma 4] the function

$$\psi(\theta) = \begin{cases} \dfrac{\frac{4}{\pi}(\theta + \sin\theta\cos\theta) - 2\sin\theta}{\cos^2\theta}, & \theta \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \\ \pm 1, & \theta = \pm\dfrac{\pi}{2} \end{cases} \tag{1.2}$$

and obtained that the function $y(\theta) = \psi(\theta)$ satisfies $\psi'(\theta) \ge 0$, the differential equation

$$y(\theta) - \sin\theta + y'\sin\theta\cos\theta - \frac{1}{2}y''(\theta)\cos^2\theta = 0, \tag{1.3}$$

and the inequality

$$0 \le \psi'(\theta)\cos\theta \le 2\left(\frac{4}{\pi} - 1\right) \tag{1.4}$$

on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. These results were ever employed in [37, p. 348, Lemma 4]. In [8, p. 3], it was pointed out that $\psi'(\theta) \ge 0$ and $|\psi(\theta)| \le 1$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. For more information, please refer to [18, Lemma 4], [23, Lemma 1 and Proposition 7], [26, Lemma 4], and [27, Proposition 3].

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Let M be a m-dimensional compact Riemannian manifold with boundary ∂M, the inner radius of M be ρ, the Ricci curvature of M be not less than $-R$, and the mean curvature of ∂M be not less than $-H_0$, where R and H_0 are positive scalars. Theorem 3 in [35, p. 331] reads that the first eigenvalue μ_1 of M under Dirichlet boundary condition satisfies

$$\mu_1 \ge \frac{\pi^2}{4\rho^2} - \frac{1}{2}R - \frac{2}{3}(m-1)H_0\frac{\pi}{\rho}. \tag{1.5}$$

For proving the inequality (1.5), the author considered the functions

$$p(\theta) = \begin{cases} \dfrac{2}{\cos^2\theta}\displaystyle\int_\theta^{\pi/2} t\cos^2 t \,\mathrm{d}\,t, & \theta \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \\ 0, & \theta = \pm\dfrac{\pi}{2} \end{cases} \tag{1.6}$$

and

$$\phi(\theta) = \begin{cases} \dfrac{1}{\cos^2\theta}\displaystyle\int_\theta^{\pi/2} \cos^2 t \,\mathrm{d}\,t, & \theta \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \\ 0, & \theta = \dfrac{\pi}{2} \end{cases} \tag{1.7}$$

and obtained in [35, pp. 338–340] that $p'(\theta) \le 0$ and $\phi'(\theta) \le 0$ on $\left[0, \frac{\pi}{2}\right]$, that

$$\int_0^{\pi/2} p(\theta)\,\mathrm{d}\,\theta = \frac{\pi}{2}, \quad \int_0^{\pi/2} \phi(\theta)\,\mathrm{d}\,\theta = \frac{1}{2}, \tag{1.8}$$

and that the function $Z(\theta) = 1 + \alpha p(\theta) + \beta\phi(\theta)$ satisfies $Z\left(\frac{\pi}{2}\right) = 1$ and

$$Z(\theta) = 1 + \alpha\cos^2\theta - Z'(\theta)\cos\theta\sin\theta + \frac{1}{2}Z''(\theta)\cos^2\theta, \quad \theta \in \left[0, \frac{\pi}{2}\right]. \tag{1.9}$$

In [18, Propositions 11 and 12], [23, Propositions 2, 3, and 5], and [27, Propositions 1 and 2], it was obtained that the function $Y(\theta) = p(\theta)$ satisfies the differential equation

$$Y''(\theta)\cos^2\theta - 2Y'(\theta)\sin\theta\cos\theta - 2Y(\theta) + 2\cos^2\theta = 0 \tag{1.10}$$

and the inequalities

$$p'(\theta)\sin\theta \le 0, \quad |p'(\theta)\cos\theta| \le \frac{8}{3}, \quad p(\theta) \le \frac{\pi^2}{8} - \frac{1}{2} \tag{1.11}$$

for $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. In [9], it was established that the function $\frac{p(\theta)}{\cos\theta}$ is increasing on $\left[0, \frac{\pi}{2}\right]$, that the function $p'(\theta)$ is decreasing, and that

$$\frac{\pi^2}{8} - \frac{1}{2} \le \frac{p(\theta)}{\cos\theta} \le \frac{\pi}{3}, \quad p(\theta) \le \frac{1}{5} + \cos^2\theta \tag{1.12}$$

on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. See also [34, p. 699].

In [13, Theorem 1.1], it was obtained that the first positive eigenvalue λ of Laplace operator on a closed n-dimensional Riemannian manifold with Ricci curvature $\mathrm{Ric}(M) \ge (n-1)K > 0$ has the lower bound

$$\lambda \ge \frac{1}{2}(n-1)K + \frac{\pi^2}{4r^2}, \tag{1.13}$$

where r is the largest interior radius of the nodal domains of eigenfunctions of the eigenvalue λ. For verifying the above conclusion, the author considered in [13, Lemma 3.1] the function $\xi(t) = -2p(t)$ and obtained some conclusions on $\xi(t)$, which may be reformulated as follows.

(1) For $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, the function $\xi(t)$ meets

$$\frac{1}{2}\xi''(t)\cos^2 t - \xi'(t)\cos t \sin t - \xi(t) = 2\cos^2 t, \tag{1.14}$$

$$\xi'(t)\cos t - 2\xi(t)\sin t = 4t\cos t; \tag{1.15}$$

(2) For $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

$$1 - \frac{\pi^2}{4} = \xi(0) \le \xi(t) \le \xi\left(\pm\frac{\pi}{2}\right) = 0 \quad \text{and} \quad \int_0^{\pi/2} \xi(t)\,\mathrm{d}\,t = -\frac{\pi}{2}; \tag{1.16}$$

(3) The derivative $\xi'(t)$ is increasing on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

$$\xi'\left(\pm\frac{\pi}{2}\right) = \pm\frac{2\pi}{3}, \quad \text{and} \quad \xi'(t)\begin{cases} < 0, & t \in \left(-\frac{\pi}{2}, 0\right), \\ > 0, & t \in \left(0, \frac{\pi}{2}\right); \end{cases} \tag{1.17}$$

(4) For $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

$$2\left(3 - \frac{\pi^2}{4}\right) \le \frac{\xi'(t)}{t} \le \frac{4}{3}, \tag{1.18}$$

and for $t \in \left(0, \frac{\pi}{2}\right)$,

$$\left[\frac{\xi'(t)}{t}\right]' > 0; \tag{1.19}$$

(5) For $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

$$\xi''(t) > 0, \quad \xi''\left(\pm\frac{\pi}{2}\right) = 2, \quad \text{and} \quad \xi''(0) = 2\left(3 - \frac{\pi^2}{4}\right); \tag{1.20}$$

(6) For $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

$$\xi'''\left(\frac{\pi}{2}\right) = \frac{8\pi}{15}, \quad \xi'''(t)\begin{cases} < 0, & t \in \left(-\frac{\pi}{2}, 0\right), \\ > 0, & t \in \left(0, \frac{\pi}{2}\right). \end{cases} \tag{1.21}$$

By calculus, it is easy to see that

$$\psi(\theta) = \begin{cases} \frac{2}{\pi}[2\theta + \sin(2\theta) - \pi\sin\theta]\sec^2\theta, & \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ \pm 1, & \theta = \pm\frac{\pi}{2}, \end{cases} \tag{1.22}$$

$$p(\theta) = \begin{cases} \left(\frac{\pi^2}{8} - \frac{1}{2}\theta^2\right)\sec^2\theta - \theta\tan\theta - \frac{1}{2}, & \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ 0, & \theta = \pm\frac{\pi}{2}, \end{cases} \tag{1.23}$$

and

$$\phi(\theta) = \begin{cases} -\frac{1}{4}[2\theta + \sin(2\theta) - \pi]\sec^2\theta, & \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \\ 0, & \theta = \frac{\pi}{2}. \end{cases} \tag{1.24}$$

See also [28, pp. 6–7]. For more information, please read [4, 5, 10, 17, 24, 25, 30, 36] and closely related references therein.

A function f is said to be completely monotonic on an interval I if it has derivatives of all orders on I and satisfies

$$0 \le (-1)^{k-1} f^{(k-1)}(x) < \infty \tag{1.25}$$

for $x \in I$ and $k \in \mathbb{N}$, where $f^{(0)}(x)$ means $f(x)$ and $\mathbb{N}$ stands for the set of all positive integers. See [14, Chapter XIII], [31, Chapter 1], or [33, Chapter IV]. The class of completely monotonic functions may be characterized by the famous Hausdorff-Bernstein-Widder theorem [33, p. 161, Theorem 12b]: A necessary and sufficient condition that $f(x)$ should be completely monotonic for $0 < x < \infty$ is that

$$f(x) = \int_0^\infty e^{-xt} \,\mathrm{d}\, \alpha(t), \tag{1.26}$$

where $\alpha(t)$ is non-decreasing and the above integral converges for $0 < x < \infty$.

Recall from [14, Chapter XIII] or [33, Chapter IV] that a function f is said to be absolutely monotonic on an interval I if it has derivatives of all orders and

$$f^{(k-1)}(t) \ge 0 \tag{1.27}$$

for $t \in I$ and $k \in \mathbb{N}$. Theorem 12c in [33, p. 162] states that a necessary and sufficient condition that $f(x)$ should be absolutely monotonic in $-\infty < x < 0$ is that

$$f(x) = \int_0^\infty e^{xt} \,\mathrm{d}\, \alpha(t), \tag{1.28}$$

where $\alpha(t)$ is non-decreasing and the integral converges for $-\infty < x < 0$.

For more information on completely and absolutely monotonic functions, please refer to [6, 7, 19, 20, 21, 22, 29] and closely related references therein.

In this paper, we will prove the following absolute and complete monotonicity of functions related to estimates of first eigenvalue of Laplace operator on Riemannian manifolds.

Theorem 1.1. *The functions $\psi(\theta)$ and $\frac{8}{\pi} - 2 - \psi'(\theta)\cos\theta$ are absolutely monotonic on $\left(0, \frac{\pi}{2}\right)$.*

Theorem 1.2. *The function $-p'(\theta)$ is absolutely monotonic on $\left(0, \frac{\pi}{2}\right)$.*

Theorem 1.3. *The function $\phi(\theta)$ is completely monotonic on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.*

2. Proofs of Theroems 1.1 to 1.3

Proof of Theorem 1.1. The function $\psi(\theta)$ may be rewritten as

$$\begin{aligned}\psi(\theta) &= \frac{4}{\pi}\tan\theta + \frac{4}{\pi}\theta\sec^2\theta - 2\tan\theta\sec\theta \\ &= \frac{4}{\pi}\tan\theta + \frac{4}{\pi}\theta(\tan\theta)' - 2(\sec\theta)' \\ &= \frac{4}{\pi}(\theta\tan\theta)' - 2(\sec\theta)'.\end{aligned}$$

It is well known [1, p. 75, 4.3.67 and 4.3.69] that the tangent $\tan x$ and the secant $\sec x$ can be expanded into power series

$$\tan z = \sum_{n=1}^\infty (-1)^{n-1} 2^{2n} (2^{2n} - 1) B_{2n} \frac{z^{2n-1}}{(2n)!} \tag{2.1}$$

and

$$\sec z = \sum_{n=0}^{\infty}(-1)^n E_{2n}\frac{z^{2n}}{(2n)!} \tag{2.2}$$

for $|z| < \frac{\pi}{2}$, where B_n for $n \ge 0$ are Bernoulli numbers which may be defined by the power series expansion

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} B_{2k}\frac{z^{2k}}{(2k)!}, \quad |z| < 2\pi \tag{2.3}$$

and E_n for $n \ge 0$ stand for Euler numbers which are integers and may be defined by

$$\frac{2e^z}{e^{2z}+1} = \sum_{n=0}^{\infty}\frac{E_n}{n!}z^n = \sum_{n=0}^{\infty} E_{2n}\frac{z^{2n}}{(2n)!}, \quad |z| < \pi, \tag{2.4}$$

see [1, p. 804, 23.1.1 and 23.1.2] or [32, p. 3, (1.1) and p. 15]. Consequently,

$$\begin{aligned}\psi(\theta) &= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}}{(2n-1)!}\theta^{2n-1} - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)!}E_{2n}\theta^{2n-1}\\ &= 2\sum_{n=1}^{\infty}\frac{1}{(2n-1)!}(-1)^{n-1}\left[\frac{2}{\pi}2^{2n}(2^{2n}-1)B_{2n} + E_{2n}\right]\theta^{2n-1}.\end{aligned}$$

In [1, p. 805, 23.1.15], it was listed that

$$\frac{4^{n+1}(2n)!}{\pi^{2n+1}} > (-1)^n E_{2n} > \frac{1}{1+3^{-1-2n}}\frac{4^{n+1}(2n)!}{\pi^{2n+1}}, \quad n \in \{0\}\cup\mathbb{N}. \tag{2.5}$$

In [2], it was obtained that the double inequality

$$\frac{2(2n)!}{(2\pi)^{2n}}\frac{1}{1-2^{\alpha-2n}} \le (-1)^{n-1}B_{2n} \le \frac{2(2n)!}{(2\pi)^{2n}}\frac{1}{1-2^{\beta-2n}} \tag{2.6}$$

holds for $n \in \mathbb{N}$ if and only if $\alpha \le 0$ and $\beta \ge 2 + \frac{\ln(1-6/\pi^2)}{\ln 2} = 0.649\ldots$. As a result,

$$\begin{aligned}&(-1)^{n-1}\left[\frac{2}{\pi}2^{2n}(2^{2n}-1)B_{2n} + E_{2n}\right]\\ &> \frac{2}{\pi}2^{2n}(2^{2n}-1)\frac{2(2n)!}{(2\pi)^{2n}}\frac{1}{1-2^{-2n}} - \frac{4^{n+1}(2n)!}{\pi^{2n+1}}\\ &= 0.\end{aligned}$$

This implies that the function $\psi(\theta)$ is absolutely monotonic on $\left[0, \frac{\pi}{2}\right]$.

Direct calculation and utilization of (2.1) and (2.2) yield

$$\begin{aligned}\frac{8}{\pi} - 2 - \psi'(\theta)\cos\theta &= 4\sec^2\theta - \frac{8}{\pi}(\theta\tan\theta\sec\theta + \sec\theta) - 4 + \frac{8}{\pi}\\ &= 4(\tan\theta)' - \frac{8}{\pi}[\theta(\sec\theta)' + \sec\theta] - 4 + \frac{8}{\pi}\\ &= 4\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(2n-1)2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}\theta^{2n-2}\\ &\quad - \frac{8}{\pi}\left[\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)!}E_{2n}\theta^{2n} + \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}E_{2n}\theta^{2n}\right] - 4 + \frac{8}{\pi}\end{aligned}$$

$$= 4\sum_{n=1}^{\infty}\frac{2n+1}{(2n)!}\left[\frac{2^{2(n+1)}(2^{2(n+1)}-1)(-1)^n B_{2(n+1)}}{2(n+1)(2n+1)}-\frac{2}{\pi}(-1)^n E_{2n}\right]\theta^{2n}.$$

Employing the inequalities (2.5) and (2.6) reveals

$$\begin{aligned}
&\frac{2^{2(n+1)}(2^{2(n+1)}-1)(-1)^n B_{2(n+1)}}{2(n+1)(2n+1)}-\frac{2}{\pi}(-1)^n E_{2n}\\
&>\frac{2^{2(n+1)}(2^{2(n+1)}-1)}{2(n+1)(2n+1)}\frac{2(2n+2)!}{(2\pi)^{2n+2}}\frac{1}{1-2^{-2n-2}}-\frac{2}{\pi}\frac{4^{n+1}(2n)!}{\pi^{2n+1}}\\
&=0.
\end{aligned}$$

This means that the function $\frac{8}{\pi}-2-\psi'(\theta)\cos\theta$ is absolutely monotonic on $\left[0,\frac{\pi}{2}\right]$. The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. Straightforward computation and utilization of (2.1) yield

$$\begin{aligned}
-p'(\theta)&=\frac{1}{2}[\theta^2(\tan\theta)']'+(\theta\tan\theta)'-\frac{\pi^2}{8}(\tan\theta)''\\
&=\frac{1}{2}\sum_{n=1}^{\infty}(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}\frac{\theta^{2n-1}}{(2n-2)!}\\
&\quad+\sum_{n=1}^{\infty}(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}\frac{\theta^{2n-1}}{(2n-1)!}\\
&\quad-\frac{\pi^2}{8}\sum_{n=2}^{\infty}(-1)^{n-1}(2n-1)(2n-2)2^{2n}(2^{2n}-1)B_{2n}\frac{\theta^{2n-3}}{(2n)!}\\
&=\sum_{n=1}^{\infty}2^{2n-1}\biggl[(2n+1)(2^{2n}-1)(-1)^{n-1}B_{2n}\\
&\quad-\frac{\pi^2}{2(n+1)}(2^{2n+2}-1)(-1)^n B_{2n+2}\biggr]\frac{\theta^{2n-1}}{(2n-1)!}.
\end{aligned}$$

Accordingly, to prove the absolute monotonicity of the function $-p'(\theta)$, it suffices to show the inequality

$$\frac{|B_{2n+2}|}{|B_{2n}|}=\frac{(-1)^n B_{2n+2}}{(-1)^{n-1}B_{2n}}\le\frac{2^{2n}-1}{2^{2n+2}-1}\frac{2(n+1)(2n+1)}{\pi^2},\quad n\in\mathbb{N}. \tag{2.7}$$

In [32, p. 5, (1.14)], it was listed that

$$B_{2n}=\frac{(-1)^{n+1}2(2n)!}{(2\pi)^{2n}}\sum_{m=1}^{\infty}\frac{1}{m^{2n}},\quad n\in\mathbb{N}. \tag{2.8}$$

Then

$$\frac{(-1)^n B_{2n+2}}{(-1)^{n-1}B_{2n}}=\frac{2(n+1)(2n+1)}{\pi^2}\frac{1}{4}\frac{\sum_{m=1}^{\infty}\frac{1}{m^{2n+2}}}{\sum_{m=1}^{\infty}\frac{1}{m^{2n}}},\quad n\in\mathbb{N}. \tag{2.9}$$

Hence, to prove the inequality (2.7), it is sufficient to verify

$$\frac{1}{4}\frac{\sum_{m=1}^{\infty}\frac{1}{m^{2n+2}}}{\sum_{m=1}^{\infty}\frac{1}{m^{2n}}}\le\frac{2^{2n}-1}{2^{2n+2}-1},\quad n\in\mathbb{N},$$

which may be rearranged as

$$\left(1-\frac{1}{2^{2n+2}}\right)\sum_{m=1}^{\infty}\frac{1}{m^{2n+2}}\le\left(1-\frac{1}{2^{2n}}\right)\sum_{m=1}^{\infty}\frac{1}{m^{2n}},\quad n\in\mathbb{N}.$$

This inequality is a special case of Lemma 2.1 in [3, 40], which may be slightly modified as follows: the sequence

$$\left(1-\frac{1}{2^{n}}\right)\sum_{m=1}^{\infty}\frac{1}{m^{n}}=\sum_{m=1}^{\infty}\frac{1}{m^{n}}-\sum_{m=1}^{\infty}\frac{1}{(2m)^{n}}=\sum_{m=1}^{\infty}\frac{1}{(2m-1)^{n}},\quad n\ge 2$$

is decreasing in n. The proof of Theorem 1.2 is complete. □

Remark 2.1. For more information on the inequality (2.7), please refer to [15, 16] and closely related references therein.

Proof of Theorem 1.3. By definition, it is easy to see that a function $f(x)$ is completely monotonic in (a,b) if and only if $f(-x)$ is absolutely monotonic in $(-b,-a)$. See [33, p. 145, Definition 2c]. Hence, it is sufficient to prove that the function $\phi(-\theta)$ is absolutely monotonic on $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

It is easy to see that

$$\begin{aligned}\phi(-\theta)&=\frac{1}{4}[2\theta+\sin(2\theta)+\pi]\sec^2\theta\\&=\frac{1}{4}[2\theta(\tan\theta)'+2\tan\theta+\pi(\tan\theta)']\\&=\frac{1}{4}[2(\theta\tan\theta)'+\pi(\tan\theta)'].\end{aligned}$$

Utilization of (2.1) leads to

$$\begin{aligned}\phi(-\theta)=\frac{1}{4}\Bigg[&2\sum_{n=1}^{\infty}2^{2n}(2^{2n}-1)(-1)^{n-1}B_{2n}\frac{\theta^{2n-1}}{(2n-1)!}\\&+\pi\sum_{n=1}^{\infty}(2n-1)2^{2n}(2^{2n}-1)(-1)^{n-1}B_{2n}\frac{\theta^{2n-2}}{(2n)!}\Bigg].\end{aligned}$$

Since $(-1)^{n-1}B_{2n}>0$ for all $n\in\mathbb{N}$, all the coefficients of θ^k for $k\ge 0$ in the power series expansion of $\phi(-\theta)$ are positive. Therefore, the function $\phi(-\theta)$ is absolutely monotonic on $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. The proof of Theorem 1.3 is complete. □

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