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GROWTH AND COMPLEX OSCILLATION OF LINEAR DIFFERENTIAL EQUATIONS WITH MEROMORPHIC COEFFICIENTS OF $[p,q]-\varphi$ ORDER

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ABSTRACT. This paper is devoted to considering the growth of solutions of complex higher order linear differential equations with meromorphic coefficients under some assumptions for $[p,q]-\varphi$ order and we obtain some results which improve and extend some previous results of H. Hu and X. M. Zheng; X. Shen, J. Tu and H. Y. Xu and others.

1. **Introduction and main results**

Throughout this paper, a meromorphic function will means meromorphic in the whole complex plane. In this paper, we assume that readers are familiar with the fundamental results and standard notations of the Nevanlinna's theory of meromorphic functions (see [9, 18]).

Consider for $n \geq 2$ the linear differential equations

$$f^{(n)} + A_{n-1}f^{(n-1)} + \cdots + A_1 f' + A_0 f = 0, \tag{1.1}$$

$$f^{(n)} + A_{n-1}f^{(n-1)} + \cdots + A_1 f' + A_0 f = F, \tag{1.2}$$

where $A_0, \cdots, A_{n-1}, F$ are meromorphic functions. In [11, 12], Juneja, Kapoor and Bajpai investigated some properties of entire functions of $[p,q]$-order and obtained some results concerning their growth. In [16], in order to maintain accordance with general definitions of the entire function f of iterated $p-$order [13, 14], Liu-Tu-Shi gave a minor modification of the original definition of the $[p,q]$-order given in [11, 12]. By this new concept of $[p,q]$-order, the $[p,q]$-order of solutions of complex linear differential equations (1.1) and (1.2) was investigated in the unit disc and in the complex plane (see e.g. [2, 3, 4, 15, 16]). In [6], I. Chyzhykov, J. Heittokangas and J. Rättyä introduced the definition of $\varphi-$order of a meromorphic function in the unit disc as follows.

Definition 1.1 ([6]) Let $\varphi : [0,1) \to (0,+\infty)$ be a non-decreasing unbounded function, the $\varphi-$order of f in the unit disc is defined by

$$\sigma(f,\varphi) = \limsup_{r\to+\infty} \frac{\log^+ T(r,f)}{\log \varphi(r)},$$

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and where in the following, $T(r,f)$ is the characteristic function of Nevanlinna.

On the basic of Definition 1.1, recently in [17], X. Shen, J. Tu and H. Y. Xu introduced the new concept of $[p,q]-\varphi$ order of meromorphic functions in the complex plane to study the growth and zeros of second order linear differential equations.

For all $r\in\mathbb{R}$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp\left(\exp_p r\right)$, $p\in\mathbb{N}$. We also define for all r sufficiently large $\log_1 r := \log r$ and $\log_{p+1} r := \log\left(\log_p r\right)$, $p\in\mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$.

Definition 1.2 [17] Let $\varphi : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function, and p,q be positive integers and satisfy $p\geq q\geq 1$. Then the $[p,q]-\varphi$ order and $[p,q]-\varphi$ lower order of a meromorphic function f are respectively defined by

$$\begin{aligned}\sigma_{[p,q]}(f,\varphi) &= \limsup_{r\to+\infty}\frac{\log_p T(r,f)}{\log_q \varphi(r)},\\ \mu_{[p,q]}(f,\varphi) &= \liminf_{r\to+\infty}\frac{\log_p T(r,f)}{\log_q \varphi(r)}.\end{aligned}$$

Definition 1.3 Let f be a meromorphic function satisfying $0<\sigma_{[p,q]}(f,\varphi)=\sigma<\infty$. Then the $[p,q]-\varphi$ type of $f(z)$ is defined by

$$\tau_{[p,q]}(f,\varphi) = \limsup_{r\to+\infty}\frac{\log_{p-1} T(r,f)}{\left[\log_{q-1}\varphi(r)\right]^{\sigma}}.$$

Definition 1.4 Let p,q be integers such that $p\geq q\geq 1$. Let f be a meromorphic function satisfying $0<\mu_{[p,q]}(f,\varphi)=\mu<\infty$. Then the lower $[p,q]-\varphi$ type of f is defined by

$$\underline{\tau}_{[p,q]}(f,\varphi) = \liminf_{r\to+\infty}\frac{\log_{p-1} T(r,f)}{\left[\log_{q-1}\varphi(r)\right]^{\mu}}.$$

Definition 1.5 ([17]) Let f be a meromorphic function. Then, the $[p,q]-\varphi$ exponent of convergence of zero-sequence (distinct zero-sequence) of f is defined by

$$\begin{aligned}\lambda_{[p,q]}(f,\varphi) &= \limsup_{r\to+\infty}\frac{\log_p n\left(r,\frac{1}{f}\right)}{\log_q \varphi(r)},\\ \bar{\lambda}_{[p,q]}(f,\varphi) &= \limsup_{r\to+\infty}\frac{\log_p \bar{n}\left(r,\frac{1}{f}\right)}{\log_q \varphi(r)}.\end{aligned}$$

And the lower exponent of distinct zero-sequence of f is defined by

$$\underline{\bar{\lambda}}_{[p,q]}(f,\varphi) = \liminf_{r\to+\infty}\frac{\log_p \bar{n}\left(r,\frac{1}{f}\right)}{\log_q \varphi(r)}.$$

Remark 1.1. If $\varphi(r)=r$ in the Definitions 1.2-1.5, then we obtain the standard definitions of the $[p,q]-$order, $[p,q]-$type and $[p,q]-$exponent of convergence.

Remark 1.2 [17] Throughout this paper, we assume that $\varphi : [0, +\infty) \to (0, +\infty)$ is a non-decreasing unbounded function and always satisfies the following two conditions :
(i) $\lim\limits_{r\to+\infty} \frac{\log_{p+1} r}{\log_q \varphi(r)} = 0$.
(ii) $\lim\limits_{r\to+\infty} \frac{\log_q \varphi(\alpha r)}{\log_q \varphi(r)} = 1$ for some $\alpha > 1$.

From Remark 1.2, we can obtain the following proposition.

Proposition 1.1 Suppose that $\varphi(r)$ satisfies the condition (i) $-$ (ii).
a) If $f(z)$ is a meromorphic function, then

$$\begin{aligned}\lambda_{[p,q]}(f,\varphi) &= \limsup_{r\to+\infty} \frac{\log_p n\left(r,\frac{1}{f}\right)}{\log_q \varphi(r)} = \limsup_{r\to+\infty} \frac{\log_p N\left(r,\frac{1}{f}\right)}{\log_q \varphi(r)},\\ \bar{\lambda}_{[p,q]}(f,\varphi) &= \limsup_{r\to+\infty} \frac{\log_p \bar{n}\left(r,\frac{1}{f}\right)}{\log_q \varphi(r)} = \limsup_{r\to+\infty} \frac{\log_p \overline{N}\left(r,\frac{1}{f}\right)}{\log_q \varphi(r)}.\end{aligned}$$

b) If $f(z)$ is a meromorphic function, then

$$\underline{\bar{\lambda}}_{[p,q]}(f,\varphi) = \liminf_{r\to+\infty} \frac{\log_p \bar{n}\left(r,\frac{1}{f}\right)}{\log_q \varphi(r)} = \liminf_{r\to+\infty} \frac{\log_p \overline{N}\left(r,\frac{1}{f}\right)}{\log_q \varphi(r)}.$$

Proof. We prove only b), for the proof of a) see [17]. We have

$$\overline{N}\left(r,\frac{1}{f}\right) = \int_0^r \frac{\bar{n}\left(t,\frac{1}{f}\right) - \bar{n}\left(0,\frac{1}{f}\right)}{t} dt + \bar{n}\left(0,\frac{1}{f}\right)\log r.$$

It follows that for $r > r_0 > 1$

$$\begin{aligned}\overline{N}\left(r,\frac{1}{f}\right) - \overline{N}\left(r_0,\frac{1}{f}\right) &= \int_0^r \frac{\bar{n}\left(t,\frac{1}{f}\right) - \bar{n}\left(0,\frac{1}{f}\right)}{t} dt + \bar{n}\left(0,\frac{1}{f}\right)\log r\\ &\quad - \left(\int_0^{r_0} \frac{\bar{n}\left(t,\frac{1}{f}\right) - \bar{n}\left(0,\frac{1}{f}\right)}{t} dt + \bar{n}\left(0,\frac{1}{f}\right)\log r_0\right)\\ &= \int_{r_0}^r \frac{\bar{n}\left(t,\frac{1}{f}\right) - \bar{n}\left(0,\frac{1}{f}\right)}{t} dt + \bar{n}\left(0,\frac{1}{f}\right)(\log r - \log r_0)\end{aligned}$$

$$= \int_{r_0}^r \frac{\bar{n}\left(t,\frac{1}{f}\right)}{t} dt \leq \bar{n}\left(r,\frac{1}{f}\right)\log\frac{r}{r_0}. \tag{1.3}$$

Then by (1.3) and $\lim\limits_{r\to+\infty} \frac{\log_{p+1} r}{\log_q \varphi(r)} = 0$, we obtain

$$\liminf_{r\to+\infty} \frac{\log_p \overline{N}\left(r,\frac{1}{f}\right)}{\log_q \varphi(r)}$$

$$\leq \max\left\{\liminf_{r\to+\infty}\frac{\log_p \bar{n}\left(r,\frac{1}{f}\right)}{\log_q \varphi(r)},\limsup_{r\to+\infty}\frac{\log_{p+1} r}{\log_q \varphi(r)}\right\}=\liminf_{r\to+\infty}\frac{\log_p \bar{n}\left(r,\frac{1}{f}\right)}{\log_q \varphi(r)}. \tag{1.4}$$

On the other hand, since $\alpha>1$, we have for $r>1$

$$\overline{N}\left(\alpha r,\frac{1}{f}\right)=\int_0^{\alpha r}\frac{\bar{n}\left(t,\frac{1}{f}\right)-\bar{n}\left(0,\frac{1}{f}\right)}{t}dt+\bar{n}\left(0,\frac{1}{f}\right)\log\alpha r$$

$$\geq\int_r^{\alpha r}\frac{\bar{n}\left(t,\frac{1}{f}\right)-\bar{n}\left(0,\frac{1}{f}\right)}{t}dt+\bar{n}\left(0,\frac{1}{f}\right)\log\alpha r$$

$$\geq\left(\bar{n}\left(r,\frac{1}{f}\right)-\bar{n}\left(0,\frac{1}{f}\right)\right)\log\alpha+\bar{n}\left(0,\frac{1}{f}\right)\log\alpha r$$

$$=\bar{n}\left(r,\frac{1}{f}\right)\log\alpha+\bar{n}\left(0,\frac{1}{f}\right)\log r\geq\bar{n}\left(r,\frac{1}{f}\right)\log\alpha. \tag{1.5}$$

By (1.5) and $\lim\limits_{r\to+\infty}\frac{\log_q\varphi(\alpha r)}{\log_q\varphi(r)}=1$, we get

$$\liminf_{r\to+\infty}\frac{\log_p \overline{N}\left(\alpha r,\frac{1}{f}\right)}{\log_q\varphi(\alpha r)}\geq\liminf_{r\to+\infty}\left(\frac{\log_p\bar{n}\left(r,\frac{1}{f}\right)}{\log_q\varphi(r)}.\frac{\log_q\varphi(r)}{\log_q\varphi(\alpha r)}\right)$$

$$\geq\liminf_{r\to+\infty}\frac{\log_p\bar{n}\left(r,\frac{1}{f}\right)}{\log_q\varphi(r)}.\liminf_{r\to+\infty}\frac{\log_q\varphi(r)}{\log_q\varphi(\alpha r)}$$

$$=\liminf_{r\to+\infty}\frac{\log_p\bar{n}\left(r,\frac{1}{f}\right)}{\log_q\varphi(r)}. \tag{1.6}$$

By (1.4) and (1.6), it is easy to see that conclusion of b) holds.

Many authors have investigated complex oscillation properties of (1.1) and obtained many results when the coefficients in (1.1) are entire or meromorphic functions under some assumptions of $[p,q]$ −order. Recently, Hu and Zheng investigated the growth of solutions of (1.1) and obtained the following results.

Theorem A ([10]) *Let p,q be integers such that $p\geq q>1$ or $p>q=1$, and let $A_0,\cdots,A_{n-1}$ be meromorphic functions. Assume that $\lambda_{[p,q]}\left(\frac{1}{A_0}\right)<\mu_{[p,q]}(A_0)<\infty$, and that $\max\left\{\sigma_{[p,q]}(A_j),j=1,\cdots,n-1\right\}\leq\mu_{[p,q]}(A_0)$ and $\max\left\{\tau_{[p,q]}(A_j):\sigma_{[p,q]}(A_j)=\mu_{[p,q]}(A_0),j\neq0\right\}<\underline{\tau}_{[p,q]}(A_0)=\tau$. If $f\,(\not\equiv0)$ is a meromorphic solution of (1.1) satisfying*

$$\frac{N(r,f)}{\overline{N}(r,f)}<\exp_{p+1}\left\{b\log_q r\right\}\quad\left(\,b\leq\mu_{[p,q]}(A_0)\,\right),$$

then we have

$$\underline{\bar{\lambda}}_{[p+1,q]}(f-\psi)=\mu_{[p+1,q]}(f)=\mu_{[p,q]}(A_0)$$
$$\leq\sigma_{[p,q]}(A_0)=\sigma_{[p+1,q]}(f)=\bar{\lambda}_{[p+1,q]}(f-\psi),$$

where $\psi(z)(\not\equiv 0)$ *is a meromorphic function with* $\sigma_{[p+1,q]}(\psi) < \mu_{[p,q]}(A_0)$.

Theorem B ([10]) *Let* p, q *be integers such that* $p \geq q > 1$ *or* $p > q = 1$, *and let* $A_0, \cdots, A_{n-1}$ *be meromorphic functions. Assume that* $\lambda_{[p,q]}\left(\frac{1}{A_0}\right) < \mu_{[p,q]}(A_0) < \infty$, *and that* $\max\left\{\sigma_{[p,q]}(A_j), j = 1, \cdots, n-1\right\} \leq \mu_{[p,q]}(A_0)$ *and* $\limsup_{r\to+\infty} \sum_{j=1}^{n-1} \frac{m(r,A_j)}{m(r,A_0)} < 1$. *If* $f(\not\equiv 0)$ *is a meromorphic solution of* (1.1) *satisfying* $\frac{N(r,f)}{\overline{N}(r,f)} < \exp_{p+1}\left\{b\log_q r\right\}$ $\left(b \leq \mu_{[p,q]}(A_0)\right)$, *then we have*

$$\underline{\bar{\lambda}}_{[p+1,q]}(f-\psi) = \mu_{[p+1,q]}(f) = \mu_{[p,q]}(A_0)$$
$$\leq \sigma_{[p,q]}(A_0) = \sigma_{[p+1,q]}(f) = \bar{\lambda}_{[p+1,q]}(f-\psi),$$

where $\psi(z)(\not\equiv 0)$ *is a meromorphic function with* $\sigma_{[p+1,q]}(\psi) < \mu_{[p,q]}(A_0)$.

For the case that the dominant coefficient A_0 is replaced by an arbitrary coefficient A_s $(s \in \{1, \cdots, n-1\})$, they obtained the following.

Theorem C ([10]) *Let* p, q *be integers such that* $p \geq q \geq 1$, *and let* $A_0, \cdots, A_{n-1}$ *be meromorphic functions. Suppose that there exists one* A_s $(0 \leq s \leq n-1)$ *with* $\lambda_{[p,q]}\left(\frac{1}{A_s}\right) < \mu_{[p,q]}(A_s) < \infty$ *and that* $\max\{\sigma_{[p,q]}(A_j), j \neq s\} \leq \mu_{[p,q]}(A_s)$ *and* $\max\{\tau_{[p,q]}(A_j) : \sigma_{[p,q]}(A_j) = \mu_{[p,q]}(A_s), j \neq s\} < \underline{\tau}_{[p,q]}(A_s) = \tau$. *Then every transcendental meromorphic solution* $f(\not\equiv 0)$ *of* (1.1) *satisfying* $\frac{N(r,f)}{\overline{N}(r,f)} < \exp_{p+1}\left\{b\log_q r\right\}$ $(b \leq \mu_{[p,q]}(A_s))$ *satisfies* $\mu_{[p+1,q]}(f) \leq \mu_{[p,q]}(A_s) \leq \mu_{[p,q]}(f)$ *and* $\sigma_{[p+1,q]}(f) \leq \sigma_{[p,q]}(A_s) \leq \sigma_{[p,q]}(f)$. *Moreover, every non-transcendental meromorphic solution* f *of* (1.1) *is a polynomial with degree* $\deg(f) \leq s-1$.

The main purpose of this paper is to make use of the concept of meromorphic functions of $[p,q]-\varphi$-order to improve the results above.

Theorem 1.1 *Let* p, q *be integers such that* $p \geq q > 1$ *or* $p > q = 1$, *and let* $A_0, \cdots, A_{n-1}$ *be meromorphic functions.*
Assume that $\lambda_{[p,q]}\left(\frac{1}{A_0}, \varphi\right) < \mu_{[p,q]}(A_0, \varphi) < \infty$, *and that* $\max\{\sigma_{[p,q]}(A_j, \varphi), j = 1, \cdots, n-1\} \leq \mu_{[p,q]}(A_0, \varphi)$ *and* $\max\{\tau_{[p,q]}(A_j, \varphi) : \sigma_{[p,q]}(A_j, \varphi) = \mu_{[p,q]}(A_0, \varphi), j \neq 0\} < \underline{\tau}_{[p,q]}(A_0, \varphi) = \tau$, *and where* φ *satisfies the conditions* $\lim_{r\to+\infty} \frac{\log_{p+1} r}{\log_q \varphi(r)} = 0$ *and* $\lim_{r\to+\infty} \frac{\log_{q-1} \varphi(\alpha r)}{\log_{q-1} \varphi(r)} = 1$ *for some* $\alpha > 1$. *If* $f(\not\equiv 0)$ *is a meromorphic solution of* (1.1) *satisfying* $\frac{N(r,f)}{\overline{N}(r,f)} < \exp_{p+1}\left\{b\log_q \varphi(r)\right\}$ $\left(b \leq \mu_{[p,q]}(A_0, \varphi)\right)$, *then we have*

$$\underline{\bar{\lambda}}_{[p+1,q]}(f-\psi, \varphi) = \mu_{[p+1,q]}(f, \varphi) = \mu_{[p,q]}(A_0, \varphi)$$
$$\leq \sigma_{[p,q]}(A_0, \varphi) = \sigma_{[p+1,q]}(f, \varphi) = \bar{\lambda}_{[p+1,q]}(f-\psi, \varphi),$$

where $\psi(z)(\not\equiv 0)$ *is a meromorphic function with* $\sigma_{[p+1,q]}(\psi, \varphi) < \mu_{[p,q]}(A_0, \varphi)$.

Theorem 1.2 *Let* p, q *be integers such that* $p \geq q > 1$ *or* $p > q = 1$, *and let* $A_0, \cdots, A_{n-1}$ *be meromorphic functions.*
Assume that $\lambda_{[p,q]}\left(\frac{1}{A_0}, \varphi\right) < \mu_{[p,q]}(A_0, \varphi) < \infty$, *and that* $\max\{\sigma_{[p,q]}(A_j, \varphi), j =$

$1, \cdots, n-1\} \leq \mu_{[p,q]}(A_0, \varphi)$ *and* $\limsup\limits_{r\to+\infty} \sum\limits_{j=1}^{n-1} \frac{m(r,A_j)}{m(r,A_0)} < 1$, *and where* φ *satisfies the conditions* (i) − (ii) *of the Remark 1.2. If* $f\,(\not\equiv 0)$ *is a meromorphic solution of* (1.1) *satisfying* $\frac{N(r,f)}{\overline{N}(r,f)} < \exp_{p+1}\left\{b \log_q \varphi(r)\right\} \left(b \leq \mu_{[p,q]}(A_0, \varphi)\right)$, *then we have*

$$\underline{\bar{\lambda}}_{[p+1,q]}(f-\psi, \varphi) = \mu_{[p+1,q]}(f, \varphi) = \mu_{[p,q]}(A_0, \varphi)$$
$$\leq \sigma_{[p,q]}(A_0, \varphi) = \sigma_{[p+1,q]}(f, \varphi) = \bar{\lambda}_{[p+1,q]}(f-\psi, \varphi),$$

where $\psi(z)\,(\not\equiv 0)$ *is a meromorphic function with* $\sigma_{[p+1,q]}(\psi, \varphi) < \mu_{[p,q]}(A_0, \varphi)$.

Theorem 1.3 *Let* p, q *be integers such that* $p \geq q \geq 1$, *and let* $A_0, \cdots, A_{n-1}$ *be meromorphic functions. Suppose that there exists one* $A_s\,(0 \leq s \leq n-1)$ *with* $\lambda_{[p,q]}\left(\frac{1}{A_s}, \varphi\right) < \mu_{[p,q]}(A_s, \varphi) < \infty$ *and that* $\max\left\{\sigma_{[p,q]}(A_j, \varphi), j \neq s\right\} \leq \mu_{[p,q]}(A_s, \varphi)$ *and* $\max\left\{\tau_{[p,q]}(A_j, \varphi) : \ \sigma_{[p,q]}(A_j, \varphi) = \mu_{[p,q]}(A_s, \varphi),\ j \neq s\right\} < \underline{\tau}_{[p,q]}(A_s, \varphi) = \tau$, *and where* φ *satisfies the conditions* (i) − (ii) *of the Remark 1.2. Then every transcendental meromorphic solution* $f\,(\not\equiv 0)$ *of* (1.1) *satisfying*
$\frac{N(r,f)}{\overline{N}(r,f)} < \exp_{p+1}\left\{b \log_q \varphi(r)\right\} \left(b \leq \mu_{[p,q]}(A_s, \varphi)\right)$
satisfies $\mu_{[p+1,q]}(f, \varphi) \leq \mu_{[p,q]}(A_s, \varphi) \leq \mu_{[p,q]}(f, \varphi)$ *and* $\sigma_{[p+1,q]}(f, \varphi) \leq \sigma_{[p,q]}(A_s, \varphi) \leq \sigma_{[p,q]}(f, \varphi)$. *Moreover, every non-transcendental meromorphic solution* $f(z)$ *of* (1.1) *is a polynomial with degree* $\deg(f) \leq s-1$.

Remark 1.3. If we put $\varphi(r) = r$ in the Theorems 1.1, 1.2, 1.3, then we obtain Theorems A, B, C.

2. Auxiliary lemmas

We need the following lemmas to obtain our results.

Lemma 2.1 ([5]) *Let* f *be a meromorphic solution of* (1.1) *assuming that not all coefficients* $A_j(z)$ *are constants. Given a real constant* $\gamma > 1$, *and denoting* $T(r) = \sum\limits_{j=0}^{n-1} T(r, A_j)$, *we have*

$$\log m(r,f) < T(r)\left\{(\log r)\log T(r)\right\}^{\gamma}, \ \textit{if } p = 0$$

and

$$\log m(r,f) < r^{2p+\gamma-1} T(r)\left\{\log T(r)\right\}^{\gamma}, \ \textit{if } p > 0,$$

outside of an exceptional set E_p *with* $\int\limits_{E_p} t^{p-1} dt < +\infty$.

Remark 2.1. *Especially, if* $p = 0$, *then the exceptional set* E_0 *has finite logarithmic measure* $\int\limits_{E_0} \frac{dt}{t} = m_l E_0$.

Lemma 2.2 ([1], [8]) *Let* $g : [0, +\infty) \to \mathbb{R}$, $h : [0, +\infty) \to \mathbb{R}$ *be monotone increasing functions. If* (i) $g(r) \leq h(r)$ *outside of an exceptional set of finite linear measure, or* (ii) $g(r) \leq h(r)$, $r \notin E_1 \cup (0, 1]$, *where* $E_1 \subset [1, \infty)$ *is a set of finite logarithmic measure, then for any* $\beta > 1$, *there exists* $r_0 = r_0(\beta) > 0$ *such that* $g(r) \leq h(\beta r)$ *for all* $r > r_0$.

Lemma 2.3 ([9]) *Let f be a transcendental meromorphic function and $n \geq 1$ be an integer. Then*

$$m\left(r, \frac{f^{(n)}}{f}\right) = O\left(\log\left(rT\left(r,f\right)\right)\right)$$

outside of a possible exceptional set E_2 of r of finite linear measure, and if f is of finite order of growth, then

$$m\left(r, \frac{f^{(n)}}{f}\right) = O\left(\log r\right).$$

Lemma 2.4 *Let p, q be integers such that $p \geq q \geq 1$, and let f be a meromorphic function satisfying $\mu_{[p,q]}\left(f,\varphi\right) = \mu < \infty$ $\left(\sigma_{[p,q]}\left(f,\varphi\right) = \sigma < \infty\right)$, where $\varphi\left(r\right)$ only satisfies $\lim\limits_{r\to+\infty} \frac{\log_q \varphi(\alpha r)}{\log_q \varphi(r)} = 1$ for some $\alpha > 1$. Then there exists a set $E_3 \subset (1,\infty)$ of infinite logarithmic measure such that for all $r \in E_3$, we have*

$$\mu = \lim_{\substack{r\to+\infty \\ r\in E_3}} \frac{\log_p T\left(r,f\right)}{\log_q \varphi\left(r\right)}, \quad \left(\sigma = \lim_{\substack{r\to+\infty \\ r\in E_3}} \frac{\log_p T\left(r,f\right)}{\log_q \varphi\left(r\right)}\right)$$

and for any given $\varepsilon > 0$ and sufficiently large $r \in E_3$

$$T\left(r,f\right) < \exp_p\left\{\left(\mu+\varepsilon\right)\log_q \varphi\left(r\right)\right\} \quad \left(T\left(r,f\right) > \exp_p\left\{\left(\sigma-\varepsilon\right)\log_q \varphi\left(r\right)\right\}\right).$$

Proof. We prove only the first assumption, for the second we use the same proof. By the Definition 1.2, there exists an increasing sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ satisfying $\left(1+\frac{1}{n+1}\right)r_n < r_{n+1}$ and

$$\mu = \mu_{[p,q]}\left(f,\varphi\right) = \lim_{r_n\to\infty} \frac{\log_p T\left(r_n,f\right)}{\log_q \varphi\left(r_n\right)}.$$

Then for any given $\varepsilon > 0$, there exists an n_1 such that for $n \geq n_1$ and any $r \in \left[r_n, \left(1+\frac{1}{n}\right)r_n\right]$, we have

$$\frac{\log_p T\left(r,f\right)}{\log_q \varphi\left(r\right)} \leq \frac{\log_p T\left(\left(1+\frac{1}{n}\right)r_n,f\right)}{\log_q \varphi\left(\left(1+\frac{1}{n}\right)r_n\right)} \frac{\log_q \varphi\left(\left(1+\frac{1}{n}\right)r_n\right)}{\log_q \varphi\left(r_n\right)}.$$

When $q \geq 1$, we have $\frac{\log_q \varphi\left(\left(1+\frac{1}{n}\right)r_n\right)}{\log_q \varphi(r_n)} \to 1$ $(n\to+\infty)$. Let

$$E_3 = \bigcup_{n=n_1}^{\infty}\left[r_n, (1+\frac{1}{n})r_n\right],$$

for any given $\varepsilon > 0$ and all $r \in E_3$, we have

$$\lim_{\substack{r\to+\infty \\ r\in E_3}} \frac{\log_p T\left(r,f\right)}{\log_q \varphi\left(r\right)} \leq \lim_{r_n\to\infty} \frac{\log_p T\left(\left(1+\frac{1}{n}\right)r_n,f\right)}{\log_q \varphi\left(\left(1+\frac{1}{n}\right)r_n\right)} = \mu_{[p,q]}\left(f,\varphi\right),$$

where $m_l E_3 = \sum\limits_{n=n_1}^{\infty} \int\limits_{r_n}^{(1+\frac{1}{n})r_n} \frac{dt}{t} = \sum\limits_{n=n_1}^{\infty} \log\left(1+\frac{1}{n}\right) = \infty$. On the other hand, we have

$$\lim_{\substack{r\to+\infty \\ r\in E_3}} \frac{\log_p T\left(r,f\right)}{\log_q \varphi\left(r\right)} \geq \liminf_{r\to+\infty} \frac{\log_p T\left(r,f\right)}{\log_q \varphi\left(r\right)} = \mu_{[p,q]}\left(f,\varphi\right).$$

Therefore,

$$\lim_{\substack{r\to+\infty \\ r\in E_3}} \frac{\log_p T(r,f)}{\log_q \varphi(r)} = \mu_{[p,q]}(f,\varphi)$$

and for any given $\varepsilon > 0$ and sufficiently large $r \in E_3$

$$T(r,f) < \exp_p\left\{(\mu+\varepsilon)\log_q \varphi(r)\right\}.$$

Lemma 2.5 *Let f_1, f_2 be meromorphic functions of $[p,q]-\varphi$ order satisfying $\sigma_{[p,q]}(f_1,\varphi) > \sigma_{[p,q]}(f_2,\varphi)$, where $\varphi(r)$ only satisfies $\lim_{r\to+\infty} \frac{\log_q \varphi(\alpha r)}{\log_q \varphi(r)} = 1$ for some $\alpha > 1$. Then there exists a set $E_4 \subset (1,+\infty)$ having infinite logarithmic measure such that for all $r \in E_4$, we have*

$$\lim_{r\to+\infty} \frac{T(r,f_2)}{T(r,f_1)} = 0.$$

Proof. Set $\sigma_1 = \sigma_{[p,q]}(f_1,\varphi)$, $\sigma_2 = \sigma_{[p,q]}(f_2,\varphi)$ $(\sigma_1 > \sigma_2)$. By Lemma 2.4, there exists a set $E_4 \subset (1,+\infty)$ having infinite logarithmic measure such that for any given $0 < \varepsilon < \frac{\sigma_1-\sigma_2}{2}$ and all sufficiently large $r \in E_4$

$$T(r,f_1) > \exp_p\left\{(\sigma_1-\varepsilon)\log_q \varphi(r)\right\}$$

and for all sufficiently large r

$$T(r,f_2) < \exp_p\left\{(\sigma_2+\varepsilon)\log_q \varphi(r)\right\}.$$

From this we can get

$$\frac{T(r,f_2)}{T(r,f_1)} < \frac{\exp_p\left\{(\sigma_2+\varepsilon)\log_q \varphi(r)\right\}}{\exp_p\left\{(\sigma_1-\varepsilon)\log_q \varphi(r)\right\}}$$

$$= \frac{1}{\exp\left\{\exp_{p-1}\left\{(\sigma_1-\varepsilon)\log_q \varphi(r)\right\} - \exp_{p-1}\left\{(\sigma_2+\varepsilon)\log_q \varphi(r)\right\}\right\}},\ r \in E_4.$$

Since $0 < \varepsilon < \frac{\sigma_1-\sigma_2}{2}$, then we have

$$\lim_{r\to+\infty} \frac{T(r,f_2)}{T(r,f_1)} = 0,\ r \in E_4.$$

Remark 2.2 *If $\mu_{[p,q]}(f_1,\varphi) > \mu_{[p,q]}(f_2,\varphi)$, then we get the same result.*

Lemma 2.6 *Let p,q be integers such that $p \geq q \geq 1$, and let $A_0, \cdots, A_{n-1}, F\,(\not\equiv 0)$ be meromorphic functions. If f is a meromorphic solution of* (1.2) *satisfying*

$$\max\left\{\sigma_{[p,q]}(F,\varphi), \sigma_{[p,q]}(A_j,\varphi), j = 0, \cdots, n-1\right\} < \mu_{[p,q]}(f,\varphi),$$

then we have

$$\overline{\lambda}_{[p,q]}(f,\varphi) = \underline{\lambda}_{[p,q]}(f,\varphi) = \mu_{[p,q]}(f,\varphi),$$

where φ satisfies the conditions (i) – (ii) *of Remark 1.2.*

Proof. By (1.2), we get

$$\frac{1}{f} = \frac{1}{F}\left(\frac{f^{(n)}}{f} + A_{n-1}\frac{f^{(n-1)}}{f} + \cdots + A_1\frac{f'}{f} + A_0\right). \tag{2.1}$$

It is easy to see that if f has a zero at z_0 of order $\alpha\,(\alpha > n)$, and $A_0, \cdots, A_{n-1}$ are analytic at z_0, then F must have a zero at z_0 of order $\alpha - n$. Hence

$$N\left(r,\frac{1}{f}\right) \leq n\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{F}\right) + \sum_{j=0}^{n-1} N\left(r, A_j\right). \tag{2.2}$$

By the Lemma 2.3 and (2.1), we have

$$m\left(r,\frac{1}{f}\right) \leq m\left(r,\frac{1}{F}\right) + \sum_{j=0}^{n-1} m\left(r, A_j\right) + O\left(\log T\left(r,f\right) + \log r\right) \quad \left(r \notin E_2\right), \tag{2.3}$$

where $E_2 \subset (1, +\infty)$ is a set of r of finite linear measure. By (2.2) and (2.3), we get

$$T\left(r,f\right) = T\left(r,\frac{1}{f}\right) + O\left(1\right) \leq n\overline{N}\left(r,\frac{1}{f}\right) + T\left(r,F\right)$$

$$+ \sum_{j=0}^{n-1} T\left(r, A_j\right) + O\left\{\log\left(rT\left(r,f\right)\right)\right\} \quad \left(r \notin E_2\right). \tag{2.4}$$

Since $\max\{\sigma_{[p,q]}\left(F,\varphi\right), \sigma_{[p,q]}\left(A_j,\varphi\right), j = 0, \cdots, n-1\} < \mu_{[p,q]}\left(f,\varphi\right)$, then

$$\max\left\{\frac{T\left(r,F\right)}{T\left(r,f\right)}, \frac{T\left(r,A_j\right)}{T\left(r,f\right)} \quad \left(j = 0, \cdots, n-1\right)\right\} \to 0, \; r \to +\infty. \tag{2.5}$$

Also, for all sufficiently large r, we have

$$\log\left(T\left(r,f\right)\right) = o\left\{T\left(r,f\right)\right\}. \tag{2.6}$$

By $(2.4) - (2.6)$, for all $|z| = r \notin E_2$, we have

$$\left(1 - o\left(1\right)\right) T\left(r,f\right) \leq n\overline{N}\left(r,\frac{1}{f}\right) + O\left(\log r\right). \tag{2.7}$$

By Definition 1.2, Proposition 1.1, Lemma 2.2 and (2.7), we get

$$\mu_{[p,q]}\left(f,\varphi\right) \leq \underline{\bar{\lambda}}_{[p,q]}\left(f,\varphi\right). \tag{2.8}$$

Since $\mu_{[p,q]}\left(f,\varphi\right) \geq \underline{\lambda}_{[p,q]}\left(f,\varphi\right) \geq \underline{\bar{\lambda}}_{[p,q]}\left(f,\varphi\right)$, then by (2.8), we have

$$\underline{\bar{\lambda}}_{[p,q]}\left(f,\varphi\right) = \underline{\lambda}_{[p,q]}\left(f,\varphi\right) = \mu_{[p,q]}\left(f,\varphi\right).$$

Using the same method above, Lemma 2.5 and Lemma 2.2 we can prove the following lemma.

Lemma 2.7 *Let p, q be integers such that $p \geq q \geq 1$, and let $A_0, \cdots, A_{n-1}$, $F\,(\not\equiv 0)$ be meromorphic functions. If f is a meromorphic solution of (1.2) satisfying*

$$\max\left\{\sigma_{[p,q]}\left(F,\varphi\right), \sigma_{[p,q]}\left(A_j,\varphi\right), j = 0, \cdots, n-1\right\} < \sigma_{[p,q]}\left(f,\varphi\right) < +\infty,$$

then we have

$$\overline{\lambda}_{[p,q]}\left(f,\varphi\right) = \lambda_{[p,q]}\left(f,\varphi\right) = \sigma_{[p,q]}\left(f,\varphi\right),$$

where φ satisfies the conditions (i) $-$ (ii) *of Remark 1.2.*

Lemma 2.8 *Let p, q be integers such that $p \geq q \geq 1$ and let $A_0, \cdots, A_{n-1}$ be meromorphic functions such that* $\max\left\{\sigma_{[p,q]}\left(A_j,\varphi\right) : j \neq s\right\} \leq \mu_{[p,q]}\left(A_s,\varphi\right) < \infty$, *where φ satisfies the conditions* (i)$-$(ii) *of Remark 1.2. If $f\,(\not\equiv 0)$ is a meromorphic* *solution of* (1.1) *satisfying* $\frac{N(r,f)}{\overline{N}(r,f)} < \exp_{p+1}\left\{b\log_q \varphi(r)\right\}$ $\left(b \leq \mu_{[p,q]}(A_s,\varphi)\right)$, *then we have*

$$\mu_{[p+1,q]}(f,\varphi) \leq \mu_{[p,q]}(A_s,\varphi).$$

Proof. By (1.1), we know that the poles of f can only occur at the poles of $A_0, \cdots, A_{n-1}$. By $\frac{N(r,f)}{\overline{N}(r,f)} < \exp_{p+1}\left\{b\log_q \varphi(r)\right\}$ $\left(b \leq \mu_{[p,q]}(A_s,\varphi)\right)$, we have

$$N(r,f) < \exp_{p+1}\left\{b\log_q \varphi(r)\right\} \overline{N}(r,f) \leq \exp_{p+1}\left\{b\log_q \varphi(r)\right\} \sum_{j=0}^{n-1} \overline{N}(r,A_j)$$

$$\leq \exp_{p+1}\left\{b\log_q \varphi(r)\right\} \sum_{j=0}^{n-1} T(r,A_j). \tag{2.9}$$

Then by (2.9), we have

$$T(r,f) \leq m(r,f) + \exp_{p+1}\left\{b\log_q \varphi(r)\right\} \sum_{j=0}^{n-1} T(r,A_j). \tag{2.10}$$

By Lemma 2.4, there exists a set E_3 of infinite logarithmic measure such that for any given $\varepsilon > 0$ and sufficiently large $r \in E_3$, we have

$$T(r,A_s) \leq \exp_p\left\{\left(\mu_{[p,q]}(A_s,\varphi) + \varepsilon\right)\log_q \varphi(r)\right\}. \tag{2.11}$$

Since $\max\left\{\sigma_{[p,q]}(A_j,\varphi) : j \neq s\right\} \leq \mu_{[p,q]}(A_s,\varphi)$, for the above $\varepsilon > 0$ and sufficiently large r, we have

$$T(r,A_j) \leq \exp_p\left\{\left(\mu_{[p,q]}(A_s,\varphi) + \varepsilon\right)\log_q \varphi(r)\right\},\ j \neq s. \tag{2.12}$$

By (2.11), (2.12), Lemma 1.1 and Remark 1.2, there exists a set E_0 of r of finite logarithmic measure such that for sufficiently large $r \in E_3 \setminus E_0$

$$m(r,f) \leq \exp\left\{\sum_{j=0}^{n-1} T(r,A_j)\left[(\log r)\log\left(\sum_{j=0}^{n-1} T(r,A_j)\right)\right]^{\gamma}\right\}$$

$$\leq \exp_{p+1}\left\{\left(\mu_{[p,q]}(A_s,\varphi) + 2\varepsilon\right)\log_q \varphi(r)\right\}. \tag{2.13}$$

From (2.10) and (2.13), we get

$$\liminf_{r\to+\infty} \frac{\log_{p+1} T(r,f)}{\log_q \varphi(r)} \leq \liminf_{\substack{r\to+\infty \\ r\in E_3\setminus E_0}} \frac{\log_{p+1} T(r,f)}{\log_q \varphi(r)} \leq \mu_{[p,q]}(A_s,\varphi) + 3\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $\mu_{[p+1,q]}(f,\varphi) \leq \mu_{[p,q]}(A_s,\varphi)$.

Lemma 2.9 *Let* p, q *be integers such that* $p \geq q > 1$ *or* $p > q = 1$ *and let* $A_0, \cdots, A_{n-1}$ *be meromorphic functions. Assume that* $\lambda_{[p,q]}\left(\frac{1}{A_0},\varphi\right) < \mu_{[p,q]}(A_0,\varphi)$ *and that* $\max\left\{\sigma_{[p,q]}(A_j,\varphi) : j = 1, \cdots, n-1\right\} \leq \mu_{[p,q]}(A_0,\varphi) = \mu$, $0 < \mu < \infty$, *and* $\max\left\{\tau_{[p,q]}(A_j,\varphi) : \sigma_{[p,q]}(A_j,\varphi) = \mu_{[p,q]}(A_0,\varphi), j \neq 0\right\} < \underline{\tau}_{[p,q]}(A_0,\varphi) = \tau$, $0 < \tau < \infty$, *where* φ *satisfies the conditions* (i) – (ii) *of Remark 1.2. If* $f(\not\equiv 0)$ *is a meromorphic solution of* (1.1), *then we have*

$$\mu_{[p+1,q]}(f,\varphi) \geq \mu_{[p,q]}(A_0,\varphi).$$

Proof. Suppose that $f\,(\not\equiv 0)$ is a meromorphic solution of (1.1). By (1.1), we obtain

$$-A_0=\frac{f^{(n)}}{f}+A_{n-1}\frac{f^{(n-1)}}{f}+\cdots+A_1\frac{f'}{f}. \tag{2.14}$$

By $\lambda_{[p,q]}\left(\frac{1}{A_0},\varphi\right)<\mu_{[p,q]}(A_0,\varphi)$, we have $N(r,A_0)=o(T(r,A_0))$, $r\to+\infty$. Then by (2.14), we get

$$T(r,A_0)=m(r,A_0)+N(r,A_0)\le\sum_{j=1}^{n-1}m(r,A_j)+\sum_{j=1}^{n-1}m\left(r,\frac{f^{(j)}}{f}\right)+o(T(r,A_0)). \tag{2.15}$$

Hence, by (2.15) and Lemma 2.3 that

$$T(r,A_0)\le O\left(\sum_{j=1}^{n-1}m(r,A_j)+\log(rT(r,f))\right), \tag{2.16}$$

for sufficiently large $r\to+\infty$, $r\notin E_2$, where E_2 is a set of r of finite linear measure. Set $b=\max\{\sigma_{[p,q]}(A_j,\varphi):\sigma_{[p,q]}(A_j,\varphi)<\mu_{[p,q]}(A_0,\varphi)=\mu,\ j=1,\cdots,n-1\}$. If $\sigma_{[p,q]}(A_j,\varphi)<\mu_{[p,q]}(A_0,\varphi)=\mu$, then for any ε $(0<2\varepsilon<\mu-b)$ and all $r\to+\infty$, we have

$$m(r,A_j)\le T(r,A_j)\le\exp_p\left\{(b+\varepsilon)\log_q\varphi(r)\right\}$$

$$<\exp_p\left\{(\mu-\varepsilon)\log_q\varphi(r)\right\}=\exp_{p-1}\left\{\left(\log_{q-1}\varphi(r)\right)^{\mu-\varepsilon}\right\}. \tag{2.17}$$

Set $\tau_1=\max\left\{\tau_{[p,q]}(A_j,\varphi):\sigma_{[p,q]}(A_j,\varphi)=\mu_{[p,q]}(A_0,\varphi),\ j\neq 0\right\}$, then $\tau_1<\tau$. If $\sigma_{[p,q]}(A_j,\varphi)=\mu_{[p,q]}(A_0,\varphi)$, $\tau_{[p,q]}(A_j,\varphi)\le\tau_1<\tau$, then for $r\to+\infty$ and any $\varepsilon\,(0<2\varepsilon<\tau-\tau_1)$, we have

$$m(r,A_j)\le T(r,A_j)<\exp_{p-1}\left\{(\tau_1+\varepsilon)\left(\log_{q-1}\varphi(r)\right)^{\mu}\right\}. \tag{2.18}$$

By the definition of the lower $[p,q]-\varphi$ type, for $r\to+\infty$, we have

$$T(r,A_0)>\exp_{p-1}\left\{(\tau-\varepsilon)\left(\log_{q-1}\varphi(r)\right)^{\mu}\right\}. \tag{2.19}$$

When $p\ge q>1$ or $p>q=1$, we have for $r\to+\infty$

$$\exp_{p-1}\left\{(\tau_1+\varepsilon)\left(\log_{q-1}\varphi(r)\right)^{\mu}\right\}=o\left(\exp_{p-1}\left\{(\tau-\varepsilon)\left(\log_{q-1}\varphi(r)\right)^{\mu}\right\}\right).$$

By substituting $(2.17)-(2.19)$ into (2.16), we obtain

$$\exp_{p-1}\left\{(\tau-2\varepsilon)\left(\log_{q-1}\varphi(r)\right)^{\mu}\right\}\le O(\log(rT(r,f))),\ r\notin E_2, r\to+\infty. \tag{2.20}$$

Then by (2.20), Remark 1.2 and Lemma 2.2, we have $\mu_{[p+1,q]}(f,\varphi)\ge\mu_{[p,q]}(A_0,\varphi)$.

Lemma 2.10 *Let p,q be integers such that $p\ge q\ge 1$ and let f be a meromorphic function with $0<\sigma_{[p,q]}(f,\varphi)<\infty$, where $\varphi(r)$ only satisfies $\lim\limits_{r\to+\infty}\frac{\log_{q-1}\varphi(\alpha r)}{\log_{q-1}\varphi(r)}=1$ for some $\alpha>1$. Then for every $\varepsilon>0$, there exists a set $E_5\subset(1,\infty)$ of infinite logarithmic measure such that*

$$\tau_{[p,q]}(f,\varphi)=\lim_{\substack{r\to+\infty\\ r\in E_5}}\frac{\log_{p-1}T(r,f)}{\left(\log_{q-1}\varphi(r)\right)^{\sigma_{[p,q]}(f,\varphi)}}.$$

Proof. By the definition of the $[p,q]-\varphi$ type, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ satisfying $\left(1+\frac{1}{n}\right)r_n < r_{n+1}$, and

$$\tau_{[p,q]}(f,\varphi) = \lim_{r_n\to\infty} \frac{\log_{p-1} T(r_n,f)}{\left(\log_{q-1}\varphi(r_n)\right)^{\sigma_{[p,q]}(f,\varphi)}}.$$

Then for any given $\varepsilon > 0$, there exists an n_1 such that for $n \geq n_1$ and any $r \in \left[r_n, \left(1+\frac{1}{n}\right)r_n\right]$, we have

$$\frac{\log_{p-1} T(r_n,f)}{\left(\log_{q-1}\varphi(r_n)\right)^{\sigma_{[p,q]}(f,\varphi)}} \left(\frac{\log_{q-1}\varphi(r_n)}{\log_{q-1}\varphi\left[\left(1+\frac{1}{n}\right)r_n\right]}\right)^{\sigma_{[p,q]}(f,\varphi)}$$

$$\leq \frac{\log_{p-1} T(r,f)}{\left(\log_{q-1}\varphi(r)\right)^{\sigma_{[p,q]}(f,\varphi)}}.$$

When $q \geq 1$, we have $\frac{\log_{q-1}\varphi(r_n)}{\log_{q-1}\varphi\left[\left(1+\frac{1}{n}\right)r_n\right]} \to 1$, $r_n \to \infty$. Set

$$E_5 = \bigcup_{n=n_1}^{\infty} \left[r_n, \left(1+\frac{1}{n}\right)r_n\right].$$

Then, we have

$$\lim_{\substack{r\to+\infty \\ r\in E_5}} \frac{\log_{p-1} T(r,f)}{\left(\log_{q-1}\varphi(r)\right)^{\sigma_{[p,q]}(f,\varphi)}} \geq \lim_{r_n\to\infty} \frac{\log_{p-1} T(r_n,f)}{\left(\log_{q-1}\varphi(r_n)\right)^{\sigma_{[p,q]}(f,\varphi)}} = \tau_{[p,q]}(f,\varphi)$$

and $\int\limits_{E_5} \frac{dr}{r} = \sum\limits_{n=n_1}^{\infty} \int\limits_{r_n}^{\left(1+\frac{1}{n}\right)r_n} \frac{dt}{t} = \sum\limits_{n=n_1}^{\infty} \log\left(1+\frac{1}{n}\right) = \infty$. Therefore, by the evident fact that

$$\lim_{\substack{r\to+\infty \\ r\in E_5}} \frac{\log_{p-1} T(r,f)}{\left(\log_{q-1}\varphi(r)\right)^{\sigma_{[p,q]}(f,\varphi)}} \leq \limsup_{\substack{r\to+\infty \\ r\in E_5}} \frac{\log_{p-1} T(r,f)}{\left(\log_{q-1}\varphi(r)\right)^{\sigma_{[p,q]}(f,\varphi)}} = \tau_{[p,q]}(f,\varphi),$$

we have

$$\tau_{[p,q]}(f,\varphi) = \lim_{\substack{r\to+\infty \\ r\in E_5}} \frac{\log_{p-1} T(r,f)}{\left(\log_{q-1}\varphi(r)\right)^{\sigma_{[p,q]}(f,\varphi)}}.$$

The proof of the following two lemmas is essentially the same as in the corresponding results for the usual order and lower order. For details, see Chapter 2 of the book by Goldberg-Ostrovskii [7] and Chapter 1 of the book by Yang-Yi [18]. So, we omit the proofs.

Lemma 2.11 *Let $p \geq q \geq 1$ be integers, and let f and g be non-constant meromorphic functions of $[p,q]-\varphi$ order. Then we have*

$$\rho_{[p,q]}(f+g,\varphi) \leq \max\left\{\rho_{[p,q]}(f,\varphi), \rho_{[p,q]}(g,\varphi)\right\}$$

and

$$\rho_{[p,q]}(fg,\varphi) \leq \max\left\{\rho_{[p,q]}(f,\varphi), \rho_{[p,q]}(g,\varphi)\right\}.$$

Furthermore, if $\rho_{[p,q]}(f,\varphi) > \rho_{[p,q]}(g,\varphi)$, *then we obtain*

$$\rho_{[p,q]}(f+g,\varphi) = \rho_{[p,q]}(fg,\varphi) = \rho_{[p,q]}(f,\varphi).$$

Lemma 2.12 *Let $p \geq q \geq 1$ be integers, and let f and g be non-constant meromorphic functions with $\rho_{[p,q]}(f,\varphi)$ as $[p,q]-\varphi$ order of f and $\mu_{[p,q]}(g,\varphi)$ as lower $[p,q]-\varphi$ order of g. Then we have*

$$\mu_{[p,q]}(f+g,\varphi) \leq \max\left\{\rho_{[p,q]}(f,\varphi), \mu_{[p,q]}(g,\varphi)\right\}$$

and

$$\mu_{[p,q]}(fg,\varphi) \leq \max\left\{\rho_{[p,q]}(f,\varphi), \mu_{[p,q]}(g,\varphi)\right\}.$$

Furthermore, if $\mu_{[p,q]}(g,\varphi) > \rho_{[p,q]}(f,\varphi)$, then we obtain

$$\mu_{[p,q]}(f+g,\varphi) = \mu_{[p,q]}(fg,\varphi) = \mu_{[p,q]}(g,\varphi).$$

3. Proof of theorems

Proof of Theorem 1.1 By Lemma 1.1 and (2.10), we have as in Lemma 2.8

$$T(r,f) \leq \exp_{p+1}\left\{\left(\sigma_{[p,q]}(A_0,\varphi)+3\varepsilon\right)\log_q \varphi(r)\right\},$$

for any $\varepsilon > 0$ and $r \notin E_0, r \to +\infty$, where E_0 is a set of r of finite logarithmic measure. By Lemma 2.2, we get $\sigma_{[p+1,q]}(f,\varphi) \leq \sigma_{[p,q]}(A_0,\varphi)$. Set $d = \max\{\sigma_{[p,q]}(A_j,\varphi) : \sigma_{[p,q]}(A_j,\varphi) < \sigma_{[p,q]}(A_0,\varphi), j = 1,\cdots,n-1\}$. If $\sigma_{[p,q]}(A_j,\varphi) < \mu_{[p,q]}(A_0,\varphi) \leq \sigma_{[p,q]}(A_0,\varphi)$ or $\sigma_{[p,q]}(A_j,\varphi) \leq \mu_{[p,q]}(A_0,\varphi) < \sigma_{[p,q]}(A_0,\varphi)$, then for any given $\varepsilon\left(0 < 2\varepsilon < \sigma_{[p,q]}(A_0,\varphi) - d\right)$ and sufficiently large r, we have

$$T(r,A_j) \leq \exp_p\left\{(d+\varepsilon)\log_q \varphi(r)\right\} = \exp_{p-1}\left\{\left(\log_{q-1}\varphi(r)\right)^{d+\varepsilon}\right\}. \tag{3.1}$$

Set $\tau_1 = \max\{\tau_{[p,q]}(A_j,\varphi) : \sigma_{[p,q]}(A_j,\varphi) = \mu_{[p,q]}(A_0,\varphi), j \neq 0\}$. If

$$\sigma_{[p,q]}(A_j,\varphi) = \mu_{[p,q]}(A_0,\varphi) = \sigma_{[p,q]}(A_0,\varphi),$$

then we have $\tau_1 < \tau \leq \tau_{[p,q]}(A_0,\varphi)$. Therefore

$$T(r,A_j) \leq \exp_{p-1}\left\{(\tau_1+\varepsilon)\left(\log_{q-1}\varphi(r)\right)^{\sigma_{[p,q]}(A_0,\varphi)}\right\}, \tag{3.2}$$

holds for any $r \to +\infty$ and any given $\varepsilon\left(0 < 2\varepsilon < \tau_{[p,q]}(A_0,\varphi) - \tau_1\right)$. By the definition of the $[p,q]-\varphi$ type and Lemma 2.10, and sufficiently large $r \in E_5$, where E_5 is a set of r of infinite logarithmic measure, we have

$$T(r,A_0) > \exp_{p-1}\left\{\left(\tau_{[p,q]}(A_0,\varphi)-\varepsilon\right)\left(\log_{q-1}\varphi(r)\right)^{\sigma_{[p,q]}(A_0,\varphi)}\right\}. \tag{3.3}$$

Then by (2.16) and (3.1) $-$ (3.3), for all sufficiently large r, $r \in E_5 \setminus E_2$ and the above ε, we obtain

$$\exp_{p-1}\left\{\left(\tau_{[p,q]}(A_0,\varphi)-2\varepsilon\right)\left(\log_{q-1}\varphi(r)\right)^{\sigma_{[p,q]}(A_0,\varphi)}\right\} \leq O\left(\log rT(r,f)\right), \tag{3.4}$$

where E_2 is a set of r of finite linear measure. Then, we have

$$\sigma_{[p+1,q]}(f,\varphi) \geq \sigma_{[p,q]}(A_0,\varphi).$$

Thus, we have $\sigma_{[p+1,q]}(f,\varphi) = \sigma_{[p,q]}(A_0,\varphi)$. By Lemmas 2.8 and 2.9, we have $\mu_{[p+1,q]}(f,\varphi) = \mu_{[p,q]}(A_0,\varphi)$. Now we need to prove $\underline{\bar{\lambda}}_{[p+1,q]}(f-\psi,\varphi) = \mu_{[p+1,q]}(f,\varphi)$ and $\bar{\lambda}_{[p+1,q]}(f-\psi,\varphi) = \sigma_{[p+1,q]}(f,\varphi)$. Setting $g = f-\psi$, since $\sigma_{[p+1,q]}(\psi,\varphi) < \mu_{[p,q]}(A_0,\varphi)$, then by Lemmas 2.11 and 2.12 we have $\sigma_{[p+1,q]}(g,\varphi) = \sigma_{[p+1,q]}(f,\varphi) = \sigma_{[p,q]}(A_0,\varphi)$, and

$$\mu_{[p+1,q]}(g,\varphi) = \mu_{[p+1,q]}(f,\varphi) = \mu_{[p,q]}(A_0,\varphi), \bar{\lambda}_{[p+1,q]}(g,\varphi) = \bar{\lambda}_{[p+1,q]}(f-\psi,\varphi)$$

and $\underline{\bar{\lambda}}_{[p+1,q]}(g,\varphi) = \underline{\bar{\lambda}}_{[p+1,q]}(f-\psi,\varphi)$. By substituting $f = g+\psi, f' = g'+\psi', \cdots, f^{(n)} = g^{(n)}+\psi^{(n)}$ into (1.1), we get

$$(3.5)\qquad g^{(n)}+A_{n-1}g^{(n-1)}+\cdots+A_0 g = -\left[\psi^{(n)}+A_{n-1}\psi^{(n-1)}+\cdots+A_0\psi\right].$$

If $F = \psi^{(n)}+A_{n-1}\psi^{(n-1)}+\cdots+A_0\psi \equiv 0$, then by Lemma 2.9, we have $\mu_{[p+1,q]}(\psi,\varphi) \geq \mu_{[p,q]}(A_0,\varphi)$, which is a contradiction. Hence $F(z) \not\equiv 0$. Since $F(z) \not\equiv 0$ and

$$\sigma_{[p+1,q]}(F,\varphi) \leq \sigma_{[p+1,q]}(\psi,\varphi) < \mu_{[p,q]}(A_0,\varphi)$$
$$= \mu_{[p+1,q]}(f,\varphi) = \mu_{[p+1,q]}(g,\varphi) \leq \sigma_{[p+1,q]}(g,\varphi) = \sigma_{[p+1,q]}(f,\varphi),$$

then by Lemma 2.7 and (3.5), we have $\bar{\lambda}_{[p+1,q]}(g,\varphi) = \lambda_{[p+1,q]}(g,\varphi) = \sigma_{[p+1,q]}(g,\varphi) = \sigma_{[p,q]}(A_0,\varphi)$, i.e., $\bar{\lambda}_{[p+1,q]}(f-\psi,\varphi) = \lambda_{[p+1,q]}(f-\psi,\varphi) = \sigma_{[p+1,q]}(f,\varphi) = \sigma_{[p,q]}(A_0,\varphi)$. By Lemma 2.6 and (3.5), we have $\underline{\bar{\lambda}}_{[p+1,q]}(g,\varphi) = \mu_{[p+1,q]}(g,\varphi)$, i.e.,

$$\underline{\bar{\lambda}}_{[p+1,q]}(f-\psi,\varphi) = \mu_{[p+1,q]}(f,\varphi) = \mu_{[p,q]}(A_0,\varphi).$$

Therefore

$$\underline{\bar{\lambda}}_{[p+1,q]}(f-\psi,\varphi) = \mu_{[p+1,q]}(f,\varphi) = \mu_{[p,q]}(A_0,\varphi)$$
$$\leq \sigma_{[p,q]}(A_0,\varphi) = \sigma_{[p+1,q]}(f,\varphi) = \bar{\lambda}_{[p+1,q]}(f-\psi,\varphi) = \lambda_{[p+1,q]}(f-\psi,\varphi).$$

The proof of the theorem is complete.

Proof of Theorem 1.2 By the first part of the proof of Theorem 1.1, we can get $\sigma_{[p+1,q]}(f,\varphi) \leq \sigma_{[p,q]}(A_0,\varphi)$. By

$$(3.6)\qquad \limsup_{r\to+\infty}\sum_{j=1}^{n-1}\frac{m(r,A_j)}{m(r,A_0)} < 1$$

we have for $r\to+\infty$

$$(3.7)\qquad \sum_{j=1}^{n-1} m(r,A_j) < \delta m(r,A_0),$$

where $\delta\in(0,1)$. By $\lambda_{[p,q]}\left(\frac{1}{A_0},\varphi\right) < \mu_{[p,q]}(A_0,\varphi)$, we have $N(r,A_0) = o(T(r,A_0))$, $r\to+\infty$. By (2.15) and (3.7), for $r\to+\infty, r\notin E_2$, we obtain

$$(3.8)\qquad T(r,A_0) = m(r,A_0)+N(r,A_0) \leq \delta T(r,A_0)+O(\log rT(r,f))+o(T(r,A_0)),$$

where E_2 is a set of r of finite linear measure. By Lemma 2.2 and (3.8), we have $\sigma_{[p+1,q]}(f,\varphi) \geq \sigma_{[p,q]}(A_0,\varphi)$. Then we have $\sigma_{[p+1,q]}(f,\varphi) = \sigma_{[p,q]}(A_0,\varphi)$. By (3.8) and Lemma 2.2, we have $\mu_{[p+1,q]}(f,\varphi) \geq \mu_{[p,q]}(A_0,\varphi)$. By Lemma 2.8, we have $\mu_{[p+1,q]}(f,\varphi) \leq \mu_{[p,q]}(A_0,\varphi)$, then we get

$$\mu_{[p+1,q]}(f,\varphi) = \mu_{[p,q]}(A_0,\varphi).$$

By using the similar proof of Theorem 1.1, we can get

$$\begin{aligned}\underline{\bar{\lambda}}_{[p+1,q]}(f-\psi,\varphi) &= \mu_{[p+1,q]}(f,\varphi) = \mu_{[p,q]}(A_0,\varphi) \leq \sigma_{[p,q]}(A_0,\varphi)\\ &= \sigma_{[p+1,q]}(f,\varphi) = \bar{\lambda}_{[p+1,q]}(f-\psi,\varphi) = \lambda_{[p+1,q]}(f-\psi,\varphi).\end{aligned}$$

The proof of the theorem is complete.

Proof of Theorem 1.3 Suppose that f is rational solution of (1.1). If f is either a rational function with a pole of multiplicity $n\geq 1$ at z_0 or a polynomial with degree

$\deg(f) \geq s$, then $f^{(s)}(z) \not\equiv 0$. If $\max\{\sigma_{[p,q]}(A_j,\varphi), j \neq s\} < \mu_{[p,q]}(A_s,\varphi) = \mu$, then we have

$$\mu_{[p,q]}(0,\varphi) = \mu_{[p,q]}\left(f^{(n)} + A_{n-1}f^{(n-1)} + \cdots + A_0 f, \varphi\right) = \mu_{[p,q]}(A_s,\varphi) = \mu > 0,$$

which is a contradiction. Set

$$\tau_1 = \max\{\tau_{[p,q]}(A_j,\varphi) : \sigma_{[p,q]}(A_j,\varphi) = \mu_{[p,q]}(A_s,\varphi), j \neq s\}.$$

If $\sigma_{[p,q]}(A_j,\varphi) = \mu_{[p,q]}(A_s,\varphi)$, $\tau_{[p,q]}(A_j,\varphi) \leq \tau_1 < \tau$, then we may choose constants δ_1, δ_2 such that $\tau_1 < \delta_1 < \delta_2 < \tau$. For sufficiently large r, we have

$$m(r,A_j) \leq T(r,A_j) \leq \exp_{p-1}\left\{\delta_1 \left(\log_{q-1}\varphi(r)\right)^{\mu}\right\}. \tag{3.9}$$

If $\sigma_{[p,q]}(A_j,\varphi) < \mu_{[p,q]}(A_s,\varphi)$, then for sufficiently large r
and any given $\varepsilon\left(0 < 2\varepsilon < \mu_{[p,q]}(A_s,\varphi) - \sigma_{[p,q]}(A_j,\varphi)\right)$, we obtain

$$m(r,A_j) \leq T(r,A_j) \leq \exp_p\left\{\left(\sigma_{[p,q]}(A_j,\varphi) + \varepsilon\right)\log_q \varphi(r)\right\}. \tag{3.10}$$

Under the assumption that $\lambda_{[p,q]}\left(\frac{1}{A_s},\varphi\right) < \mu_{[p,q]}(A_s,\varphi)$, for sufficiently large r, we have

$$N(r,A_s) = o(T(r,A_s)). \tag{3.11}$$

By the definition of the lower $[p,q]-\varphi$ type, for sufficiently large r, we get

$$T(r,A_s) \geq \exp_{p-1}\left\{\delta_2 \left(\log_{q-1}\varphi(r)\right)^{\mu}\right\}. \tag{3.12}$$

By (1.1), we have

$$T(r,A_s) \leq N(r,A_s) + \sum_{j\neq s} m(r,A_j) + O(\log r), \tag{3.13}$$

for sufficiently large r. Hence, by substituting (3.9), (3.10) and (3.11) into (3.13) we have the contradiction. Therefore, if f is a non-transcendental meromorphic solution, then it must be a polynomial with degree $\deg(f) \leq s-1$.

Now, we assume that f is a transcendental meromorphic solution of (1.1). By (1.1), we have

$$-A_s = \frac{f}{f^{(s)}}\left[\frac{f^{(n)}}{f} + \cdots + A_{s+1}\frac{f^{(s+1)}}{f} + A_{s-1}\frac{f^{(s-1)}}{f} + \cdots + A_0\right]. \tag{3.14}$$

Noting that

$$\begin{aligned} m\left(r,\frac{f}{f^{(s)}}\right) &\leq T(r,f) + T\left(r,\frac{1}{f^{(s)}}\right) = T(r,f) + T\left(r,f^{(s)}\right) + O(1) \\ &\leq T(r,f) + (s+1)T(r,f) + o(T(r,f)) + O(1) \\ &= (s+2)T(r,f) + o(T(r,f)) + O(1). \end{aligned} \tag{3.15}$$

By Lemma 2.3, (3.14) and (3.15), we obtain

$$\begin{aligned} T(r,A_s) &= m(r,A_s) + N(r,A_s) \\ &\leq N(r,A_s) + \sum_{j\neq s} m(r,A_j) + (s+3)T(r,f) + O(\log(rT(r,f))), \end{aligned} \tag{3.16}$$

for sufficiently large $r \notin E_2$, where E_2 is a set of r of finite linear measure. Then by $(3.9)-(3.12)$, (3.16) and Lemma 2.2, we can get $\mu_{[p,q]}(f,\varphi) \geq \mu_{[p,q]}(A_s,\varphi)$ and $\sigma_{[p,q]}(f,\varphi) \geq \sigma_{[p,q]}(A_s,\varphi)$. By Lemma 1.1 and (2.10), we have

$$T(r,f) \leq \exp_{p+1}\left\{\left(\sigma_{[p,q]}(A_s,\varphi)+3\varepsilon\right)\log_q \varphi(r)\right\}, \tag{3.17}$$

for any $\varepsilon > 0$, and $r \notin E_0$, $r \to +\infty$, where E_0 is a set of r of linear logarithmic measure. Then by (3.17) and Lemma 2.2, we have $\sigma_{[p+1,q]}(f,\varphi) \leq \sigma_{[p,q]}(A_s,\varphi)$. By Lemma 2.8, we obtain $\mu_{[p+1,q]}(f,\varphi) \leq \mu_{[p,q]}(A_s,\varphi)$. Then we get $\sigma_{[p+1,q]}(f,\varphi) \leq \sigma_{[p,q]}(A_s,\varphi) \leq \sigma_{[p,q]}(f,\varphi)$ and

$$\mu_{[p+1,q]}(f,\varphi) \leq \mu_{[p,q]}(A_s,\varphi) \leq \mu_{[p,q]}(f,\varphi).$$

The proof of the theorem is complete.

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