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INDEX FORMULAS FOR COUNTABLY $\varphi-$SET CONTRACTION

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Abstract. In this paper, we study the index formulas for a class of bounded linear operators, namely $\varphi-$set contractions, acting on a Banach space and we discuss some application of this class of operators to the theory of bifurcation points. In particular our results generalize and improve some recent results mentioned in the literature.

1. Introduction and Preliminary

Finding necessary and sufficient conditions for the appearance of nontrivial solutions arbitrary close to some points (called bifurcation points) of the trivial branch, with assumption of existence of a known (trivial) branch of solutions for a parametrized family of an equation, is one of the oldest problems of mathematics which have created bifurcation theory. One of the most important role in bifurcation theory is played by index formulas for suitable kind of operators. In recent years, many authors have focused on set-contractive operators and obtained a lot of valuable results (see [12, 7, 10]). Nussbaum (1969) [14] developed degree theory for k-set contractive operators ($0 \leq k < 1$), which was first introduced by Kuratowski at 1930 [11], Stuart, Toland [18] and Amann (1976) [2] established the index formula for k-set contractions and condensing operators. Kim (2008)[10] presented an index formula for countably k-set contractive bounded linear operators in a real Banach space, by using a degree theory for countably condensing operators. In this paper, we continue to study set-contractive operators and investigate the conditions under which the topological degrees can be defined for a larger class of k-set contraction, namely $\varphi-$set-contractive operators. Moreover, we introduce generalized φ_k-set-contractive operators. It should be noted that this class of operators, as special cases, includes linear bounded operators, nonexpansive operators, completely continuous operators, k- set-contractive operators, condensing operators and $1-$set contractive operators. Correspondingly, we can obtain some new bifurcation theorems of these operators, which improve and extend many famous theorems such as the Hetzer's theorem, Nussbaum's theorem, Kim's theorem, etc.

Before proceeding to the main results of this paper, we must recall some notations, definitions and theorems we shall need.
A function $\gamma : \{B \subset X : B \textit{ is bounded}\} \to [0, \infty)$ is said to be a measure of noncompactness on a Banach space X, if it satisfies the following conditions:

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(1)(invariance under closure and convex hull): $\gamma(c\bar{o}B) = \gamma(B)$,
(2)(regularity): $\gamma(B) = 0$ if and only if B is relatively compact,
(3)(semi-additivity): $\gamma(B_1 \cup B_2) = \max\{\gamma(B_1), \gamma(B_2)\}$,
(4)(algebraic semi-additivity): $\gamma(B_1 + B_2) \leq \gamma(B_1) + \gamma(B_2)$
(5)(semi-homogeneity): $\gamma(\alpha B) = |\alpha|\gamma(B)$ for all $\alpha \in R$, and
(6)(Lipschitzianity): $|\gamma(B_1) - \gamma(B_2)| \leq L_\gamma \rho(B_1, B_2)$, where ρ denotes the Hausdorff semi-metric, that is,

$$\rho(B_1, B_2) = \inf\{\epsilon > 0 : B_2 \subset B_1 + \epsilon\bar{B}(0,1), B_1 \subset B_2 + \epsilon\bar{B}(0,1)\}.$$

The most important examples of measures of noncompactness are the Kuratowski measure of noncompactness (or set measure of noncompactness)

$$\alpha(\Omega) = \inf\{r > 0 : X \text{may be covered by finitely many sets of diameter} \leq r\},$$

and the Hausdorff measure of noncompactness (or ball measure of noncompactness)

$$\beta(\Omega) = \inf\{r > 0 : \text{there exists a finite } r\text{-net for} \Omega \ \ in \ \ X\}.$$

A detailed account of theory and applications of measures of noncompactness may be found in the monographs [1, 3]. Note that the Kuratowski measure of noncompactness and the Hausdorff measure of noncompactness have the above properties; see [1, 19].

A function $\varphi : R^+ \to R^+$ is said to be a
• comparison function if $\varphi(0) = 0$ and $\varphi(t) < t$ for each $t > 0$,
• semi-comparison function if $\varphi(0) = 0$ and $\varphi(t) < kt$ for each $t > 0$ and some $k \geq 0$, Denote

$$\Phi_k = \{\varphi : R^+ \to R^+, \varphi(t) < kt \ \ for \ \ t > 0, \ \ \varphi(0) = 0\},$$

for $k = 1$, we denote Φ_1 by Φ.
Let $k \geq 1$. A continuous operator $T : X \to X$ (with respect to γ) is said to be
• countably $k-$set contractive [10]:
if $\gamma(T(C)) \leq k\gamma(C)$ for each countable bounded set $C \subseteq X$.
• countably $k-$set contraction [10]:
if $\gamma(T(C)) \leq k\gamma(C)$ for each countable bounded set $C \subseteq X$ and $0 \leq k < 1$.
• $k-$set contraction [18]:
if $\gamma(T(C)) \leq k\gamma(C)$ for all bounded sets $C \subset X$.
and $1-$set contraction, if $k = 1$ [6].
• countably condensing [10]:
if $\gamma(T(C)) < \gamma(C)$ for each countable bounded set $C \subset X$ with $\gamma(C) > 0$.
• countably $\varphi-$set contractive :
if $\gamma(T(C)) \leq \varphi(\gamma(C))$ for some $\varphi \in \Phi_k$ and each countable bounded set $C \subseteq X$.
•countably $\varphi-$set contraction [5]:
if $\gamma(T(C)) \leq \varphi(\gamma(C))$ for some $\varphi \in \Phi$ and each countable bounded set $C \subseteq X$.

Remark 1.1. *Clearly, every $k-$set contractive mapping is a $\varphi-$set contractive and every bounded linear operator is a $k-$set contractive mapping by property* (6) *of measure of noncompactness γ, so every bounded linear operator is a $\varphi-$set contractive.*

Let T be a bounded linear operator of a Banach space X into a Banach space Y . The null space and the range of T are denoted by $N(T)$ and $R(T)$, respectively.

T is said to be a Fredholm operator if the dimension of $N(T)$ is finite and the codimension of $R(T)$ is finite. If the dimension of $N(T)$ is finite and $R(T)$ is closed, then T is said to be a semi-Fredholm operator. In this case, $i(T) = dimN(T) - codimR(T)$ is called the index of T.

2. Main Results

In this section, we present a necessary condition for the existence of bifurcation points of

$$u = \lambda Gu,$$

where X is a Banach space, $u \in X$, $\lambda \in R$ and $G : X \to X$ is a countably generalized $\varphi-$set contraction.
This result can be regarded as a generalization of the [9] and [8]. Moreover, we extend the index formula for countably $\varphi-$set contractive, comutable bounded linear operators as follows:

Theorem 2.1. *Let $T : X \to X$ be a countably $\varphi-$set contractive bounded linear comutable operator on a real Banach space X. If 1 is not an eigenvalue of T, then $ind(T, 0) = (-1)^\nu$, where ν is the sum of the multiplicities of the eigenvalues $\lambda > 1$ of T.*

So the results from [10] can be obtained as consequences of our result.
In order to show the main theorem above, we shall need some notations, definitions and lemmas to show that the decomposition is possible for countably generalized $\varphi-$set contractive bounded linear operators.

Now let us state our main definition which determines an important class of operators including linear bounded operators, nonexpansive operators, completely continuous operators, k- set-contractive operators, condensing operators and $1-$set contractive operators.

Definition 2.2. *A continuous operator $T : \Omega \to X$ is said to be countably generalized $\varphi-$set contractive if*

$$\gamma(T^n(C)) \leq \varphi(\gamma(C))$$

for some $n \in N$, $\varphi \in \Phi_k$, and each countable bounded set $C \subseteq X$, and is said to be countably generalized $\varphi-$set contraction if $\varphi \in \Phi$.

The following provides two examples; one is an example of a $\varphi-$set contraction which is not a k-set contraction for any $0 < k < 1$, the other is an example of a generalized $\varphi-$set contraction which is not a $\varphi-$set contraction.

Example 2.3. *Let $X = [0, 1] \cup \{2, 3, 4, ...\}$ For the metric, let*

$$\rho(x, y) = \begin{cases} |x - y| & if x, y \in [0, 1], \\ \\ x + y & if \ \ one \ \ of \ \ x, y \notin [0, 1]. \end{cases}$$

It is apparent that (X, ρ) is a complete metric space Define the mapping $T : X \to X$ by

$$Tx = \begin{cases} x - \frac{1}{2}x^2 & if x, y \in [0, 1], \\ \\ x - 1 & if x \in \{2, 3, ...\}. \end{cases}$$

Then, for $x, y \in [0,1]$ with $x?y = t > 0$,

$$\rho(x,y) = (x-y)(1-\frac{1}{2}(x+y)) \leq t(l-\frac{1}{2}t)$$

and, if $x \in \{2,3,4,...\}$ with $x > y$, then

$$\rho(x,y) = Tx + Ty < x - 1 + y = \rho(x,y) - 1.$$

Thus, if we define $\psi(t)$ by

$$\psi(t) = \begin{cases} t - \frac{1}{2}t^2 & if t \in [0,1], \\ \\ t-1 & if t > 1. \end{cases}$$

then $\psi \in \Phi$ and T is a $\varphi-$set contraction because it is easy to prove that every $\varphi-$contraction mapping is a $\varphi-$set contraction with respect to the Kuratowskii measure of noncompactness.
However, as $n \to \infty$, $\rho(Tn,0)/\rho(n,0) \to 1$ so there can be no $0 \leq k < 1$ for which T is a $k-$set contraction.

Example 2.4. *Let $X = \{1\} \cup \{2n, 3n : n \in N\}$ is equipped with the discrete metric and $T : X \to X$ be defined by*

$$T(x) = \begin{cases} \{1\} & if x = 1, \\ \\ \{3(2n+1)\} & if x = 2n, \\ \\ \{1\} & if x = 3n \quad and \quad n \quad is \quad odd. \end{cases}$$

If α is the Kuratowski measure of noncompactness, then $\alpha(T^2X) = 0$. Therefore, T is a generalized $\varphi-$set contraction. But if $C = \{2n : n \in N\}$, then $\alpha(T(C)) = \alpha(C) = 1$ and so T could not be a $\varphi-$set contraction since $\varphi(1)$ would have to be one.

Lemma 2.5. *Let X be a real or complex Banach space and $T : X \to X$ a bounded linear operator which is countably generalized $\varphi-$set contraction. Let I denote the identity operator in X. Then for any relatively compact set $M \subset X$ and for any bounded countable set $C \subset X$, $M_1 = \{x \in C : x - Tx \in M\}$ is relatively compact.*

Proof. Let M be a relatively compact set in X and C a bounded countable set in X and $M_1 = \{x \in C : x - Tx \in M\}$. We will show that $\gamma(M_1) = 0$. Suppose that $x \in M_1$, so that $x = Tx + z$ for some $z \in M$. Substituting for x on the right, $x = T^2x + Tz + z$, and continuing in this way we find

$$x = T^n x + (\Sigma_{j=0}^{n-1} T^j)z.$$

If we write $M_2 = (\Sigma_{j=0}^{n-1} T^j)(M)$, the set M_2 is relatively compact because it is the continuous image of a relatively compact set. Furthermore, the above equality implies that $M_1 \subset T^n(M_1) + M_2$, so that $\gamma(M_1) \leq \gamma(T^n M_1)$. Since T is generalized countably $\varphi-$set contraction we have $\gamma(M_1) \leq \varphi(\gamma(M_1))$. Since $\varphi \in \Phi$, we have $\varphi(\gamma(M_1)) < \gamma(M_1)$, which yields a contradiction. It follows that $\gamma(M_1) = 0$ and thus M_1 is relatively compact.

□

Proposition 2.6. *Let X be a real or complex Banach space and $T : X \to X$ a bounded linear operator which is countably generalized $\varphi-$set contraction, then $I - T$ is a semi-Fredholm operator.*

Proof. We must show that, the null space of $I - T$ is finite dimensional and the range of $I - T$ is closed in X. But it follows by previous Lemma and Proposition 2.1 in [10]. □

Remark 2.7. *Let X be a real or complex Banach space and $T : X \to X$ a bounded linear operator which is countably generalized $\varphi-$set contraction, then $I - \lambda T$ is a semi-Fredholm operator for any $\lambda \in [0, 1]$, since*

$$\gamma((\lambda T)^n(C)) = \lambda^n \gamma(T^n(C)) \leq \lambda^n \varphi(\gamma(C)) \leq \varphi(\gamma(C)),$$

so λT is a countably generalized $\varphi-$set contraction too and by last Proposition $I - \lambda T$ is a semi-Fredholm operator.

Proposition 2.8. *Let X be a real or complex Banach space and $T : X \to X$ a bounded linear operator which is countably generalized $\varphi-$set contraction, then $I - T$ is a Fredholm operator of index zero.*

Proof. By remark $I - \lambda T$ is a semi-Fredholm operator for any $\lambda \in [0, 1]$. Since the index for semi-Fredholm operators is constant on its connected components, and $\{I - \lambda T : \lambda \in [0, 1]\}$ is connected, we have

$$i(I - T) = i(I) = 0.$$

Therefore, $I - T$ is a Fredholm operator of index zero. □

Proposition 2.9. *Let X be a real or complex Banach space and $T : X \to X$ a bounded linear operator, so by Remark* 1.1 *we have*

$$\gamma(T^n(C)) \leq \varphi(\gamma(C)), \tag{2.1}$$

for some $n \in N$ and some $\varphi \in \Phi_k$.
Let $\mu \in R$ be such that $\mu^n < k^{-1}$ for the same n and k in 2.1. *Then $I - \mu T$ is a Fredholm operator of index zero.*

Proof. Since T is a linear operator and satisfying , we have

$$\gamma((\mu T)^n(C)) = \mu^n \gamma(T^n(C)) \leq \mu^n \varphi(\gamma(C)) \leq \mu^n k(\gamma(C)) = \psi(\gamma(C)),$$

where $\psi(t) = \mu^n kt$ is belong to ϕ_k, thus μT is a countably generalized $\varphi-$set contraction and therefore $I - \mu T$ is a Fredholm operator of index zero. □

Let $(X, ||.||)$ be a real Banach space and $E = R \times X$ a Banach space with the norm

$$||(\lambda, u)|| = (|\lambda|^2 + ||u||^2)^{\frac{1}{2}} \quad for \quad (\lambda, u) \in E.$$

Consider the following nonlinear equation:

$$u = \lambda G u, \quad (\lambda, u) \in E. \tag{2.2}$$

Assume that the operator $G : X \to X$ satisfies the following conditions:
(H1) $Gu = Lu + Hu$ for all $u \in X$.

(H2) $L : X \to X$ is a bounded linear operator (so it satisfies 2.1).
(H3) $H : X \to X$ is a continuous operator such that

$$\frac{||Hu||}{||u||} \to 0 \quad as \quad ||u|| \to 0.$$

(H4) $G : X \to X$ is a generalized $\varphi-$set contraction.

A real number λ is called a characteristic value of L if there exists a nonzero vector u in X such that $u = \lambda Lu$. We call the line $\{(\lambda, 0) : \lambda \in R\}$ the set of trivial solutions of 2.2. Let S denote the subset of E consisting of all nontrivial solutions of 2.2. A point $(\mu, 0)$ is called a bifurcation point of 2.2 if, given any $\epsilon > 0$, there exists an element $(\lambda, u) \in S$ such that $|\lambda - \mu|^2 + ||u||^2 < \epsilon^2$.
Now we give a necessary condition for the existence of bifurcation points of the above equation, for the case of countably generalized $\varphi-$set operators, which extends [10].

Theorem 2.10. *Let X be a real Banach space and let $G : X \to X$ satisfy the hypotheses $(H1), (H2)$ and $(H3)$. Suppose that $\mu \in R$ be such that $\mu^n \leq k^{-1}$ and that μ is not a characteristic value of L. Then $(\mu, 0)$ is not a bifurcation point of* 2.2.

Proof. Suppose that μ is not a characteristic value of L. Then $I - \mu L$ is injective. From $\mu^n \leq k^{-1}$ it follows by Proposition 2.8 that $I - \mu L$ is Fredholm of index zero. Hence $codimR(I - \mu L) = dimN(I - \mu L) = 0$ and so $R(I - \mu L) = X$. Since $I - \mu L$ is a bijective bounded linear operator on X, the bounded inverse theorem implies that $(I - \mu L)^{-1}$ is bounded. Assume on the contrary that $(\mu, 0)$ is a bifurcation point of 2.2. Then there exist a sequence $\{u_n\}$ in $X \setminus \{0\}$ and a sequence $\{\mu_n\}$ in R such that $u_n = \mu_n G u_n$, $u_n \neq 0$ and $\mu_n \neq \mu$ as $n \to \infty$. Hence we have

$$\begin{aligned} ||u_n|| \leq & \quad ||(I - \mu L)^{-1}||||(I - \mu L)u_n|| \\ = & \quad ||(I - \mu L)^{-1}||||\mu_n H u_n + (\mu - \lambda)Lu_n|| \\ \leq & \quad ||(I - \mu L)^{-1}||(|\mu_n|||Hu_n|| + |\mu_n - \mu|||L||||u_n||). \end{aligned}$$

Therefore

$$1 \leq ||(I - \mu L)^{-1}||(|\mu_n|\frac{||Hu_n||}{||u_n||} + |\mu_n - \mu|||L||).$$

Since the right-hand side of the last inequality tends to zero as $n \to \infty$ by $(H3)$, this is a contradiction. We conclude that $(\mu, 0)$ is not a bifurcation point of 2.2. $\square$

Definition 2.11. *A bounded linear operator $T : X \to X$ on a complex Banach space X is said to be commutable if there exists a finite-dimensional linear operator F such that F commutes with T and $I - T - F$ is a one-to-one operator of X onto X. When we say an operataor is finite dimensional, we shall mean its range is finite dimensional and when we say that a linear operator F commutes with T we shall mean (i) the domain of F, $D(F)$, contains the domain of T, (ii) $F(x) \in D(T)$ whenever $x \in D(T)$, (iii) and $TFx = FTx$ for $x \in D(T^2)$.*

Browder defined the essential spectrum of a densely defined closed linear operator T on a Banach space, in symbols $ess(T)$, to be the set of $\lambda \in \sigma(T)$ such that at least one of the following conditions holds:
(i) $R(\lambda I - T)$, the range of $\lambda I - T$, is not closed.
(ii) λ is a limit point of $\sigma(T)$.

(iii) $\cup_{\nu=1}^{\infty} N(\lambda I - T)^{\nu}$ is infinite dimensional, where $N(\lambda I - T)^{\nu}$ denotes the null space of $(\lambda I - T)^{\nu}$.

If T is a densely defined closed linear operator on X, define $r_e(T)$, the essential spectral radius of T, by

$$r_e(T) := \sup\{|\lambda| : \lambda \in ess(T)\}.$$

Note that $ess(T)$ is the largest subset of the spectrum $\sigma(T)$ which remains invariant under perturbations of T by compact operators which commute with T, i.e.

$$\begin{aligned} ess(T) = \{\lambda : \quad & \lambda \in \sigma(T + C) \text{for every compact operator } C \text{ such that} \\ & C(D(T)) \subset C, \quad and \quad TCx = CTx \quad for \quad x \in D(T^2)\} \end{aligned}$$

Now, we may recall another notion of the essential spectrum, introduced by Schechter [17], as follows:

If one takes the essential spectrum to be the largest subset of the spectrum which remains invariant under arbitrary compact perturbations it yeilds to Schechter's definition. Let T be a closed linear operator on a Banach space X.

The Schechter essential spectrum of the operator T is defined by

$$\sigma_s(T) = \cap_{K \in K(X)} \sigma(T + K),$$

where $K(X)$ denote the set of all compact linear operators. It is clear that $ess(T)$ includes properly $\sigma_s(T)$ and if we add to $\sigma_s(T)$, all limit points of the spectrum, then it will be equivalent to one given by Browder, i.e $ess(T)$.

The following proposition gives a characterization of the Schechter essential spectrum by means of Fredholm operators:

Proposition 2.12. *([16], Theorem 5.4, p. 180]). Let X be a Banach space and $T : X \to X$ be a closed, densely defined linear operator. Then*

$$\lambda \notin \sigma_s(T) \quad \textit{if and only if} \quad \lambda I - T \textit{is a Fredholm operator of index zero.}$$

Lemma 2.13. *(Nussbaum [13]). Let T be an operator on X and $r > r_e(T)$. Then there exists a finite dimensional operator F on X, which commutes with T, such that $\sigma(T + F) \subset \{\lambda \in C : |\lambda| \leq r\}$.*

Lemma 2.14. *Let $T : X \to X$ be a bounded linear operator on a complex Banach space X. If $r_e(T) < 1$ then T is comutable.*

Proof. By Lemma 2.13 there exists a finite dimensional operator F on X, which commutes with T, such that $I - T - F$ is invertible operator of X onto X, so T is commutable. □

Corollary 2.15. *Let $T : X \to X$ be a densely defined closed linear operator on a complex Banach space X which is countably generalized $\varphi-$set contraction. Then $r_e(T) < 1$ if and only if 1 is not belong to limit points of spectrum of T.*

Proof. Since T is a linear operator satisfying , for $|t| \leq 1$, tT is generalized $\varphi-$ set contraction and therefore by 2.8, $I - tT$ is a Fredholm operator of index zero, so $\lambda I - T$ for $\{\lambda : |\lambda| \geq 1\}$ is a Fredholm operator of index zero.
Thus by Proposition 2.12, $\lambda \notin \sigma_s(T)$ for $\lambda \in C$ with $|\lambda| \geq 1$, and since $\{\lambda \in C : |\lambda| = 1\}$ is not a limit point of spectrum of T, therefore

$$\sigma_e(T) \subset \{\lambda \in C : |\lambda| < 1\}.$$

Consequently $r_e(T) < 1$. $\square$

Theorem 2.16. *Let X be a real or complex Banach space and $T : X \to X$ a bounded linear comutable operator which is countably generalized $\varphi-$set contraction. Then there exist a finite-dimensional subspace N and a closed subspace E of finite codimension such that $X = N \oplus E$, N and E are both invariant under T , and $(I - tT)|_E$ is a homeomorphism of E onto itself for each $t \in [0, 1]$.*

Proof. Since T is a countably generalized $\varphi-$set contraction, $I - T$ is a Fredholm operator of index zero. If C is a closed subspace of X such that $I - T|_C : C \to C$ is one-to-one, this implies that $I - T|_C$ is one-to-one and onto C.
Let ζ be the complexification of B and let $\mathbf{T}$ be the natural extension of T to ζ:

$$\mathbf{T}(x + iy) = Tx + iTy \quad for \quad x, y \in C.$$

Since $\mathbf{T}$ is comutable, there exists a finite-dimensional complex linear operator F such that $I - \mathbf{T} - F$ is a one-to-one operator of ζ onto ζ and F commutes with $\mathbf{T}$. Now applying Theorem 2.7 in [10], the desired result is obtained. $\square$

Corollary 2.17. *Let $T : X \to X$ be a comutable, countably generalized $\varphi-$set contractive bounded linear operator. Then the sum of the multiplicities of the eigenvalues $\lambda \geq 1$ of T is finite.*

Proof. Let $\lambda \geq 1$ be any eigenvalue of T. Suppose that $x \in X$ is a nonzero vector such that $(\lambda I - T)^n x = 0$ for some positive integer n. Let $x = z + w$, where $z \in N$ and $w \in E$, and $X = N \oplus E$ is the decomposition described in Theorem 2.16. Since N and E are invariant under T , we have $(\lambda I - T)^n z = -(\lambda I - T)^n w \in N \cap E = \{0\}$, and so $(\lambda I - T)^n w = 0$. Since $(I - \lambda^{-1}T)|_E$ is one-to-one by Theorem 2.16, we have $w = 0$, and therefore $x = z \in N$. Since N is finite dimensional, the conclusion follows. $\square$

Let Ω be a nonempty bounded open set in a Banach space X. If $T : X \to X$ is a countably γ-condensing operator that has no fixed points on the boundary $\partial\Omega$, one may define the degree of $I - T$ on Ω as an integer, denoted by $deg(I - T, \Omega, 0)$ More details of this definition are given in [20].

The above degree has the following basic properties, see [18, Theorem 1.3] and [18, Corollary 2.1].

Lemma 2.18. *Let Ω be a nonempty bounded open set in a Banach space X and $T : \bar{\Omega} \to X$ a countably γ-condensing operator such that T has no fixed points on $\partial\Omega$. Then the following statements hold:*
(1) *If $deg(I - T, \Omega, 0) = 0$, then T has a fixed point in Ω.*
(2) *If $0 \in \Omega$, then $deg(I, \Omega, 0) = 1$.*
(3) *(Homotopy invariance) If $H : [0, 1] \times \bar{\Omega} \to X$ is a countably γ-condensing homotopy such that $H(t, x) \neq x$ for all $(t, x) \in [0, 1] \times \partial\Omega$, then*

$$deg(I - H(0, .), \Omega, 0) = deg(I - H(1, .), \Omega, 0).$$

Remark 2.19. *If T is a generalized $\varphi-$set contraction, then T^n is a condensing mapping, so we can use the above properties for it.*

Definition 2.20. *Let Ω be an open subset of a Banach space X and $T : \Omega \to X$ a $\varphi-$set contractive operator. If x_0 is an isolated fixed point of T, then the index of x_0 for T is defined by*

$$ind(T, x_0) = deg(I - T, B(x_0, r), 0);$$

where $B(x_0, r)$ is an open ball in X centered at x_0 with radius r so small that T has no fixed points other than x_0 in $B(x_0, r)$.

Now, it is time to mention our main theorem which shows that the index formula holds for countably φ-set contractive bounded linear operators.

Proof. of 2.1:
Since 1 is not an eigenvalue of T , then 0 is the only fixed point of T and

$$ind(T, x_0) = deg(I - T, B(x_0, r), 0).$$

Let $X = N \oplus E$ be the decomposition of X introduced in Theorem 2.16. Define an operator $S : X \to X$ by $S = T \circ P$, where P denotes the projection onto N. Since N is finite dimensional, we obtain that P is compact and so is S. Now consider a continuous homotopy $H : [0, 1] \times X \to X$ defined by

$$H(t, x) = tSx + (1 - t)Tx \quad for \quad (t, x) \in [0, 1] \times X.$$

Then H is countably γ-condensing on $[0, 1] \times \bar{B}(0, 1)$. In fact, for each countable set $C \subset \bar{B}(0, 1)$ with $\gamma(C) > 0$ we have

$$\begin{aligned} \gamma(H([0, 1] \times C)) \leq & \quad \gamma(co(S(C) \cup T(C))) \\ \leq & \quad \max\{\gamma(S(C)), \gamma(T(C))\} \\ \leq & \quad \varphi(\gamma(C)) \\ < & \quad \gamma(C) \end{aligned}$$

because S is compact and T is countably $\varphi-$set contraction. We claim that $H(t, x) \neq x$ for all $(t, x) \in [0, 1] \times \partial B(0, 1)$. Indeed, suppose that $H(t_0, x_0) = x_0$ for some $(t_0, x_0) \in [0, 1] \times \partial B(0, 1)$. Let $x_0 = z + w$, where $z \in N$ and $w \in E$. Then $z + w = t_0 Tz + (1 - t_0)Tz + (1 - t_0)Tw$. By the invariance of N and E under T , we have $z = Tz$ and $w = (1 - t_0)Tw$. Since 1 is not an eigenvalue of T and T is $\varphi-$set contraction, $I - (1 - t_0)T|E$ is one-to-one by Theorem 2.7, we have $z = 0$ and $w = 0$ and hence $x_0 = 0$, which contradicts the assumption that $x_0 \in \partial B(0, 1)$. Lemma 3.1 implies that $deg(I - T, B(0, 1), 0)$ and $deg(I - S, B(0, 1), 0)$ are equal. Since S is compact, $deg(I - S, B(0, 1), 0)$ is equal to the LeraySchauder degree. Using the LeraySchauder formula for a compact linear operator (see e.g. [3, Theorem 8.10]), we have

$$ind(T, 0) = deg(I - S, B(0, 1), 0) = (-1)^{\nu},$$

where ν is the sum of the multiplicities of the eigenvalues $\lambda > 1$ of S. It remains to show that $V_n = U_n$ for every positive integer n, where

$$\begin{aligned} V_n = & \quad \{(\lambda, x) : \lambda > 1, x \in X, x \neq 0 \;\; and \;\; (I - T)^n x = 0\}; \\ U_n = & \quad \{(\lambda, x) : \lambda > 1, x \in X, x \neq 0 \;\; and \;\; (I - S)^n x = 0\}; \end{aligned}$$

Suppose that $(\lambda, x) \in V_n$ and put $x = z + w$, where $z \in N$ and $w \in E$. Then $(\lambda I - T)^n z = -(\lambda I - T)^n w \in N \cap E = \{0\}$. Since $(I - \lambda^{-1}T)|E$ is one-to-one, we have $w = 0$ and so $Tx = T(Pz) = Sx$. This shows that $V_n \subset U_n$. Now let $(\lambda, x) \in U_n$. Since $S = T$ on N and $S = 0$ on E, we have $(\lambda I - S)^n z = -(\lambda I - S)^n w = -\lambda^n w = 0$, where $x = z + w$ with $z \in N$ and $w \in E$. Hence $w = 0$ and $Sx = Tz = Tx$. This shows that $U_n \subset V_n$. This completes the proof. $\square$

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