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ON ESTIMATES FOR THE DUNKL TRANSFORM IN THE SPACE $L_{2,\alpha}(\mathbb{R})$

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ABSTRACT. In this paper, we study two estimates useful in applications are proved for the Dunkl transform in a Hilbert space $L_{2,\alpha}(\mathbb{R}) = L_2(\mathbb{R}, |x|^{2\alpha+1}dx)$, $\alpha > -\frac{1}{2}$ as applied to some classes of functions characterized by a generalized modulus of continuity.

1. Introduction and preliminaries

Dunkl operators are differential-difference operators introduced in 1989, by Dunkl [5]. On the real line, these operators, which are denote by D_α, depend on a real parameter $\alpha > -\frac{1}{2}$ and they are associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. For $\alpha > -\frac{1}{2}$, Dunkl kernel e_α is defined as the unique solution of a differential-difference equation related to D_α and satisfying $e_\alpha(0) = 1$. This kernel is used to define Dunkl transform which was introduced by Dunkl in [6]. More complete results concerning this transform were later obtained by de Jeu [7]. Rösler in [8] shows that Dunkl kernels verify a product formula. This allows us to define Dunkl translation operators $T_h,\ h \in \mathbb{R}$.

The Dunkl operator on $\mathbb{R}$ of index $(\alpha + \frac{1}{2})$ is defined in [5] by

$$Df(x) = D_\alpha f(x) = \frac{df(x)}{dx} + (\alpha + \frac{1}{2})\frac{f(x) - f(-x)}{x}, \quad \alpha > \frac{-1}{2}.$$

These operators are very important in mathematics and physics.

In this paper, we prove two useful estimates in certain classes of functions characterized by a generalized continuity modulus and connected with the Dunkl transform in $L_{2,\alpha}(\mathbb{R})$, For this purpose, we use a translation operator in [4]. We point out that similar results have been established in the context of Fourier transform in real line (see [2]).

Assume that $L_{2,\alpha}(\mathbb{R})$, is stand for the Hilbert space which consists of measurable functions $f(x)$ is defined on $\mathbb{R}$ with the finite norm

$$\|f\| = \|f\|_{2,\alpha} = \left(\int_{-\infty}^{+\infty} |f(x)|^2 |x|^{2\alpha+1} dx\right)^{\frac{1}{2}},$$

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Given a function $f \in \mathrm{L}_{2,\alpha}(\mathbb{R})$, the Dunkl transform [4] of order α is defined as

$$\widehat{f}(\lambda) = \int_{-\infty}^{+\infty} f(x)e_\alpha(\lambda x)|x|^{2\alpha+1}dx, \quad \lambda \in \mathbb{R},$$

where $e_\alpha(x)$ Dunkl kernel is defined by

$$e_\alpha(x) = j_\alpha(x) + i(2\alpha+2)^{-1}xj_{\alpha+1}(x). \tag{1}$$

The function $y = e_\alpha(x)$ satisfies the equation $\mathrm{D}y = iy$ with the initial condition $y(0) = 1$, $j_\alpha(x)$ is a normalized Bessel function of the first kind, i.e

$$j_\alpha(x) = \frac{2^\alpha \Gamma(\alpha+1)J_\alpha(x)}{x^\alpha}, \tag{2}$$

where $J_\alpha(x)$ is a Bessel function of the first kind ([3], chap7) the function j_α is infinitely differentiable and even, in addition, $j_\alpha(0) = 1$.

From formula (1), we have

$$|1 - j_\alpha(x)| \leq |1 - e_\alpha(x)| \tag{3}$$

The inverse Dunkl transform is defined by the formula

$$f(x) = \frac{1}{(2^{\alpha+1}\Gamma(\alpha+1))^2} \int_{-\infty}^{+\infty} \widehat{f}(\lambda)e_\alpha(-\lambda x)|\lambda|^{2\alpha+1}d\lambda.$$

In $\mathrm{L}_{2,\alpha}(\mathbb{R})$, we define the operator of the generalized Dunkl translation (see [9])

$$\begin{aligned}\mathrm{T}_h f(x) = C(\int_0^\pi f_e(G(x,h,\varphi))h^e(\quad x \quad, h, \varphi)sin^{2\alpha}\varphi d\varphi \\ + \quad \int_0^\pi f_0(G(x,h,\varphi))h^0(x,h,\varphi)sin^{2\alpha}\varphi d\varphi)\end{aligned}$$

where

$$C = \frac{\Gamma(\alpha+1)}{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{1}{2})}, \quad G(x,h,\varphi) = \sqrt{x^2+h^2-2|xh|cos\varphi}$$

$$h^e(x,h,\varphi) = 1 - sgn(xh)cos\varphi$$

and

$$\begin{cases} h^0(x,h,\varphi) = \frac{(x+h)h^e(x,h,\varphi)}{G(x,h,\varphi)} & \text{for} \quad (x,h) \neq (0,0) \\ h^0(x,h,\varphi) = 0 & \text{for } (x,h) = (0,0) \end{cases}$$

$$f_e(x) = \frac{1}{2}(f(x) + f(-x)), \quad f_0(x) = \frac{1}{2}(f(x) - f(-x)).$$

Lemma 1.1. [4] *Let $f \in \mathrm{L}_{2,\alpha}(\mathbb{R})$, then the following equality is true for any $h \in \mathbb{R}$*

$$\widehat{(\mathrm{T}_h f)}(\lambda) = e_\alpha(\lambda h)\widehat{f}(\lambda).$$

The first-and higher order finite differences of f(x) are defined as follows

$$\Delta_h f(x) = \mathrm{T}_h f(x) - f(x) = (\mathrm{T}_h - \mathrm{I})f(x),$$

where I is the identity operator in $\mathrm{L}_{2,\alpha}(\mathbb{R})$.

$$\Delta_h^k f(x) = \Delta_h(\Delta_h^{k-1} f(x)) = (\mathrm{T}_h - \mathrm{I})^k f(x) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \mathrm{T}_h^i f(x),$$

where $\mathrm{T}_h^0 f(x) = f(x)$, $\mathrm{T}_h^i f(x) = \mathrm{T}_h(\mathrm{T}_h^{i-1} f(x))$ for i=1,2,....,k and k=1,2,.....

The kth order generalized modulus of continuity of function $f \in \mathrm{L}_{2,\alpha}(\mathbb{R})$ such that

$$\Omega_k(f,\delta) = \sup_{0<h\leq\delta} \|\Delta_h^k f(x)\|.$$

Let $\mathrm{W}_{2,\alpha}^m$ be the Sobolev space constructed by the operator D such that

$$\mathrm{W}_{2,\alpha}^m = \{f \in \mathrm{L}_{2,\alpha}(\mathbb{R}), \quad \mathrm{D}^j f \in \mathrm{L}_{2,\alpha}(\mathbb{R}), \ j = 1, 2, ..., m\}$$

$\mathrm{W}_{2,\phi}^{m,k}(\mathrm{D})$ denote the class of functions $f \in \mathrm{W}_{2,\alpha}^m$, satistying the estimate

$$\Omega_k(\mathrm{D}^m f, \delta) = O(\phi(\delta^m)),$$

where $\phi(t)$ is any nonnegative function given on $[0,\infty)$ and $\phi(0) = 0$, for the Dunkl operator D, we have $\mathrm{D}^0 f = f$, $\mathrm{D}^m f = \mathrm{D}(\mathrm{D}^{m-1} f), m = 1, 2, ...$

From lemma 1.1, we have

$$\mathrm{T}_h f(x) = \frac{1}{(2^{\alpha+1}\Gamma(\alpha+1))^2} \int_{-\infty}^{+\infty} e_\alpha(\lambda h)\widehat{f}(\lambda)e_\alpha(-\lambda x)|\lambda|^{2\alpha+1} d\lambda$$

Therefore, combining the relation

$$\mathrm{T}_h f(x) - f(x) = \frac{1}{(2^{\alpha+1}\Gamma(\alpha+1))^2} \int_{-\infty}^{+\infty} (e_\alpha(\lambda h) - 1)\widehat{f}(\lambda)e_\alpha(-\lambda x)|\lambda|^{2\alpha+1} d\lambda$$

Parseval's identity gives (see [4])

$$\|\mathrm{T}_h f(x) - f(x)\|^2 = A \int_{-\infty}^{+\infty} |e_\alpha(\lambda h) - 1|^2 |\widehat{f}(\lambda)|^2 |\lambda|^{2\alpha+1} d\lambda,$$

where

$$A = (2^{\alpha+1}\Gamma(\alpha+1))^{-2}.$$

Hence, for any function $f \in \mathrm{W}_{2,\phi}^{m,k}(\mathrm{D})$, we obtain

$$\|\Delta_h^k \mathrm{D}^m f(x)\|^2 = A \int_{-\infty}^{+\infty} |e_\alpha(\lambda h) - 1|^{2k} |\lambda|^{2m} |\widehat{f}(\lambda)|^2 |\lambda|^{2\alpha+1} d\lambda. \tag{4}$$

2. Estimates for the Dunkl transform

Taking into account what was said in section 1, for some classes of functions characterized by the generalized modulus of continuity, we can prove two estimates for the integral

$$\int_{|\lambda|\geq R} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda,$$

which are useful in applications.

Lemma 2.1. *For $x \in \mathbb{R}$ the following inequalities are fulfilled:*

(1) $|e_\alpha(x)| \leq 1$,
(2) $|1 - e_\alpha(x)| \leq 2|x|$.

Proof. (see[4]) ■

Lemma 2.2. *The following inequalities are fulfilled:*

(1) $1 - j_p(x) = O(1),\ x \geq 1$,
(2) $1 - j_p(x) = O(x^2),\ 0 \leq x \leq 1$,
(3) $\sqrt{hx}J_p(hx) = O(1),\ hx \geq 0$.

Proof. (see [1]) ■

Theorem 2.3. *For $f(x) \in \mathrm{L}_{2,\alpha}(\mathbb{R})$*

$$\sup_{\mathrm{W}^{m,k}_{2,\phi}(\mathrm{D})} \sqrt{\int_{|\lambda|\geq R} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda} = O(R^{-m}\phi(\frac{c}{R})^k),$$

where m=0,1,....; k=1,2....; $c > 0$ is a fixed constant, and $\phi(t)$ is any nonnegative function defined on $[0, \infty)$.

Proof. Let $f \in \mathrm{W}^{m,k}_{2,\phi}(\mathrm{D})$. Taking into account the Hölder inequality yields

$$\begin{aligned}
&\int\limits_{|\lambda|\geq R} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda - \int_{|\lambda|\geq R} j_\alpha(\lambda h)|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda\\
= &\int\limits_{|\lambda|\geq R} (1-j_\alpha(\lambda h))|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda\\
= &\int\limits_{|\lambda|\geq R} (1-j_\alpha(\lambda h))(|\widehat{f}(\lambda)||\lambda|^{\alpha+\frac{1}{2}})^2d\lambda\\
= &\int\limits_{|\lambda|\geq R} (1-j_\alpha(\lambda h))(|\widehat{f}(\lambda)||\lambda|^{\alpha+\frac{1}{2}})^{2-\frac{1}{k}}(|\widehat{f}(\lambda)||\lambda|^{\alpha+\frac{1}{2}})^{\frac{1}{k}}d\lambda\\
\leq &\ (\int\limits_{|\lambda|\geq R} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda)^{\frac{2k-1}{2k}}(\int_{|\lambda|\geq R} |1-j_\alpha(\lambda h)|^{2k}|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda)^{\frac{1}{2k}}\\
\leq &\ (\int_{|\lambda|\geq R} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda)^{\frac{2k-1}{2k}}(\int\limits_{|\lambda|\geq R} |\lambda|^{-2m}|1-e_\alpha(\lambda h)|^{2k}|\lambda|^{2m}|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda)^{\frac{1}{2k}}\\
\leq &\ R^{\frac{-m}{k}}(\int_{|\lambda|\geq R} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda)^{\frac{2k-1}{2k}}(\int\limits_{|\lambda|\geq R} |1-e_\alpha(\lambda h)|^{2k}|\lambda|^{2m}|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda)^{\frac{1}{2k}}
\end{aligned}$$

In view of (4), we have

$$\int\limits_{|\lambda|\geq R} |\lambda|^{2m}|1-e_\alpha(\lambda h)|^{2k}|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda \leq \frac{1}{A}\|\Delta_h^k \mathrm{D}^m f(x)\|^2.$$

Therefore

$$\begin{aligned}
\int\limits_{|\lambda|\geq R} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda \quad &\leq \int\limits_{|\lambda|\geq R} j_\alpha(\lambda h)|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda\\
&+ \frac{1}{A}R^{\frac{-m}{k}}(\int\limits_{|\lambda|\geq R} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda)^{\frac{2k-1}{2k}}.\|\Delta_h^k \mathrm{D}^m f(x)\|^{\frac{1}{k}}
\end{aligned}$$

In view of formulas (2) and (2) in lemma 2.2,

$$j_\alpha(\lambda h) = O((|\lambda h|)^{-\alpha-\frac{1}{2}}).$$

consequently

$$\begin{aligned}\int\limits_{|\lambda|\geq R}|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda &= O(\int\limits_{|\lambda|\geq R}|h\lambda|^{-\alpha-\frac{1}{2}}|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda)\\ &\quad +R^{\frac{-m}{k}}(\int\limits_{|\lambda|\geq R}|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda)^{\frac{2k-1}{2k}}\|\Delta_h^k\mathrm{D}^m f(x)\|^{\frac{1}{k}}\\ &= O((Rh)^{-\alpha-\frac{1}{2}})\int\limits_{|\lambda|\geq R}|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda\\ &\quad +R^{\frac{-m}{k}}(\int\limits_{|\lambda|\geq R}|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda)^{\frac{2k-1}{2k}}\|\Delta_h^k\mathrm{D}^m f(x)\|^{\frac{1}{k}}\end{aligned}$$

or

$$(1-O(Rh)^{-\alpha-\frac{1}{2}})\int\limits_{|\lambda|\geq R}|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda$$

$$=O(R^{\frac{-m}{k}})(\int\limits_{|\lambda|\geq R}|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda)^{\frac{2k-1}{2k}}\|\Delta_h^k\mathrm{D}^m f(x)\|^{\frac{1}{k}}$$

Setting $h=\frac{c}{R}$ in the last inequality and choosing $c>0$ such that

$$1-O(c^{-\alpha-\frac{1}{2}})\geq\frac{1}{2}$$

We obtain

$$\int\limits_{|\lambda|\geq R}|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda=O(R^{\frac{-m}{k}})(\int\limits_{|\lambda|\geq R}|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda)^{\frac{2k-1}{2k}}\phi^{\frac{1}{k}}((\frac{c}{R})^k)$$

we have

$$\int\limits_{|\lambda|\geq R}|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda=O(R^{-2m}\phi^2((\frac{c}{R})^k)).$$

the theorem is proved. ■

Theorem 2.4. *Let $\phi(t)=t^\nu$, then*

$$\left(\int\limits_{|\lambda|\geq R}|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda\right)^{\frac{1}{2}}=O(R^{-m-k\nu})\Longleftrightarrow f\in\mathrm{W}^{m,k}_{2,\phi}(\mathrm{D})$$

where m=0,1,...; k=1,2,....; $0<\nu<2$.

Proof. Sufficiency by Theorem 2.3 let $f\in\mathrm{W}^{m,k}_{2,t^\nu}(\mathrm{D})$ we have

$$\left(\int\limits_{|\lambda|\geq R}|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda\right)^{\frac{1}{2}}=O(R^{-m-k\nu})$$

Necessity: Let

$$\sqrt{\int\limits_{|\lambda|\geq R} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda} = O(R^{-m-k\nu})$$

that is

$$\int\limits_{|\lambda|\geq R} |\widehat{f}(\xi)|^2|\lambda|^{2\alpha+1}d\lambda = O(R^{-2m-2k\nu})$$

It is easy to prove, that there exists a function $f \in L_{2,\alpha}(\mathbb{R})$ such that $D^m f \in L_{2,\alpha}(\mathbb{R})$ and

$$D^m f(x) = \frac{(-i)^m}{(2^{\alpha+1}\Gamma(\alpha+1))^2}\int_{-\infty}^{+\infty} \lambda^m \widehat{f}(\lambda)e_\alpha(-\lambda x)|\lambda|^{2\alpha+1}d\lambda$$

Then, we have the equality

$$\|\Delta_h^k D^m f(x)\|^2 = A\int_{-\infty}^{+\infty} |1-e_\alpha(\lambda h)|^{2k}|\lambda|^{2m}|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda$$

This integral is divided into two:

$$\int_{-\infty}^{+\infty} = \int_{|\lambda|<R} + \int_{|\lambda|\geq R} = I_1 + I_2$$

where $R = [h^{-1}]$. and estimate each of them.

From (1) in lemma 2.1, we have

$$
\begin{aligned}
\mathrm{I}_2 &= \int\limits_{|\lambda|\geq R} |1-e_\alpha(\lambda h)|^{2k}|\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+2m+1}d\lambda \\
&= O\left(\int\limits_{|\lambda|\geq R} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+2m+1}d\lambda\right) \\
&= O\left(\sum_{n=R}^{\infty}\int_n^{n+1} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+2m+1}d\lambda\right) \\
&= O\left(\sum_{n=R}^{\infty} n^{2m}\int_n^{n+1} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda\right) \\
&= O\left(\sum_{n=R}^{\infty} n^{2m}\left[\int_n^{\infty} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda - \int_{n+1}^{\infty} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda\right]\right) \\
&= O\left(\sum_{n=R}^{\infty} n^{2m}\int_n^{\infty} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda - \sum_{n=R}^{\infty} n^{2m}\int_{n+1}^{\infty} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda\right) \\
&= O\left(R^{2m}\int_R^{\infty} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda\right. \\
&+ \left.\sum_{n=R+1}^{\infty} n^{2m}\int_n^{\infty} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda - \sum_{n=R}^{\infty} n^{2m}\int_{n+1}^{\infty} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda\right) \\
&= O(R^{2m}\int_R^{\infty} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda + \sum_{n=R}^{\infty}((n+1)^{2m}-n^{2m})\int_n^{\infty} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda) \\
&= O(R^{2m}\int_R^{\infty} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda + \sum_{n=R}^{\infty} n^{2m-1}\int_n^{\infty} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda) \\
&= O(R^{2m}R^{-2m-2k\nu}) + O(\sum_{n=R}^{\infty} n^{2m-1}n^{-2m-2k\nu}) \\
&= O(R^{-2k\nu}) + O(R^{-2k\nu}) = O(h^{2k\nu})
\end{aligned}
$$

i.e

$$\mathrm{I}_2 = O(h^{2k\nu}).$$

We estimate I_1, since (2) in lemma 2.1, we have

$$
\begin{aligned}
\mathrm{I}_1 &= \int\limits_{|\lambda|<R} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+2m+1}|1-e_\alpha(\lambda h)|^{2k}d\lambda \\
&= O(h^{2k}) \int\limits_{|\lambda|<R} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+2m+2k+1}d\lambda \\
&= O(h^{2k}) \sum_{n=0}^{R} \int\limits_{n\le|\lambda|<n+1} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+2m+2k+1}d\lambda \\
&= O(h^{2k}) \sum_{n=0}^{R}(n+1)^{2m+2k} \int\limits_{n\le|\lambda|<n+1} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda \\
&= O(h^{2k}) \sum_{n=0}^{R}(n+1)^{2m+2k} \left[\int\limits_{|\lambda|\ge n} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda - \int\limits_{|\lambda|\ge n+1} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda\right] \\
&= O(h^{2k}) \left[1+\sum_{n=0}^{R}((n+1)^{2m+2k}-n^{2m+2k}) \int\limits_{|\lambda|\ge n} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda\right] \\
&= O(h^{2k}) \left[1+\sum_{n=0}^{R} n^{2m+2k-1} \int\limits_{|\lambda|\ge n} |\widehat{f}(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda\right] \\
&= O(h^{2k}) \left[1+\sum_{n=0}^{R} n^{2m+2k-1}n^{-2m-2k\nu}\right] \\
&= O(h^{2k}) \left[1+\sum_{n=0}^{R} n^{2k-2k\nu-1}\right] \\
&= O(h^{2k})O(R^{2k-2k\nu}) = O(h^{2k\nu})
\end{aligned}
$$

that is

$$\mathrm{I}_1 = O(h^{2k\nu})$$

Combining the estimates for I_1 and I_2 gives

$$\|\Delta_h^k \mathrm{D}^m f(x)\| = O(h^{k\nu})$$

The necessity is proved ■

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