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Some Properties of Generalized (Λ, α)-Closed Sets

Chawalit Boonpok, Montri Thongmoon*

Mathematics and Applied Mathematics Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham, 44150, Thailand

**Corresponding author:* montri.t@msu.ac.th

Abstract. The aim of this paper is to introduce the concept of generalized (Λ, α)-closed sets. Moreover, we investigate some characterizations of Λ_α-$T_{\frac{1}{2}}$-spaces, (Λ, α)-normal spaces and (Λ, α)-regular spaces by utilizing generalized (Λ, α)-closed sets.

1. Introduction

The concept of generalized closed sets was first introduced by Levine [7]. Moreover, Levine defined a separation axiom called $T_{\frac{1}{2}}$ between T_0 and T_1. Dontchev and Ganster [3] introduced the notion of $T_{\frac{3}{4}}$-spaces which are situated between T_1 and $T_{\frac{1}{2}}$ and showed that the digital line or the Khalimsky line [5] $(\mathbb{Z}, \kappa)$ lies between T_1 and $T_{\frac{3}{4}}$. As a modification of generalized closed sets, Palaniappan and Rao [10] introduced and studied the notion of regular generalized closed sets. As the further modification of regular generalized closed sets, Noiri and Popa [9] introduced and investigated the concept of regular generalized α-closed sets. Park et al. [11] obtained some characterizations of $T_{\frac{3}{4}}$ spaces. Dungthaisong et al. [4] characterized $\mu_{(m,n)}$-$T_{\frac{1}{2}}$ spaces by utilizing the concept of $\mu_{(m,n)}$-closed sets. Torton et al. [12] introduced and studied the notions of $\mu_{(m,n)}$-regular spaces and $\mu_{(m,n)}$-normal spaces. Buadong et al. [1] introduced and investigated the notions of T_1-GTMS spaces and T_2-GTMS spaces. Caldas et al. [2] by considering the concepts of α-open sets and α-closed sets, introduced and investigated Λ_α-sets, (Λ, α)-closed sets, (Λ, α)-open sets and the (Λ, α)-closure operator. Khampakdee and Boonpok [6] studied some properties of (Λ, α)-open sets. In the present paper, we introduce the concept of generalized (Λ, α)-closed sets. Furthermore, some properties of

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generalized (Λ, α)-closed sets are discussed. In particular, several characterizations of Λ_α-$T_{\frac{1}{2}}$-spaces, (Λ, α)-normal spaces and (Λ, α)-regular spaces are established.

2. Preliminaries

Let A be a subset of a topological space (X, τ). The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A subset A of a topological space (X, τ) is said to be *α-open* [8] if $A \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$. The complement of an α-open set is called *α-closed*. The family of all α-open sets in a topological space (X, τ) is denoted by $\alpha(X, \tau)$. A subset $\Lambda_\alpha(A)$ [2] is defined as follows:

$$\Lambda_\alpha(A) = \cap\{O \in \alpha(X, \tau) | A \subseteq O\}.$$

Lemma 2.1. [2] *For subsets A, B and $A_i (i \in I)$ of a topological space (X, τ), the following properties hold:*

(1) $A \subseteq \Lambda_\alpha(A)$.
(2) *If* $A \subseteq B$, *then* $\Lambda_\alpha(A) \subseteq \Lambda_\alpha(B)$.
(3) $\Lambda_\alpha(\Lambda_\alpha(A)) = \Lambda_\alpha(A)$.
(4) $\Lambda_\alpha(\cap\{A_i | i \in I\}) \subseteq \cap\{\Lambda_\alpha(A_i) | i \in I\}$.
(5) $\Lambda_\alpha(\cup\{A_i | i \in I\}) = \cup\{\Lambda_\alpha(A_i) | i \in I\}$.

Recall that a subset A of a topological space (X, τ) is said to be a *Λ_α-set* [2] if $A = \Lambda_\alpha(A)$.

Lemma 2.2. [2] *For subsets A and $A_i (i \in I)$ of a topological space (X, τ), the following properties hold:*

(1) $\Lambda_\alpha(A)$ *is a* Λ_α*-set.*
(2) *If A is α-open, then A is a Λ_α-set.*
(3) *If A_i is a Λ_α-set for each $i \in I$, then $\cap_{i \in I} A_i$ is a Λ_α-set.*
(4) *If A_i is a Λ_α-set for each $i \in I$, then $\cup_{i \in I} A_i$ is a Λ_α-set.*

A subset A of a topological space (X, τ) is called *(Λ, α)-closed* [2] if $A = T \cap C$, where T is a Λ_α-set and C is an α-closed set. The complement of a (Λ, α)-closed set is called *(Λ, α)-open*. The collection of all (Λ, α)-open (resp. (Λ, α)-closed) sets in a topological space (X, τ) is denoted by $\Lambda_\alpha O(X, \tau)$ (resp. $\Lambda_\alpha C(X, \tau)$). Let A be a subset of a topological space (X, τ). A point $x \in X$ is called a *(Λ, α)-cluster point* of A [2] if for every (Λ, α)-open set U of X containing x we have $A \cap U \neq \emptyset$. The set of all (Λ, α)-cluster points of A is called the *(Λ, α)-closure* of A and is denoted by $A^{(\Lambda,\alpha)}$.

Lemma 2.3. [2] *Let A and B be subsets of a topological space (X, τ). For the (Λ, α)-closure, the following properties hold:*

(1) $A \subseteq A^{(\Lambda,\alpha)}$ *and* $[A^{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} = A^{(\Lambda,\alpha)}$.
(2) $A^{(\Lambda,\alpha)} = \cap\{F | A \subseteq F$ *and F is (Λ, α)-closed*$\}$.
(3) *If* $A \subseteq B$, *then* $A^{(\Lambda,\alpha)} \subseteq B^{(\Lambda,\alpha)}$.

(4) *A is* (Λ, α)*-closed if and only if* $A = A^{(\Lambda,\alpha)}$.

(5) $A^{(\Lambda,\alpha)}$ *is* (Λ, α)*-closed.*

Definition 2.1. [6] *Let A be a subset of a topological space* (X, τ)*. The union of all* (Λ, α)*-open sets of X contained in A is called the* (Λ, α)*-interior of A and is denoted by* $A_{(\Lambda,\alpha)}$.

Lemma 2.4. [6] *Let A and B be subsets of a topological space* (X, τ)*. For the* (Λ, α)*-interior, the following properties hold:*

(1) $A_{(\Lambda,\alpha)} \subseteq A$ *and* $[A_{(\Lambda,\alpha)}]_{(\Lambda,\alpha)} = A_{(\Lambda,\alpha)}$.

(2) *If* $A \subseteq B$*, then* $A_{(\Lambda,\alpha)} \subseteq B_{(\Lambda,\alpha)}$.

(3) *A is* (Λ, α)*-open if and only if* $A_{(\Lambda,\alpha)} = A$.

(4) $A_{(\Lambda,\alpha)}$ *is* (Λ, α)*-open.*

(5) $[X - A]^{(\Lambda,\alpha)} = X - A_{(\Lambda,\alpha)}$.

(6) $[X - A]_{(\Lambda,\alpha)} = X - A^{(\Lambda,\alpha)}$.

3. Generalized (Λ, α)-closed sets

In this section, we introduce the notion of generalized (Λ, α)-closed sets. Moreover, some properties of generalized (Λ, α)-closed sets are discussed.

Definition 3.1. *A subset A of a topological space* (X, τ) *is said to be generalized* (Λ, α)*-closed (briefly g-*(Λ, α)*-closed) if* $A^{(\Lambda,\alpha)} \subseteq U$ *and U is* (Λ, α)*-open in* (X, τ)*. The complement of a generalized* (Λ, α)*-closed set is said to be generalized* (Λ, α)*-open (briefly g-*(Λ, α)*-open).*

Definition 3.2. *A topological space* (X, τ) *is said to be* Λ_α*-symmetric if for x and y in X,* $x \in \{y\}^{(\Lambda,\alpha)}$ *implies* $y \in \{x\}^{(\Lambda,\alpha)}$.

Theorem 3.1. *A topological space* (X, τ) *is* Λ_α*-symmetric if and only if* $\{x\}$ *is g-*(Λ, α)*-closed for each* $x \in X$.

Proof. Assume that $x \in \{y\}^{(\Lambda,\alpha)}$ but $y \notin \{x\}^{(\Lambda,\alpha)}$. This implies that the complement of $\{x\}^{(\Lambda,\alpha)}$ contains y. Therefore, the set $\{y\}$ is a subset of the complement of $\{x\}^{(\Lambda,\alpha)}$. This implies that $\{y\}^{(\Lambda,\alpha)}$ is a subset of the complement of $\{x\}^{(\Lambda,\alpha)}$. Now the complement of $\{x\}^{(\Lambda,\alpha)}$ contains x which is a contradiction.

Conversely, suppose that $\{x\} \subseteq V \in \Lambda_\alpha O(X, \tau)$, but $\{x\}^{(\Lambda,\alpha)}$ is not a subset of V. This means that $\{x\}^{(\Lambda,\alpha)}$ and the complement of V are not disjoint. Let y belongs to their intersection. Now, we have $x \in \{y\}^{(\Lambda,\alpha)}$ which is a subset of the complement of V and $x \notin V$. This is a contradiction. □

Theorem 3.2. *A subset A of a topological space* (X, τ) *is g-*(Λ, α)*-closed if and only if* $A^{(\Lambda,\alpha)} - A$ *contains no nonempty* (Λ, α)*-closed set.*

Proof. Let F be a (Λ,α)-closed subset of $A^{(\Lambda,\alpha)} - A$. Now, $A \subseteq X - F$ and since A is g-(Λ,α)-closed, we have $A^{(\Lambda,\alpha)} \subseteq X - F$ or $F \subseteq X - A^{(\Lambda,\alpha)}$. Thus, $F \subseteq A^{(\Lambda,\alpha)} \cap [X - A^{(\Lambda,\alpha)}] = \emptyset$ and hence F is empty.

Conversely, suppose that $A \subseteq U$ and U is (Λ,α)-open. If $A^{(\Lambda,\alpha)} \not\subseteq U$, then $A^{(\Lambda,\alpha)} \cap (X - U)$ is a nonempty (Λ,α)-closed subset of $A^{(\Lambda,\alpha)} - A$. □

Definition 3.3. *Let A be a subset of a topological space (X,τ). The (Λ,α)-frontier of A, $\Lambda_\alpha Fr(A)$, is defined as follows: $\Lambda_\alpha Fr(A) = A^{(\Lambda,\alpha)} \cap [X - A]^{(\Lambda,\alpha)}$.*

Theorem 3.3. *Let A be a subset of a topological space (X,τ). If A is g-(Λ,α)-closed and*

$$A \subseteq V \in \Lambda_\alpha O(X,\tau),$$

then $\Lambda_\alpha Fr(V) \subseteq [X - A]_{(\Lambda,\alpha)}$.

Proof. Let A be g-(Λ,α)-closed and $A \subseteq V \in \Lambda_\alpha O(X,\tau)$. Then, $A^{(\Lambda,\alpha)} \subseteq V$. Suppose that $x \in \Lambda_\alpha Fr(V)$. Since $V \in \Lambda_\alpha O(X,\tau)$, $\Lambda_\alpha Fr(V) = V^{(\Lambda,\alpha)} - V$. Therefore, $x \notin V$ and $x \notin A^{(\Lambda,\alpha)}$. Thus, $x \in [X - A]_{(\Lambda,\alpha)}$ and hence $\Lambda_\alpha Fr(V) \subseteq [X - A]_{(\Lambda,\alpha)}$. □

Theorem 3.4. *Let (X,τ) be a topological space. For each $x \in X$, either $\{x\}$ is (Λ,α)-closed or g-(Λ,α)-open.*

Proof. Suppose that $\{x\}$ is not (Λ,α)-closed. Then, $X - \{x\}$ is not (Λ,α)-open and the only (Λ,α)-open set containing $X - \{x\}$ is X itself. Thus, $[X - \{x\}]^{(\Lambda,\alpha)} \subseteq X$ and hence $X - \{x\}$ is g-(Λ,α)-closed. Therefore, $\{x\}$ is g-(Λ,α)-open. □

Theorem 3.5. *Let A be a subset of a topological space (X,τ). Then, A is g-(Λ,α)-open if and only if $F \subseteq A_{(\Lambda,\alpha)}$ whenever $F \subseteq A$ and F is (Λ,α)-closed.*

Proof. Suppose that A is g-(Λ,α)-open. Let $F \subseteq A$ and F be (Λ,α)-closed. Then, we have

$$X - A \subseteq X - F \in \Lambda_\alpha O(X,\tau)$$

and $X - A$ is g-(Λ,α)-closed. Thus, $X - A_{(\Lambda,\alpha)} = [X - A]^{(\Lambda,\alpha)} \subseteq X - F$ and hence $F \subseteq A_{(\Lambda,\alpha)}$.

Conversely, let $X - A \subseteq U$ and $U \in \Lambda_\alpha O(X,\tau)$. Then, $X - U \subseteq A$ and $X - U$ is (Λ,α)-closed. By the hypothesis, $X - U \subseteq A_{(\Lambda,\alpha)}$ and hence $[X - A]^{(\Lambda,\alpha)} = X - A_{(\Lambda,\alpha)} \subseteq U$. This shows that $X - A$ is g-(Λ,α)-closed. Thus, A is g-(Λ,α)-open. □

Theorem 3.6. *A subset A of a topological space (X,τ) is g-(Λ,α)-closed if and only if $A \cap \{x\}^{(\Lambda,\alpha)} \neq \emptyset$ for every $x \in A^{(\Lambda,\alpha)}$.*

Proof. Let A be a g-(Λ,α)-closed set and suppose that there exists $x \in A^{(\Lambda,\alpha)}$ such that $A \cap \{x\}^{(\Lambda,\alpha)} = \emptyset$. Therefore, $A \subseteq X - \{x\}^{(\Lambda,\alpha)}$ and so $A^{(\Lambda,\alpha)} \subseteq X - \{x\}^{(\Lambda,\alpha)}$. Hence $x \notin A^{(\Lambda,\alpha)}$, which is a contradiction.

Conversely, suppose that the condition of the theorem holds and let U be any (Λ, α)-open set containing A. Let $x \in A^{(\Lambda,\alpha)}$. Then, by the hypothesis $A \cap A^{(\Lambda,\alpha)} \neq \emptyset$, so there exists $y \in A \cap \{x\}^{(\Lambda,\alpha)}$ and so $y \in A \subseteq U$. Thus, $\{x\} \cap U \neq \emptyset$. Hence $x \in U$, which implies that $A^{(\Lambda,\alpha)} \subseteq U$. This shows that A is g-(Λ, α)-closed. □

Definition 3.4. *A subset A of a topological space (X, τ) is said to be locally (Λ, α)-closed if $A = U \cap F$, where $U \in \Lambda_\alpha O(X, \tau)$ and F is a (Λ, α)-closed set.*

Theorem 3.7. *For a subset A of a topological space (X, τ), the following properties are equivalent:*

(1) *A is locally (Λ, α)-closed;*
(2) *$A = U \cap A^{(\Lambda,\alpha)}$ for some $U \in \Lambda_\alpha O(X, \tau)$;*
(3) *$A^{(\Lambda,\alpha)} - A$ is (Λ, α)-closed;*
(4) *$A \cup [X - A^{(\Lambda,\alpha)}] \in \Lambda_\alpha O(X, \tau)$;*
(5) *$A \subseteq [A \cup [X - A^{(\Lambda,\alpha)}]]_{(\Lambda,\alpha)}$.*

Proof. (1) ⇒ (2): Suppose that $A = U \cap F$, where $U \in \Lambda_\alpha O(X, \tau)$ and F is a (Λ, α)-closed set. Since $A \subseteq F$, we have $A^{(\Lambda,\alpha)} \subseteq F^{(\Lambda,\alpha)} = F$. Since $A \subseteq U$, $A \subseteq U \cap A^{(\Lambda,\alpha)} \subseteq U \cap F = A$. Thus, $A = U \cap A^{(\Lambda,\alpha)}$ for some $U \in \Lambda_\alpha O(X, \tau)$.

(2) ⇒ (3): Suppose that $A = U \cap A^{(\Lambda,\alpha)}$ for some $U \in \Lambda_\alpha O(X, \tau)$. Then, we have

$$A^{(\Lambda,\alpha)} - A = [X - U \cap A^{(\Lambda,\alpha)}] \cap A^{(\Lambda,\alpha)} = (X - U) \cap A^{(\Lambda,\alpha)}.$$

Since $(X - U) \cap A^{(\Lambda,\alpha)}$ is (Λ, α)-closed, $A^{(\Lambda,\alpha)} - A$ is (Λ, α)-closed.

(3) ⇒ (4): Since $X - [A^{(\Lambda,\alpha)} - A] = [X - A^{(\Lambda,\alpha)}] \cup A$ and by (3), $A \cup [X - A^{(\Lambda,\alpha)}] \in \Lambda_\alpha O(X, \tau)$.

(4) ⇒ (5): By (4), we obtain $A \subseteq A \cup [X - A^{(\Lambda,\alpha)}] = [A \cup [X - A^{(\Lambda,\alpha)}]]_{(\Lambda,\alpha)}$.

(5) ⇒ (1): We put $U = [A \cup [X - A^{(\Lambda,\alpha)}]]_{(\Lambda,\alpha)}$. Then, $U \in \Lambda_\alpha O(X, \tau)$ and

$$A = A \cap U \subseteq U \cap A^{(\Lambda,\alpha)} \subseteq [A \cup [X - A^{(\Lambda,\alpha)}]]_{(\Lambda,\alpha)} \cap A^{(\Lambda,\alpha)} = A \cap A^{(\Lambda,\alpha)} = A.$$

Thus, $A = U \cap A^{(\Lambda,\alpha)}$, where $U \in \Lambda_\alpha O(X, \tau)$ and $A^{(\Lambda,\alpha)}$ is a (Λ, α)-closed set. This shows that A is locally (Λ, α)-closed. □

Theorem 3.8. *A subset A of a topological space (X, τ) is (Λ, α)-closed if and only if A is locally (Λ, α)-closed and g-(Λ, α)-closed.*

Proof. Let A be (Λ, α)-closed. Then, A is g-(Λ, α)-closed. Since $X \in \Lambda_\alpha O(X, \tau)$ and $A = X \cap A$, A is locally (Λ, α)-closed.

Conversely, suppose that A is locally (Λ, α)-closed and g-(Λ, α)-closed. Since A is locally (Λ, α)-closed, by Theorem 3.7, $A \subseteq [A \cup [X - A^{(\Lambda,\alpha)}]]_{(\Lambda,\alpha)}$. Since $[A \cup [X - A^{(\Lambda,\alpha)}]]_{(\Lambda,\alpha)} \in \Lambda_\alpha O(X, \tau)$ and A is g-(Λ, α)-closed, $A^{(\Lambda,\alpha)} \subseteq [A \cup [X - A^{(\Lambda,\alpha)}]]_{(\Lambda,\alpha)} \subseteq A \cup [X - A^{(\Lambda,\alpha)}]$ and hence $A^{(\Lambda,\alpha)} = A$. Thus, by Lemma 2.3, A is (Λ, α)-closed. □

4. Applications of generalized (Λ, α)-closed sets

We begin this section by introducing the concept of Λ_α-$T_{\frac{1}{2}}$-spaces.

Definition 4.1. *A topological space (X, τ) is called a Λ_α-$T_{\frac{1}{2}}$-space if every g-(Λ, α)-closed set of X is (Λ, α)-closed.*

Lemma 4.1. *Let (X, τ) be a topological space. For each $x \in X$, the singleton $\{x\}$ is (Λ, α)-closed or $X - \{x\}$ is g-(Λ, α)-closed.*

Proof. Let $x \in X$ and the singleton $\{x\}$ be not (Λ, α)-closed. Then, $X - \{x\}$ is not (Λ, α)-open and X is the only (Λ, α)-open set which contains $X - \{x\}$ and $X - \{x\}$ is g-(Λ, α)-closed. □

Let A be a subset of a topological space (X, τ). A subset $\Lambda_{(\Lambda,\alpha)}(A)$ [6] is defined as follows:

$$\Lambda_{(\Lambda,\alpha)}(A) = \cap\{U \mid A \subseteq U, U \in \Lambda_\alpha O(X, \tau)\}.$$

Lemma 4.2. [6] *For subsets A, B of a topological space (X, τ), the following properties hold:*

1. *$A \subseteq \Lambda_{(\Lambda,\alpha)}(A)$.*
2. *If $A \subseteq B$, then $\Lambda_{(\Lambda,\alpha)}(A) \subseteq \Lambda_{(\Lambda,\alpha)}(B)$.*
3. *$\Lambda_{(\Lambda,\alpha)}[\Lambda_{(\Lambda,\alpha)}(A)] = \Lambda_{(\Lambda,\alpha)}(A)$.*
4. *If A is (Λ, α)-open, $\Lambda_{(\Lambda,\alpha)}(A) = A$.*

A subset A of a topological space (X, τ) is called a $\Lambda_{(\Lambda,\alpha)}$-*set* if $A = \Lambda_{(\Lambda,\alpha)}(A)$. The family of all $\Lambda_{(\Lambda,\alpha)}$-sets of (X, τ) is denoted by $\Lambda_{(\Lambda,\alpha)}(X, \tau)$ (or simply $\Lambda_{(\Lambda,\alpha)}$).

Definition 4.2. *A subset A of a topological space (X, τ) is called a generalized $\Lambda_{(\Lambda,\alpha)}$-set (briefly g-$\Lambda_{(\Lambda,\alpha)}$-set) if $\Lambda_{(\Lambda,\alpha)}(A) \subseteq F$ whenever $A \subseteq F$ and F is (Λ, α)-closed.*

Lemma 4.3. *Let (X, τ) be a topological space. For each $x \in X$, the singleton $\{x\}$ is (Λ, α)-open or $X - \{x\}$ is g-$\Lambda_{(\Lambda,\alpha)}$-set.*

Proof. Let $x \in X$ and the singleton $\{x\}$ be not (Λ, α)-open. Then, $X - \{x\}$ is not (Λ, α)-closed and X is the only (Λ, α)-closed set which contains $X - \{x\}$ and $X - \{x\}$ is g-$\Lambda_{(\Lambda,\alpha)}$-set. □

Theorem 4.1. *For a topological space (X, τ), the following properties are equivalent:*

1. *(X, τ) is a Λ_α-$T_{\frac{1}{2}}$-space.*
2. *For each $x \in X$, the singleton $\{x\}$ is (Λ, α)-open or (Λ, α)-closed.*
3. *Every g-$\Lambda_{(\Lambda,\alpha)}$-set is a $\Lambda_{(\Lambda,\alpha)}$-set.*

Proof. (1) ⇒ (2): By Lemma 4.1, for each $x \in X$, the singleton $\{x\}$ is (Λ, α)-closed or $X - \{x\}$ is g-(Λ, α)-closed. Since (X, τ) is a Λ_α-$T_{\frac{1}{2}}$-space, we have $X - \{x\}$ is (Λ, α)-closed and hence $\{x\}$ is (Λ, α)-open in the latter case. Thus, the singleton $\{x\}$ is (Λ, α)-open or (Λ, α)-closed.

(2) ⇒ (3): Suppose that there exists a g-$\Lambda_{(\Lambda,\alpha)}$-set A which is not a $\Lambda_{(\Lambda,\alpha)}$-set. Then, there exists $x \in \Lambda_{(\Lambda,\alpha)}(A)$ such that $x \notin A$. In case the singleton $\{x\}$ is (Λ,α)-open, $A \subseteq X - \{x\}$ and $X - \{x\}$ is (Λ,α)-closed. Since A is a g-$\Lambda_{(\Lambda,\alpha)}$-set, $\Lambda_{(\Lambda,\alpha)}(A) \subseteq X - \{x\}$. This is a contradiction. In case the singleton $\{x\}$ is (Λ,α)-closed, $A \subseteq X - \{x\}$ and $X - \{x\}$ is (Λ,α)-open. By Lemma 4.2,

$$\Lambda_{(\Lambda,\alpha)}(A) \subseteq \Lambda_{(\Lambda,\alpha)}(X - \{x\}) = X - \{x\}.$$

This is a contradiction. Therefore, every g-$\Lambda_{(\Lambda,\alpha)}$-set is a $\Lambda_{(\Lambda,\alpha)}$-set.

(3) ⇒ (1): Suppose that (X,τ) is not a Λ_α-$T_{\frac{1}{2}}$-space. There exists a g-(Λ,α)-closed set A which is not (Λ,α)-closed. Since A is not (Λ,α)-closed, there exists a point $x \in A^{(\Lambda,\alpha)}$ such that $x \notin A$. By Lemma 4.3, the singleton $\{x\}$ is (Λ,α)-open or $X - \{x\}$ is a g-$\Lambda_{(\Lambda,\alpha)}$-set. (a) In case $\{x\}$ is (Λ,α)-open, since $x \in A^{(\Lambda,\alpha)}$, $\{x\} \cap A \neq \emptyset$ and $x \in A$. This is a contradiction. (b) In case $X - \{x\}$ is a $\Lambda_{(\Lambda,\alpha)}$-set, if $\{x\}$ is not (Λ,α)-closed, $X - \{x\}$ is not (Λ,α)-open and $\Lambda_{(\Lambda,\alpha)}(X - \{x\}) = X$. Thus, $X - \{x\}$ is not a $\Lambda_{(\Lambda,\alpha)}$-set. This contradicts (3). If $\{x\}$ is (Λ,α)-closed, $A \subseteq X - \{x\} \in \Lambda_\alpha O(X,\tau)$ and A is g-(Λ,α)-closed. Thus, $A^{(\Lambda,\alpha)} \subseteq X - \{x\}$. This contradicts that $x \in A^{(\Lambda,\alpha)}$. Therefore, (X,τ) is a Λ_α-$T_{\frac{1}{2}}$-space. □

Definition 4.3. *A topological space (X,τ) is said to be (Λ,α)-normal if for any pair of disjoint (Λ,α)-closed sets F and H, there exist disjoint (Λ,α)-open sets U and V such that $F \subseteq U$ and $H \subseteq V$.*

Lemma 4.4. *Let (X,τ) be a topological space. If U is a (Λ,α)-open set, then $U^{(\Lambda,\alpha)} \cap A \subseteq [U \cap A]^{(\Lambda,\alpha)}$ for every subset A of X.*

Theorem 4.2. *For a topological space (X,τ), the following properties are equivalent:*

1. *(X,τ) is (Λ,α)-normal.*
2. *For every pair of (Λ,α)-open sets U and V whose union is X, there exist (Λ,α)-closed sets F and H such that $F \subseteq U$, $H \subseteq V$ and $F \cup H = X$.*
3. *For every (Λ,α)-closed set F and every (Λ,α)-open set G containing F, there exists a (Λ,α)-open set U such that $F \subseteq U \subseteq U^{(\Lambda,\alpha)} \subseteq G$.*
4. *For every pair of disjoint (Λ,α)-closed sets F and H, there exist disjoint (Λ,α)-open sets U and V such that $F \subseteq U$ and $H \subseteq V$ and $U^{(\Lambda,\alpha)} \cap V^{(\Lambda,\alpha)} = \emptyset$.*

Proof. (1) ⇒ (2): Let U and V be a pair of (Λ,α)-open sets such that $X = U \cup V$. Then, $X - U$ and $X - V$ are disjoint (Λ,α)-closed sets. Since (X,τ) is (Λ,α)-normal, there exist disjoint (Λ,α)-open sets G and W such that $X - U \subseteq G$ and $X - V \subseteq W$. Put $F = X - G$ and $H = X - W$. Then, F and H are (Λ,α)-closed sets such that $F \subseteq U$, $H \subseteq V$ and $F \cup H = X$.

(2) ⇒ (3): Let F be a (Λ,α)-closed set and G be a (Λ,α)-open set containing F. Then, $X - F$ and G are (Λ,α)-open sets whose union is X. Then by (2), there exist (Λ,α)-closed sets M and N such that $M \subseteq X - F$, $N \subseteq G$ and $M \cup N = X$. Then, $F \subseteq X - M$, $X - G \subseteq X - N$ and $(X - M) \cap (X - N) = \emptyset$. Put $U = X - M$ and $V = X - N$. Then U and V are disjoint (Λ,α)-open

sets such that $F \subseteq U \subseteq X - V \subseteq G$. As $X - V$ is a (Λ, α)-closed set, we have $U^{(\Lambda,\alpha)} \subseteq X - V$ and hence $F \subseteq U \subseteq U^{(\Lambda,\alpha)} \subseteq G$.

(3) $\Rightarrow$ (4): Let F and H be two disjoint (Λ, α)-closed sets of X. Then, $F \subseteq X - H$ and $X - H$ is (Λ, α)-open and hence there exists a (Λ, α)-open set U of X such that $F \subseteq U \subseteq U^{(\Lambda,\alpha)} \subseteq X - H$. Put $V = X - U^{(\Lambda,\alpha)}$. Then, U and V are disjoint (Λ, α)-open sets of X such that $F \subseteq U$, $H \subseteq V$ and $U^{(\Lambda,\alpha)} \cap V^{(\Lambda,\alpha)} = \emptyset$.

(4) $\Rightarrow$ (1): The proof is obvious. □

Theorem 4.3. *For a topological space (X, τ), the following properties are equivalent:*

1. *(X, τ) is (Λ, α)-normal.*
2. *For every pair of disjoint (Λ, α)-closed sets F and H of X, there exist disjoint g-(Λ, α)-open sets U and V of X such that $F \subseteq U$ and $H \subseteq V$.*
3. *For each (Λ, α)-closed set F and each (Λ, α)-open set G containing F, there exists a g-(Λ, α)-open set U such that $F \subseteq U \subseteq U^{(\Lambda,\alpha)} \subseteq G$.*
4. *For each (Λ, α)-closed set F and each g-(Λ, α)-open set G containing F, there exists a (Λ, α)-open set U such that $F \subseteq U \subseteq U^{(\Lambda,\alpha)} \subseteq G_{(\Lambda,\alpha)}$.*
5. *For each (Λ, α)-closed set F and each g-(Λ, α)-open set G containing F, there exists a g-(Λ, α)-open set U such that $F \subseteq U \subseteq U^{(\Lambda,\alpha)} \subseteq G_{(\Lambda,\alpha)}$.*
6. *For each g-(Λ, α)-closed set F and each (Λ, α)-open set G containing F, there exists a (Λ, α)-open set U such that $F^{(\Lambda,\alpha)} \subseteq U \subseteq U^{(\Lambda,\alpha)} \subseteq G$.*
7. *For each g-(Λ, α)-closed set F and each (Λ, α)-open set G containing F, there exists a g-(Λ, α)-open set U such that $F^{(\Lambda,\alpha)} \subseteq U \subseteq U^{(\Lambda,\alpha)} \subseteq G$.*

Proof. (1) $\Rightarrow$ (2): The proof is obvious.

(2) $\Rightarrow$ (3): Let F be a (Λ, α)-closed set and G be a (Λ, α)-open set containing F. Then, we have F and $X - G$ are two disjoint (Λ, α)-closed sets. Hence by (2), there exist disjoint g-(Λ, α)-open sets U and V of X such that $F \subseteq U$ and $X - G \subseteq V$. Since V is g-(Λ, α)-open and $X - G$ is (Λ, α)-closed, by Theorem 3.5, $X - G \subseteq V_{(\Lambda,\alpha)}$. Thus, $[X - V]^{(\Lambda,\alpha)} = X - V_{(\Lambda,\alpha)} \subseteq G$ and hence $F \subseteq U \subseteq U^{(\Lambda,\alpha)} \subseteq G$.

(3) $\Rightarrow$ (1): Let F and H be two disjoint (Λ, α)-closed sets of X. Then, F is a (Λ, α)-closed set and $X - H$ is a (Λ, α)-open set containing F. Thus by (3), there exists a g-(Λ, α)-open set U such that $F \subseteq U \subseteq U^{(\Lambda,\alpha)} \subseteq X - H$. By Theorem 3.5, $F \subseteq U_{(\Lambda,\alpha)}$, $H \subseteq X - U^{(\Lambda,\alpha)}$, where $U_{(\Lambda,\alpha)}$ and $X - U^{(\Lambda,\alpha)}$ are two disjoint (Λ, α)-open sets.

(4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (2): The proofs are obvious.

(6) $\Rightarrow$ (7) and (7) $\Rightarrow$ (3): The proofs are obvious.

(3) $\Rightarrow$ (5): Let F be a (Λ, α)-closed set and G be a g-(Λ, α)-open set containing F. Since G is g-(Λ, α)-open and F is (Λ, α)-closed, by Theorem 3.5, $F \subseteq G_{(\Lambda,\alpha)}$ and by (3), there exists a g-(Λ, α)-open set U such that $F \subseteq U \subseteq U^{(\Lambda,\alpha)} \subseteq G_{(\Lambda,\alpha)}$.

(5) ⇒ (6): Let F be a g-(Λ,α)-closed set and G be a (Λ,α)-open set containing F. Then, $F^{(\Lambda,\alpha)} \subseteq G$. Since G is g-(Λ,α)-open, by (6), there exists a g-(Λ,α)-open set U such that $F^{(\Lambda,\alpha)} \subseteq U \subseteq U^{(\Lambda,\alpha)} \subseteq G$. Since U is g-(Λ,α)-open and $F^{(\Lambda,\alpha)} \subseteq U$, by Theorem 3.5, $F^{(\Lambda,\alpha)} \subseteq U_{(\Lambda,\alpha)}$. Put $V = U_{(\Lambda,\alpha)}$. Then, V is (Λ,α)-open and $F^{(\Lambda,\alpha)} \subseteq V \subseteq V^{(\Lambda,\alpha)} = [U_{(\Lambda,\alpha)}]^{(\Lambda,\alpha)} \subseteq U^{(\Lambda,\alpha)} \subseteq G$.

(6) ⇒ (4): Let F be a (Λ,α)-closed set and G be a g-(Λ,α)-open set containing F. Then by Theorem 3.5, $F^{(\Lambda,\alpha)} = F \subseteq G_{(\Lambda,\alpha)}$. Since F is g-(Λ,α)-closed and $G_{(\Lambda,\alpha)}$ is (Λ,α)-open, by (6), there exists a (Λ,α)-open set U such that $F^{(\Lambda,\alpha)} = F \subseteq U \subseteq U^{(\Lambda,\alpha)} \subseteq G_{(\Lambda,\alpha)}$. □

Definition 4.4. *A topological space (X,τ) is said to be (Λ,α)-regular if for each (Λ,α)-closed set F of X not containing x, there exist disjoint (Λ,α)-open sets U and V such that $x \in U$ and $F \subseteq V$.*

Theorem 4.4. *For a topological space (X,τ), the following properties are equivalent:*

(1) *(X,τ) is (Λ,α)-regular.*
(2) *For each $x \in X$ and each $U \in \Lambda_\alpha O(X,\tau)$ with $x \in U$, there exists $V \in \Lambda_\alpha O(X,\tau)$ such that $x \in V \subseteq V^{(\Lambda,\alpha)} \subseteq U$.*
(3) *For each (Λ,α)-closed set F of X, $\cap\{V^{(\Lambda,\alpha)} \mid F \subseteq V \in \Lambda_\alpha O(X,\tau)\} = F$.*
(4) *For each subset A of X and each $U \in \Lambda_\alpha O(X,\tau)$ with $A\cap U \neq \emptyset$, there exists $V \in \Lambda_\alpha O(X,\tau)$ such that $A \cap V \neq \emptyset$ and $V^{(\Lambda,\alpha)} \subseteq U$.*
(5) *For each nonempty subset A of X and each (Λ,α)-closed set F of X with $A \cap F = \emptyset$, there exist $V, W \in \Lambda_\alpha O(X,\tau)$ such that $A \cap V \neq \emptyset$, $F \subseteq W$ and $V \cap W = \emptyset$.*
(6) *For each (Λ,α)-closed set F of X and $x \notin F$, there exist $U \in \Lambda_\alpha O(X,\tau)$ and a g-(Λ,α)-open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \emptyset$.*
(7) *For each subset A of X and each (Λ,α)-closed set F with $A\cap F = \emptyset$, there exist $U \in \Lambda_\alpha O(X,\tau)$ and a g-(Λ,α)-open set V such that $A \cap U \neq \emptyset$, $F \subseteq V$ and $U \cap V = \emptyset$.*

Proof. (1) ⇒ (2): Let $G \in \Lambda_\alpha O(X,\tau)$ and $x \notin X - G$. Then, there exist disjoint $U, V \in \Lambda_\alpha O(X,\tau)$ such that $X - G \subseteq U$ and $x \in V$. Thus, $V \subseteq X - U$ and so $x \in V \subseteq V^{(\Lambda,\alpha)} \subseteq X - U \subseteq G$.

(2) ⇒ (3): Let $X - F \in \Lambda_\alpha O(X,\tau)$ with $x \in X - F$. Then by (2), there exists $U \in \Lambda_\alpha O(X,\tau)$ such that $x \in U \subseteq U^{(\Lambda,\alpha)} \subseteq X - F$. Thus, $F \subseteq X - U^{(\Lambda,\alpha)} = V \in \Lambda_\alpha O(X,\tau)$ and hence $U \cap V = \emptyset$. Then, we have $x \notin V^{(\Lambda,\alpha)}$. This shows that $F \supseteq \cap\{V^{(\Lambda,\alpha)} \mid F \subseteq V \in \Lambda_\alpha O(X,\tau)\}$.

(3) ⇒ (4): Let A be a subset of X and $U \in \Lambda_\alpha O(X,\tau)$ such that $A\cap U \neq \emptyset$. Let $x \in A\cap U$. Then, $x \notin X - U$. Hence by (3), there exists $W \in \Lambda_\alpha O(X,\tau)$ such that $X - U \subseteq W$ and $x \notin W^{(\Lambda,\alpha)}$. Put $V = X - W^{(\Lambda,b)}$ which is a (Λ,α)-open set containing x and $A \cap V \neq \emptyset$. Now, $V \subseteq X - W$ and so $V^{(\Lambda,\alpha)} \subseteq X - W \subseteq U$.

(4) ⇒ (5): Let A be a nonempty subset of X and F be a (Λ,α)-closed set such that $A \cap F = \emptyset$. Then, $X - F \in \Lambda_\alpha O(X,\tau)$ with $A \cap (X - F) \neq \emptyset$ and hence by (4), there exists $V \in \Lambda_\alpha O(X,\tau)$ such that $A \cap V \neq \emptyset$ and $V^{(\Lambda,\alpha)} \subseteq X - F$. If we put $W = X - V^{(\Lambda,\alpha)}$, then $F \subseteq W$ and $W \cap V = \emptyset$.

(5) ⇒ (1): Let F be a (Λ, α)-closed set not containing x. Then, $F \cap \{x\} = \emptyset$. Thus by (5), there exist $V, W \in \Lambda_\alpha O(X, \tau)$ such that $x \in V, F \subseteq W$ and $V \cap W = \emptyset$.

(1) ⇒ (6): The proof is obvious.

(6) ⇒ (7): Let A be a subset of X and F be a (Λ, α)-closed set such that $A \cap F = \emptyset$. Then, for $x \in A$, $x \notin F$ and by (6), there exist $U \in \Lambda_\alpha O(X, \tau)$ and a g-(Λ, α)-open set V such that $x \in U$, $F \subseteq V$ and $U \cap V = \emptyset$. Thus, $A \cap U \neq \emptyset$, $F \subseteq V$ and $U \cap V = \emptyset$.

(7) ⇒ (1): Let F be a (Λ, α)-closed set such that $x \notin F$. Since $\{x\} \cap F = \emptyset$, by (7), there exist $U \in \Lambda_\alpha O(X, \tau)$ and a g-(Λ, α)-open set W such that $x \in U$, $F \subseteq W$ and $U \cap W = \emptyset$. Since W is g-(Λ, α)-open, by Theorem 3.5, we have $F \subseteq W_{(\Lambda,\alpha)} = V \in \Lambda_\alpha O(X, \tau)$ and hence $U \cap V = \emptyset$. This shows that (X, τ) is (Λ, α)-regular. □

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