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New Fixed Point Results in Neutrosophic b-Metric Spaces With Application

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Abstract. In this manuscript, we establish the notion of neutrosophic b-metric spaces as a generalization of fuzzy b-metric spaces, intuitionistic fuzzy b-metric spaces and neutrosophic metric spaces in which three symmetric properties plays an important role for membership, non-membership and neutral functions as well we derive some common fixed point and coincident point results for contraction mappings. Also, we provide several non-trivial examples with graphical views of neutrosophic b-metric spaces and contraction mappings by using computational techniques. Our results are more generalized with respect to the existing ones in the literature. At the end of the paper, we provide an application to test the validity of the main result.

1. Introduction and Prelimainaries

Fixed-point theorems in metric spaces (and their different generalizations) have made exquisite theoretical progress and have a variety of practical applications. These advancements over the last three decades were fantastic. The majority of scholars based their reference findings on Banach's

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contraction Theorem [1]. By applying nonlinear equations to similar fixed-point (FP) applications, numerous problems in engineering and economics can be resolved. A fixed point $Fx = x$ can be established for an operator sum $Gx = 0$, where F is a self-mapping in some relevant disciplines. For resolving issues brought on by a variety of mathematical inspection origins, such as split feasibility concerns, supporting problems, equilibrium problems, and matching and selection issues, FP theory has a number of key modes. Studying the notion of FPs is both exciting and interesting. This idea has already been demonstrated to be an amazing attempt to condense nonlinear analysis into a short timeframe.

The idea of fuzzy sets (FSs), that is utilized to characterize/manipulate information and data having non-statistical uncertainty, was first introduced by Zadeh [2] in 1965. The idea of FSs seeks to address issues where errors and a high degree of uncertainty are present by providing logical and set hypothetical tools. Later, in 1986, Atanassov [3] proposed the concept of intuitionistic FSs. This set theory, which is a broader version of FS theory, defines both the degree of membership and the degree of non-membership. Many authors used this idea in various branches of mathematics. This theory has been applied to groups and its properties by Gulzar et al. [4-6]. Akber [7] established the intuitionistic fuzzy mappings and developed standard FPs for particular types of mappings. The important sign that the notion of a distance function plays in approximation theory has led to the application of FSs to the fundamental notion of metric as well. A number of publications [8–10] have taken steps in this direction by introducing the applications of metric spaces to fuzzy circumstances. Kramosil and Michalek [11] introduced the idea of fuzzy metric spaces (FMSs) in 1975, and in 1994, George and Veeramani [12] established a Hausdorff topology utilizing fuzzy metric. In fuzzy cone metric spaces, Rehmanand and Aydi [13] established their findings. The idea of b-metric space, which is a broader category than metric space, was initially put forth by Bakhtin [14]. Later, Saleem et al. [15] coined the concept graphical of FMSs. The concept of fuzzy b-metric space (FbMS), was put up by Nadaban [16] in 2016, generalization of FMSs. Ishtiaq et al. [17] derived several FP results in generalizations of FMSs. In fuzzy strong b-metric spaces, Shazia et al. [18] identified FPs for a number of nonlinear contraction mappings. In 2004, Park [19] used the idea of intuitionistic FSs, continuous t-norm (CTN), and continuous t-conorm (CTCN) to established intuitionistic fuzzy metric spaces (IFMSs) as a generalization of FMSs. Banach's contraction principle was improved by Jungck [20] in 1976 by looking into coincidence and common FPs in commuter mappings. In 1986, Jungck [21] introduced the idea of common FPs as well as compatible maps for a pair of self-mappings. Jungck's common FP theorems were generalized by Turkoglu et al. [22] in IFMSs in 2006. Weakly compatible (w-compatible) mappings were first described by Jungck and Rhoades [23] in 2006. Since any pair of compatible mappings is w-compatible, but the converse is not true in general, weakly compatible mappings are more general. Grabiec [24] derived the Banach's FP results in the context of FMSs in 1988. Schweizer and Sklar [25] introduced statistical metric spaces. Kanwal et al. [26] derived some new FP results in intuitionistic fuzzy b-metric spaces (IFbMS).

In 2005, Smarandache [27] proposed the concept of neutrosophic sets (NSs), as a generalization of IFSs. In 2019, Kirişci and Şimşek [28] established the notion of neutrosophic metric spaces (NMSs) and discussed a topological structure. In NMSs, membership (M), non-membership (N) and neutral functions (O) are used and they establish the following three symmetric properties for these functions:

$$M(\omega, \upsilon, \iota) = M(\omega, \upsilon, \iota), \text{ for all } \iota > 0,$$
$$N(\omega, \upsilon, \iota) = N(\omega, \upsilon, \iota), \text{ for all } \iota > 0,$$
$$O(\omega, \upsilon, \iota) = O(\omega, \upsilon, \iota), \text{ for all } \iota > 0.$$

Şimşek and Kirişci [29] derived numerous FP results for contraction mappings in the context of NMS. Ishtiaq et al. [30] generalized the notion of NMS and introduced the notion orthogonal NMSs and proved some new types of FP theorems for contraction mappings. Debnath [31] worked on a mathematical model using fixed point theorem for two-choice behavior of rhesus monkeys in a noncontingent environment. Authors in [32] and [33] did amazing work in the direction of fixed point theory.

In this manuscript, we aim to introduce the notion of neutrosophic b-metric space (NbMS) as a generalization of NMS and we use the above defined three symmetric properties to introduce the notion of NbMS. We established some coincident point (c-point) and common FP results in which symmetric properties of NbMS plays a very significant role. We coined several non-trivial examples and graphical views via computational techniques. Also, we provide an application to support our main result.

We start with some definitions that are helpful for readers to understand the main results.

Definition 1.1 [25] A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is called CTN if the below circumstances are fulfilled:

(a1) $*$ is associative and commutative,

(a2) $*$ is continuous,

(a3) $\lambda * 1 = \lambda, \text{ for all } \lambda \in [0,1],$

(a4) If $\lambda \leq k$ and $a \leq l$ with $\lambda, a, k, l \in [0,1]$, then $\lambda * a \leq k * l$.

Definition 1.2 [3] A binary operation $\circ: [0,1] \times [0,1] \to [0,1]$ is called CTCN if the below circumstances are fulfilled:

(b1) $*$ is associative and commutative,

(b2) $\circ$ is continuous,

(b3) $\lambda \circ 0 = \lambda,$ for all $\lambda \in [0,1]$,

(b4) If $\lambda \leq k$ and $a \leq l$ with $\lambda, a, k, l \in [0,1]$, then $\lambda * a \leq k * l$.

Definition 1.3 [16] Suppose ζ be a non-empty set. Let $s \in \mathbb{R}, s \geq 1$ and $*$ be CTN. A FS M on $\zeta \times \zeta \times [0, +\infty)$ is known as fuzzy b-metric if, for all $\omega, \upsilon, z \in \zeta$ the below conditions are verified:

(bM1) $M(\omega, \upsilon, 0) = 0$,

(bM2) $M(\omega, \upsilon, \iota) = 1,$ for all $\iota \geq 0 \iff S\omega = \upsilon$,

(bM3) $M(\omega, \upsilon, \iota) = M(\omega, \upsilon, \iota),$ for all $\iota \geq 0$,

(bM4) $M(\omega, z, s(\iota + \theta)) \geq M(\omega, \upsilon, \iota) * M(\upsilon, z, \theta)d, for\ all\ \iota, \theta \geq 0$,

(bM5) $\lim_{\iota \to +\infty} M(\omega, \upsilon, \iota) = 1$ and $M(\omega, \upsilon, .): [0, +\infty] \to [0,1]$ is left continuous.

Then $(\zeta, M, *, s)$ is called a fuzzy b-metric space.

Definition 1.4 [27] Let a set $\zeta \neq \emptyset$ and $\vartheta \in X$. A neutrosophic set (NS) G in ζ is categorized by three components

(i) truth-membership function $M(\vartheta)$,

(ii) indeterminacy-membership function $N(\vartheta)$,

(iii) falsity-membership function $O(\vartheta)$.

The functions $M(\vartheta)$, $N(\vartheta)$ and $O(\vartheta)$ are real standard or non-standard subsets of $]0^-, 1^+[$, that is $M(\vartheta): \zeta \to]0^-, 1^+[$, $N(\vartheta): \zeta \to]0^-, 1^+[$ and $O(\vartheta): \zeta \to]0^-, 1^+[$ such that

$$0^- \leq \sup M(\vartheta) + \sup N(\vartheta) + \sup O(\vartheta) \leq 3^+.$$

Definition 1.5 [28] A 6-tuple $(\zeta, M, N, O, *, \circ)$ is known as a NMS if ζ is an arbitrary set, $*$ and $\circ$ are CTN and CTCN respectively, M, N, O are NSs on $\zeta^2 \times [0, +\infty)$ verifying the bellow circumstances $for\ all\ \omega, \upsilon, z \in \zeta$,

(N1) $M(\omega, \upsilon, \iota) + N(\omega, \upsilon, \iota) + O(\omega, \upsilon, \iota) \leq 3$,

(N2) $M(\omega, \upsilon, 0) = 0$,

(N3) $M(\omega, \upsilon, \iota) = 1,$ for all $\iota > 0$ iff $\omega = \upsilon$,

(N4) $M(\omega, \upsilon, \iota) = M(\omega, \upsilon, \iota),$ for all $\iota > 0$, (N5) $M(\omega,z,\iota+\theta) \geq M(\omega,v,\iota) * M(v,z,\theta)$, for all $\iota,\theta > 0$,

(N6) $M(\omega,v,.)\colon [0,+\infty) \to [0,1]$ is left continuous and $\lim_{\iota\to+\infty} M(\omega,v,\iota) = 1$,

(N7) $N(\omega,v,0) = 1$,

(N8) $N(\omega,v,\iota) = 0$, for all $\iota > 0$ iff $\omega = v$,

(N9) $N(\omega,v,\iota) = N(\omega,v,\iota)$, for all $\iota > 0$,

(N10) $N(\omega,z,\iota+\theta) \leq N(\omega,v,\iota) \circ N(v,z,\theta)$, for all $\iota,\theta > 0$,

(N11) $\lim_{\iota\to+\infty} N(\omega,v,\iota) = 0$ and $N(\omega,v,.) : [0,+\infty) \to [0,1]$ is right continuous,

(N12) $O(\omega,v,0) = 1$,

(N13) $O(\omega,v,\iota) = 0$, for all $\iota > 0$ iff $\omega = v$,

(N14) $O(\omega,v,\iota) = O(\omega,v,\iota)$, for all $\iota > 0$,

(N15) $O(\omega,z,\iota+\theta) \leq O(\omega,v,\iota) \circ O(v,z,\theta)$, for all $\iota,\theta > 0$,

(N16) $\lim_{\iota\to+\infty} O(\omega,v,\iota) = 0$ and $O(\omega,v,.) : [0,+\infty) \to [0,1]$ is right continuous.

Then $(\zeta,M,N,O,*,\circ,s)$ said to be a NMS.

Definition 1.6 [26] A 6-tuple $(\zeta,M,N,*,\circ,s)$ is known as an IFbMS if $\zeta \neq \phi$, $s \geq 1$ is a given real number, $*$ and $\circ$ are CTN and CTCN, respectively, M and N are FSs on $\zeta^2 \times [0,+\infty)$ verifying the below circumstances $for\ all\ \omega,v,z \in \zeta$,

(IFB1) $M(\omega,v,\iota) + N(\omega,v,\iota) \leq 1$,

(IFB2) $M(\omega,v,0) = 0$,

(IFB3) $M(\omega,v,\iota) = 1$, for all $\iota > 0$ iff $\omega = v$,

(IFB4) $M(\omega,v,\iota) = M(\omega,v,\iota)$, for all $\iota > 0$,

(IFB5) $(\omega,z,s(\iota+\theta)) \geq M(\omega,v,\iota) * M(v,z,\theta)$, for all $\iota,\theta > 0$,

(IFB6) $M(\omega,v,.)\colon [0,+\infty) \to [0,1]$ is left continuous and $\lim_{\iota\to+\infty} M(\omega,v,\iota) = 1$,

(IFB7) $N(\omega,v,0) = 1$,

(IFB8) $N(\omega,v,\iota) = 0$, for all $\iota > 0$ iff $\omega = v$,

(IFB9) $N(\omega,v,\iota) = N(\omega,v,\iota), for\ all\ \iota > 0$,

(IFB10) $N(\omega,z,s(\iota+\theta)) \leq N(\omega,v,\iota) \circ N(v,z,\theta)$, for all $\iota,\theta > 0$,

(IFB11) $\lim_{\iota\to+\infty} N(\omega,v,\iota) = 0$,

(IFB12) $N(\omega,v,.) : [0,+\infty) \to [0,1]$ is right continuous.

2. NEUTROSOPHIC b-METRIC SPACES

In this section, we will establish the notion of NbMS and several non-trivial examples with their graphical structures.

Definition 2.1 A 7-tuple $(\zeta, M, N, O, *, \circ, s)$ known to be an NbMS if $\zeta \neq \phi$, $s \geq 1$ is a given real number, $*$ and $\circ$ are CTN and CTCN, respectively, and M, N, O are NSs on $\zeta^2 \times [0, +\infty)$ verifying the below circumstances $for\ all\ \omega, \upsilon, z \in \zeta$,

(NBM1) $M(\omega, \upsilon, \iota) + N(\omega, \upsilon, \iota) + O(\omega, \upsilon, \iota) \leq 3$,

(NBM2) $M(\omega, \upsilon, 0) = 0$,

(NBM3) $M(\omega, \upsilon, \iota) = 1, \text{for all} \iota > 0 \text{ iff } \omega = \upsilon$,

(NBM4) $M(\omega, \upsilon, \iota) = M(\omega, \upsilon, \iota), \text{ for all } \iota > 0$,

(NBM5) $M(\omega, z, s(\iota + \theta)) \geq M(\omega, \upsilon, \iota) * M(\upsilon, z, \theta), \text{ for all } \iota, \theta > 0$,

(NBM6) $M(\omega, \upsilon, .): [0, +\infty) \to [0,1]$ is left continuous and $\lim_{\iota \to +\infty} M(\omega, \upsilon, \iota) = 1$,

(NBM7) $N(\omega, \upsilon, 0) = 1$,

(NBM8) $N(\omega, \upsilon, \iota) = 0, \text{ for all } \iota > 0 \text{ iff } \omega = \upsilon$,

(NBM9) $N(\omega, \upsilon, \iota) = N(\omega, \upsilon, \iota), for\ all\ \iota > 0$,

(NBM10) $N(\omega, z, s(\iota + \theta)) \leq N(\omega, \upsilon, \iota) \circ N(\upsilon, z, \theta), \text{ for all } \iota, \theta > 0$,

(NBM11) $\lim_{\iota \to +\infty} N(\omega, \upsilon, \iota) = 0$ and $N(\omega, \upsilon, .) : [0, +\infty) \to [0,1]$ is right continuous,

(NBM12) $O(\omega, \upsilon, 0) = 1$,

(NBM13) $O(\omega, \upsilon, \iota) = 0, \text{ for all } \iota > 0 \text{ iff } \omega = \upsilon$,

(NBM14) $O(\omega, \upsilon, \iota) = O(\omega, \upsilon, \iota), \text{for all } \iota > 0$,

(NBM15) $O(\omega, z, s(\iota + \theta)) \leq O(\omega, \upsilon, \iota) \circ N(\upsilon, z, \theta), \text{ for all } \iota, \theta > 0$,

(NBM16) $\lim_{\iota \to +\infty} O(\omega, \upsilon, \iota) = 0$ and $O(\omega, \upsilon, .) : [0, +\infty) \to [0,1]$ is right continuous.

Then $(\zeta, M, N, O, *, \circ, s)$ said to be a NBMS.

Remark 2.1 If, we let $s = 1$ in the above definition, then it will become NMS. So, every NMS is an NbMS, but the converse is not generally true.

Example 2.1 Suppose $(\zeta, \varpi, \mathrm{s})$ be a b-metric space and $a * \mathrm{b} = \min\{a, b\}, a \circ b = \max\{a, b\}, for\ all\ a, b \in [0,1]$, and let M_ϖ, N_ϖ and O_ϖ be NSs on $\zeta^2 \times [0, +\infty)$, defined as follows:

$$M_\varpi(\omega, \upsilon, \iota) = \begin{cases} \dfrac{\iota}{\iota + \varpi(\omega, \upsilon)}, & \text{if } \iota > 0, \\ 0, & \text{if } \iota = 0 \end{cases}$$

$$N_{\varpi}(\omega,\upsilon,\iota)=\begin{cases}\dfrac{\varpi(\omega,\upsilon)}{\iota+\varpi(\omega,\upsilon)}, & \text{if } \iota>0,\\ 1, & \text{if } \iota=0\end{cases}$$

and

$$O_{\varpi}(\omega,\upsilon,\iota)=\begin{cases}\dfrac{\varpi(\omega,\upsilon)}{\iota}, & \text{if } \iota>0,\\ 1, & \text{if } \iota=0.\end{cases}$$

We verify the axioms (NBM5), (NBM10) and (NBM15) of definition 2.1 others; are obvious. Let $\omega,\upsilon,z\in\zeta$ and $\iota,\sigma>0.$ without loss of the generality, we suppose that

$$M_{\varpi}(\omega,\upsilon,\iota)\leq M_{\varpi}(\upsilon,z,\sigma)$$
$$N_{\varpi}(\omega,\upsilon,\iota)\geq N_{\varpi}(\upsilon,z,\sigma),$$

and

$$O_{\varpi}(\omega,\upsilon,\iota)\geq O_{\varpi}(\upsilon,z,\sigma),$$

Thus,

$$\frac{\iota}{\iota+\varpi(\omega,\upsilon)}\leq\frac{\sigma}{\sigma+\varpi(\omega,\upsilon)}$$
$$\frac{\varpi\ (\omega,\upsilon)}{\iota+\varpi(\omega,\upsilon)}\geq\frac{\varpi(\upsilon,z)}{\sigma+\varpi(\omega,\upsilon)}$$
$$\iota\varpi(\upsilon,z)\leq\sigma\varpi(\upsilon,z)$$

On the contrary,

$$M_{\varpi}\big(\omega,z,s(\iota+\sigma)\big)=\frac{s(\iota+\sigma)}{s(\iota+\sigma)+\varpi(\omega,z)}$$
$$\geq\frac{s(\iota+\sigma)}{s(\iota+\sigma)+s[\varpi(\omega,\upsilon)+\varpi(\upsilon,z)]}$$
$$=\frac{\iota+\sigma}{\iota+\sigma+\varpi(\omega,\upsilon)+\varpi(\upsilon,z)}.$$

Also,

$$N_{\varpi}\big(\omega,z,s(\iota+\sigma)\big)=\frac{\varpi(\omega+z)}{s(\iota+\sigma)+\varpi(\omega,z)}$$
$$\leq\frac{s[\varpi(\omega,\upsilon)+\omega(\upsilon,z)]}{s(\iota+\sigma)+s[\varpi(\omega,\upsilon)+\varpi(\upsilon,z)]}\ ,$$
$$=\frac{\varpi(\omega,\upsilon)+\varpi(\upsilon,z)}{\iota+\sigma+\varpi(\omega,\upsilon)+\varpi(\upsilon,z)}$$
$$\frac{\varpi(\omega,\upsilon)+\varpi(\upsilon,z)}{\iota+\sigma+\varpi(\omega,\upsilon)+\varpi(\upsilon,z)}\leq\frac{\varpi(\omega,\upsilon)}{\iota+\varpi(\omega,\upsilon)}$$

and

$$O_\varpi\big(\omega, z, s(\iota+\sigma)\big) = \frac{\varpi(\omega+z)}{s(\iota+\sigma)}$$
$$\leq \frac{s[\varpi(\omega,\upsilon)+\omega(\upsilon,z)]}{s(\iota+\sigma)},$$
$$= \frac{\varpi(\omega,\upsilon)+\varpi(\upsilon,z)}{\iota+\sigma}$$
$$\frac{\varpi(\omega,\upsilon)+\varpi(\upsilon,z)}{\iota+\sigma} \leq \frac{\varpi(\omega,\upsilon)}{\iota}.$$

Hence,

$$M_\varpi\big(\omega, z, s(\iota+\sigma)\big) \geq \quad M_\varpi(\omega,\upsilon,\iota) = M_\varpi(\omega,\upsilon,\iota) * M_\varpi(\upsilon,z,\sigma)$$
$$N_\varpi\big(\omega, z, s(\iota+\sigma)\big) \leq \ N_\varpi(\omega,\upsilon,\iota) = N_\varpi(\omega,\upsilon,\iota) \circ N_\varpi(\upsilon,z,\sigma),$$

and

$$O_\varpi\big(\omega, z, s(\iota+\sigma)\big) \leq O_\varpi(\omega,\upsilon,\iota) = O_\varpi(\omega,\upsilon,\iota) \circ O_\varpi(\upsilon,z,\sigma).$$

Now

$$\frac{\iota+\sigma}{\iota+\sigma+\varpi(\omega,\upsilon)+\varpi(\upsilon,z)} \geq \frac{\iota}{\iota+\varpi(\omega,\upsilon)}$$
$$\Leftrightarrow \iota^2+\sigma\iota+\iota\varpi(\omega,\upsilon)+\sigma\varpi(\omega,\upsilon) \geq \iota^2+\sigma\iota+\iota\varpi(\omega,\upsilon)+\iota\varpi(\upsilon,z)$$
$$\Leftrightarrow \sigma\varpi(\omega,\upsilon) \geq \iota\varpi(\upsilon,z),$$

which is true.

Also

$$\frac{\varpi(\omega,\upsilon)+\varpi(\upsilon,z)}{\iota+\sigma+\varpi(\omega,\upsilon)+\varpi(\upsilon,z)} \leq \frac{\varpi(\omega,\upsilon)}{\iota+\varpi(\omega,\upsilon)}$$
$$\Leftrightarrow \iota\varpi(\omega,\upsilon)+\iota\varpi(\upsilon,z)+\varpi(\omega,\upsilon)+\varpi(\upsilon,z)+\big(\varpi(\omega,\upsilon)\big)^2$$
$$\leq \iota\varpi(\omega,\upsilon)+\sigma\varpi(\omega,\upsilon)+\ \varpi(x,\upsilon)\varpi(\upsilon,z)+\big(\varpi(\omega,\upsilon)\big)^2$$
$$\Leftrightarrow \iota\varpi(\upsilon,z) \leq \sigma\varpi(\omega,\upsilon),$$

and

$$\frac{\varpi(\omega,\upsilon)+\varpi(\upsilon,z)}{\iota+\sigma} \leq \frac{\varpi(\omega,\upsilon)}{\iota}$$
$$\Leftrightarrow \iota\varpi(\omega,\upsilon)+\iota\varpi(\upsilon,z) \leq \iota\varpi(\omega,\upsilon)+\sigma\varpi(\omega,\upsilon)$$
$$\Leftrightarrow \iota\varpi(\upsilon,z) \leq \sigma\varpi(\omega,\upsilon),$$

which is true. Hence, $(\zeta, M_{\varpi,} N_\varpi, O_\varpi, *, \circ, s)$ is an NbMS.

Remark 2.2 Let $\zeta = [0,1]$ and $\alpha(\omega,\upsilon) = |\omega-\upsilon|^s$ with $s \geq 1$ be a b-metric space. Consider the above example, we have the graphical views for M_ϖ in figure 1, N_ϖ in figure 2 and O_ϖ in Figure 3.

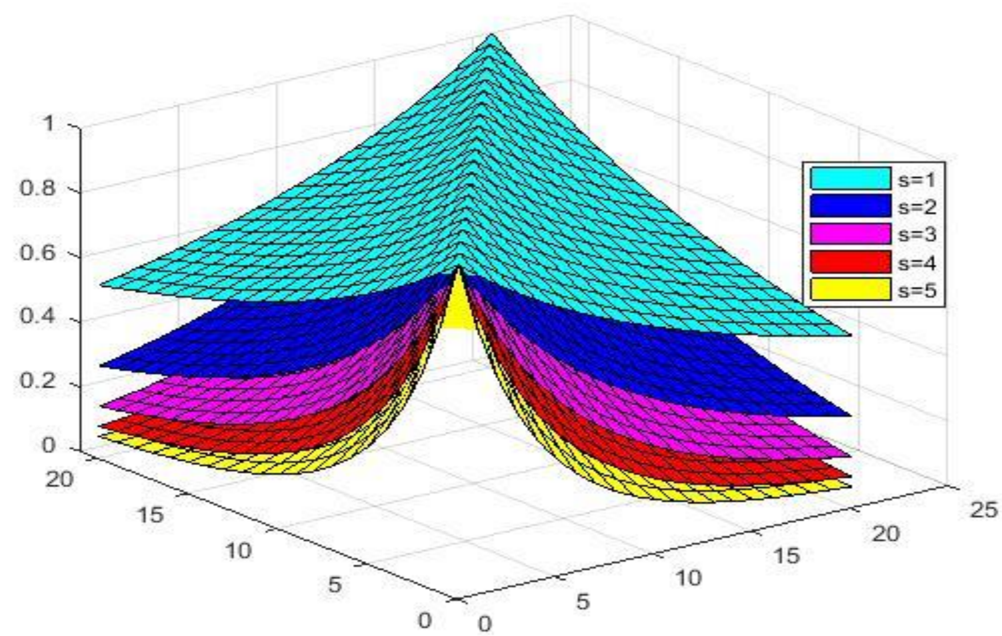

Figure 1 shows the graphical behavior of M_{ϖ} for $s=1, s=2, s=3, s=4$ and $s=5$.

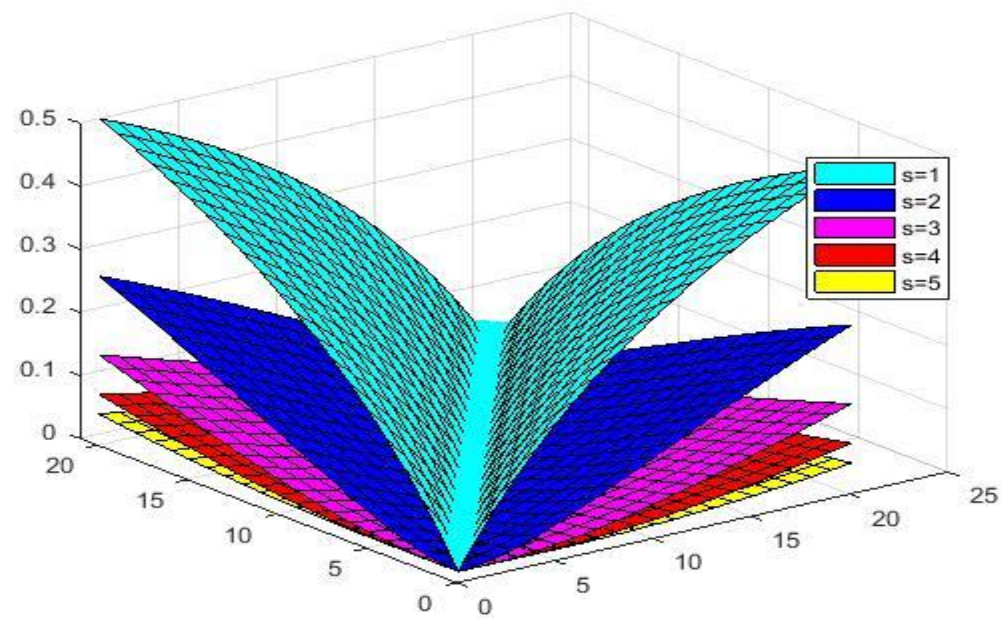

Figure 2 shows the graphical behavior of N_{ϖ} for $s=1, s=2, s=3, s=4$ and $s=5$.

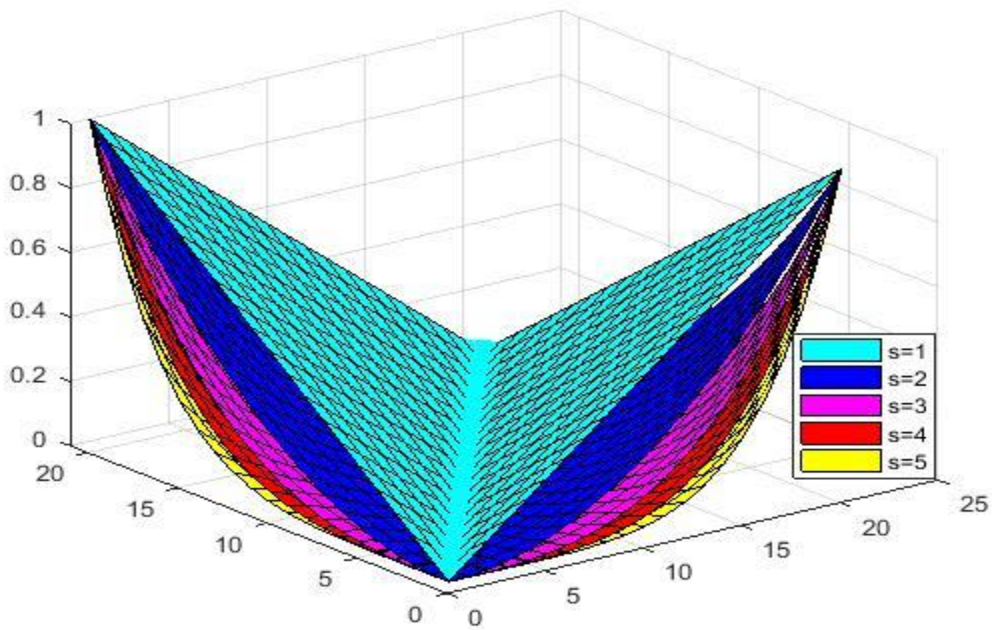

Figure 3 shows the graphical behavior of O_{ϖ} for $s = 1, s = 2, s = 3, s = 4$ and $s = 5$.

Definition 2.2 Let $s \geq 1$ be a given real number. A function $Q: R \rightarrow R$ is said to be an s-non-decreasing if $\iota < \sigma$ implies that $Q(\iota) \leq Q(s\sigma)$ and Q is said to be s-non-increasing if $\iota < \sigma$ implies that $Q(\iota) \geq Q(s\sigma)$.

Proposition 2.1 In an NbMS $(\zeta, M_b, N_b, O_b, *, \circ, s)$, $M(\omega, \upsilon, .): [0, +\infty) \rightarrow [0,1]$ is s-non-decreasing, $N(\omega, \upsilon, .): [0, +\infty) \rightarrow [0,1]$ is s-non-increasing and $O(\omega, \upsilon, .): [0, +\infty) \rightarrow [0,1]$ is s-non-increasing, for all $\omega, \upsilon \in \zeta$.

Proof: For $0 < \iota < \sigma$, we get

$$M(\omega, \upsilon, s\sigma) = M(\omega, \upsilon, s(\sigma - \iota + \iota))$$

$$\geq M(\omega, \omega, \sigma - \iota) * M(\omega, \upsilon, \iota)$$

$$= 1 * M(\omega, \upsilon, \iota) = M(\omega, \upsilon, \iota).$$

Also

$$N(\omega, \upsilon, s\sigma) = N\big(\omega, \upsilon, s(\iota - \iota + \iota)\big)$$

$$\leq N(\omega, \omega, \sigma - \iota) \circ N(\omega, \upsilon, \iota)$$

$$= 0 \circ N(\omega, \upsilon, \iota) = N(\omega, \upsilon, \iota)$$

and

$$O(\omega, \upsilon, s\sigma) = O\big(\omega, \upsilon, s(\iota - \iota + \iota)\big)$$

$$\leq O(\omega, \omega, \sigma - \iota) \circ O(\omega, \upsilon, \iota)$$

$$= 0 \circ O(\omega, \upsilon, \iota) = O(\omega, \upsilon, \iota).$$

Definition 2.3 Suppose $(\zeta, M, N, O, *, \circ, s)$ be an NbMS. An open ball $B(\omega, r, \iota)$ with the center $\omega \in \zeta$ and radius $r, 0 < r < 1$, and $\iota > 0$ is defined as $B(\omega, r, \iota) = \{\upsilon \in \zeta : M(\omega, r, \iota) > 1 - r,\ N(\omega, r, \iota) < r \text{ and } O(\omega, r, \iota) < r\}$.

Definition 2.4 Let $(\zeta, M, N, O, *, \circ, s)$ be an NbMS and a subset A of ζ. If for each $\omega \in A$, there is an open ball $B(\omega, r, \iota)$ contained in A, then A is called an open in ζ.

Definition 2.5 Suppose $(\zeta, M, N, O, *, \circ, s)$ be an NbMS. Define $\tau_{M,N,O}$ as $\tau_{M,N,O} = \{A \in P(\zeta) : \omega \in A \text{ iff there exists } \iota > 0 \text{ and } r \in (0,1) : B(\omega, r, \iota \subset A)\}$ then $\tau_{M,N,O}$ is a topology on ζ, where $P(\zeta)$ is the power set of ζ.

Definition 2.6 Suppose $(\zeta, M, N, O, *, \circ, s)$ be an NbMS.

(a) Any sequence ω_n in ζ is said to be convergent if there exist $\omega \in \zeta$ such that $\lim_{n \to +\infty} M(\omega_n, \omega, \iota) = 1$, $\lim_{n \to +\infty} N(\omega_n, \omega, \iota) = 0$ and $\lim_{n \to +\infty} O(\omega_n, \omega, \iota) = 0$ $for\ all \iota > 0$. A point ω is said to be the limit of the sequence ω_n and it is described as $\lim_{n \to +\infty} \omega_n = \omega$, or $\omega_n \to \omega$.

(b) Any sequence ω_n in $(\zeta, M, N, O, *, \circ, s)$ is said to be a Cauchy sequence if, for every ε in $(0,1)$, there is $n_0 \in N$ such that $M(\omega_n, \omega_m, \iota) > 1 - \varepsilon, N(\omega_n, \omega_m, \iota) < \varepsilon$ and $O(\omega_n, \omega_m, \iota) < \varepsilon$ for all $m, n \geq n_0$ and $\iota > 0$.

(c) ζ is known to be complete if every Cauchy sequence in ζ is convergent in ζ.

3. MAIN RESULTS

In this section, we will derive several coincident point and common FP results in the context of NbMS.

Definition 3.1 Suppose $\zeta \neq \phi$ and $\Delta, \sigma : \zeta \to \zeta$ be two mappings on ζ.

(i) An element $\omega \in \zeta$ is said to be a c-point of Δ and σ if $\Delta(\omega) = \sigma(\omega)$.

(ii) An element $\upsilon \in \zeta$ is said to be a c-point of Δ and σ if there exists $\omega \in \zeta$ such that if $\upsilon = \Delta(\omega) = \sigma(\omega)$.

(iii) An element $z \in \zeta$ is called a common FP of Δ and σ if $z = \Delta(z) = \sigma(z)$.

Definition 3.2 Two self-mappings $\Delta, \sigma : \zeta \to \zeta$ are called w-compatible if $\Delta\sigma(\omega) = \sigma\Delta(\omega)$ when $\Delta(\omega) = \sigma(\omega)$.

Theorem 3.1 Suppose $\zeta \neq \phi, Y \neq \phi$, and $(Y, M, N, O, *, \circ, s)$ be an NbMS and $\Delta, \sigma : \zeta \longrightarrow Y$ be mappings verifying the below circumstances:

(i) $\sigma(\zeta) \subseteq \Delta(\omega)$;

(ii) There is k, such that $0 \leq k \leq 1$, for all $\omega, \upsilon \in \zeta$

$$M(\sigma(\omega), \sigma(\upsilon), k\iota) \geq M(\Delta(\omega), \Delta(\upsilon), \iota)$$

$$N(\sigma(\omega), \sigma(\upsilon), k\iota) \leq N(\Delta(\omega), \Delta(\upsilon), \iota).$$

and

$$O(\sigma(\omega), \sigma(\upsilon), k\iota) \leq O(\Delta(\omega), \Delta(\upsilon), \iota).$$

If $\Delta(\zeta)$ or $\sigma(\zeta)$ is complete, then there exists an element $z \in \zeta$ such that $\Delta(\zeta) = \sigma(\zeta)$. Furthermore, Δ and σ have a unique c-point.

Proof: Suppose $\omega_0 \in \zeta$. Applying (i), we can deduce $\omega_1 \in \zeta$ such that $\Delta(\omega_1) = \sigma(\omega_0)$, for $k = 0$, we have

$$M(\sigma(\omega_0), \sigma(\omega_1), 0\iota) \geq M(\Delta(\omega_0), \Delta(\omega_1), \iota),$$
$$N(\sigma(\omega_0), \sigma(\omega_1), 0\iota) \leq N(\Delta(\omega_0), \Delta(\omega_1), \iota)$$

and

$$O(\sigma(\omega_0), \sigma(\omega_1), 0\iota) \leq O(\Delta(\omega_0), \Delta(\omega_1), \iota)$$
$$M(\sigma(\omega_0), \sigma(\omega_1), 0\iota) = 1,$$
$$N(\sigma(\omega_0), \sigma(\omega_1), 0\iota) = 0$$

and

$$O(\sigma(\omega_0), \sigma(\omega_1), 0\iota) = 0.$$

That is,

$$\sigma(\omega_0) = \sigma(\omega_1)$$
$$\Delta(\omega_1) = \sigma(\omega_1).$$

Hence, ω_1 is the c-point of Δ and σ. For $k \neq 0$, by induction, we have a sequence $\{\omega_n\}$ in ζ, such that

$$\Delta(\omega_n) = \sigma(\omega_{n-1}): M(\Delta(\omega_n), \Delta(\omega_{n+1}), \iota) = M(\sigma(\omega_{n-1}), \sigma(\omega_n), \iota)$$
$$\geq M(\Delta(\omega_{n-1}), \Delta(\omega_n), \iota/k) \geq \cdots \geq M(\Delta(\omega_0), \Delta(\omega_1), \iota/k^n).$$

Clearly, $1 \geq M(\Delta(\omega_n), \Delta(\omega_{n+1}), \iota) \geq M(\Delta(\omega_0), \Delta(\omega_1), \iota/k^n) \to 1$, when $n \to +\infty$.

Thus,

$$\lim_{n\to+\infty} N(\Delta(\omega_n), \Delta(\omega_{n+1}), \iota) = N(\sigma(\omega_{n-1}), \sigma(\omega_n), \iota)$$
$$\leq N(\Delta(\omega_{n-1}), \Delta(\omega_n), \iota/k) \leq \cdots \leq N(\Delta(\omega_0), \Delta(\omega_1), \iota/k^n).$$

Clearly,

$$0 \le N(\Delta(\omega_n), \Delta(\omega_{n+1}), \iota) \le N(\Delta(\omega_0), \Delta(\omega_1), \iota/k^n) \to 0, \text{when } n \to +\infty.$$

That is,

$$\lim_{n\to+\infty} N(\Delta(\omega_n), \Delta(\omega_{n+1}), \iota) = 0.$$

Also,

$$\lim_{n\to+\infty} O(\Delta(\omega_n), \Delta(\omega_{n+1}), \iota) = O(\sigma(\omega_{n-1}), \sigma(\omega_n), \iota)$$
$$\le O(\Delta(\omega_{n-1}), \Delta(\omega_n), \iota/k) \le \cdots \le O(\Delta(\omega_0), \Delta(\omega_1), \iota/k^n).$$

Clearly,

$$0 \le O(\Delta(\omega_n), \Delta(\omega_{n+1}), \iota) \le O(\Delta(\omega_0), \Delta(\omega_1), \iota/k^n) \to 0, \text{when } n \to +\infty.$$

That is,

$$\lim_{n\to+\infty} O(\Delta(\omega_n), \Delta(\omega_{n+1}), \iota) = 0.$$

Let

$$\tau_n(\iota) = M(\Delta(\omega_n), \Delta(\omega_{n+1}), \iota),$$
$$\mu_n(\iota) = N(\Delta(\omega_n), \Delta(\omega_{n+1}), \iota),$$
$$h_n(\iota) = O(\Delta(\omega_n), \Delta(\omega_{n+1}), \iota) \text{ for all } n \in \mathbb{N} \cup \{0\}, \iota > 0.$$

To show that $\Delta(\omega_n)$ is a Cauchy sequence, assume it is not, then there exists $0 < \varepsilon < 1$ and two sequences $p(\eta)$ and $q(\eta)$ such that for every $\eta \in \mathbb{N} \cup \{0\}, \iota > 0, p(\eta) > q(\eta) \ge \eta$,

$$M\left(\Delta\left(\omega_p(\eta), \omega_q(\eta), \iota\right)\right) \le 1 - \varepsilon,$$
$$N\left(\Delta\omega_p(\eta), \left(\Delta\omega_q(\eta), \iota\right)\right) \ge \varepsilon,$$
$$O\left(\Delta\omega_p(\eta), \left(\Delta\omega_q(\eta), \iota\right)\right) \ge \varepsilon.$$

Then

$$M\left(\Delta\left(\omega_{p(\eta)-1}\right), \Delta\left(\omega_{q(\eta)-1}\right), \iota\right) > 1 - \varepsilon$$
$$M\left(\Delta\left(\omega_{p(\eta)-1}\right), \Delta\left(\omega_{q(\eta)}\right), \iota\right) > 1 - \varepsilon,$$
$$N\left(\Delta\left(\omega_{p(\eta)-1}\right), \Delta\left(\omega_{q(\eta)-1}\right), \iota\right) < \varepsilon$$
$$N\left(\Delta\left(\omega_{p(\eta)-1}\right), \Delta\left(\omega_{q(\eta)}\right), \iota\right) < \varepsilon,$$

and

$$O\left(\Delta\left(\omega_{p(\eta)-1}\right), \Delta\left(\omega_{q(\eta)-1}\right), \iota\right) < \varepsilon$$
$$O\left(\Delta\left(\omega_{p(\eta)-1}\right), \Delta\left(\omega_{q(\eta)}\right), \iota\right) < \varepsilon.$$

Now,

$$1-\varepsilon \geq M\big(\Delta\big(\omega_{p(\eta)}\big),\Delta\big(\omega_{q(\eta)}\big),\iota\big)$$
$$\geq M\big(\Delta\big(\omega_{p(\eta)-1}\big),\Delta\big(\omega_{p(\eta)}\big),\iota/2s\big) * M\big(\Delta\big(\omega_{p(\eta)-1}\big),\Delta\big(\omega_{q(\eta)}\big),\iota/2s\big)$$
$$> \tau_{p(\eta)-1}(\iota/2s) * 1-\varepsilon$$
$$\varepsilon \leq N\big(\Delta\big(\omega_{p(\eta)}\big),\Delta\big(\omega_{q(\eta)}\big),\iota\big)$$
$$\leq N\big(\Delta\big(\omega_{p(\eta)-1}\big),\Delta\big(\omega_{p(\eta)}\big),\iota/2s\big) \circ N\big(\Delta\big(\omega_{p(\eta)-1}\big),\Delta\big(\omega_{q(\eta)}\big),\iota/2s\big)$$
$$< \mu_{p(\eta)-1}(\iota/2s) \circ \varepsilon$$

and

$$\varepsilon \leq O\big(\Delta\big(\omega_{p(\eta)}\big),\Delta\big(\omega_{q(\eta)}\big),\iota\big)$$
$$\leq O\big(\Delta\big(\omega_{p(\eta)-1}\big),\Delta\big(\omega_{p(\eta)}\big),\iota/2s\big) \circ O\big(\Delta\big(\omega_{p(\eta)-1}\big),\Delta\big(\omega_{q(\eta)}\big),\iota/2s\big)$$
$$< h_{p(\eta)-1}(\iota/2s) \circ \varepsilon.$$

Since, $\tau_{p(\eta)-1}(\iota/2s) \to 1$ as $\eta \to +\infty, \mu_{p(\eta)-1}(\iota/2s) \to 0$ as $\eta \to +\infty$ and $h_{p(\eta)-1}(\iota/2s) \to 0$ as $\eta \to +\infty$ for every ι, supposing that $\eta \to +\infty$, we have

$$1-\varepsilon \geq \; M\big(\Delta\big(\omega_{p(\eta)}\big),\Delta\big(\omega_{q(\eta)}\big),\iota\big) < \varepsilon.$$

Hence, it is a contradiction. That is, $\Delta(\omega_n)$ is a Cauchy sequence in $\Delta(\zeta)$.

Case1: Let $\Delta(\zeta)$ is complete. Then, there exists an element $\upsilon \in \Delta(\zeta)$ such that $\lim_{n\to+\infty} \Delta(\omega_n) = \upsilon$. This shows that there exists $z \in \zeta$ such that $\upsilon = \Delta(z)$.

$$M(\Delta(z),\sigma(z),\iota) \geq M(\Delta(z),\Delta(\omega_n),\iota/2s) * M(\Delta(\omega_n),\sigma(z),\iota/2s)$$
$$= M(\Delta(z),\Delta(\omega_n),\iota/2s) * M(\sigma(\omega_{n-1}),\sigma(z),\iota/2s) \geq M(\Delta(z),\Delta(\omega_n),\iota/2s) * M(\Delta(\omega_{n-1}),\Delta(z),\iota/2sk)$$
$$\geq 1 * 1 = 1 \text{ as } n \to +\infty,$$
$$N(\Delta(z),\sigma(z),\iota) \leq N(\Delta(z),\Delta(\omega_n),\iota/2s) \circ N(\Delta(\omega_n),\sigma(z),\iota/2s)$$
$$= N(\Delta(z),\Delta(\omega_n),\iota/2s) \circ N(\sigma(\omega_{n-1}),\sigma(z),\iota/2s) \leq N(\Delta(z),\Delta(\omega_n),\iota/2s) \circ N(\Delta(\omega_{n-1}),\Delta(z),\iota/2sk)$$
$$\leq 0 \circ 0 = 0 \text{ as } n \to +\infty$$

and

$$O(\Delta(z),\sigma(z),\iota) \leq O(\Delta(z),\Delta(\omega_n),\iota/2s) \circ O(\Delta(\omega_n),\sigma(z),\iota/2s)$$
$$= O(\Delta(z),\Delta(\omega_n),\iota/2s) \circ O(\sigma(\omega_{n-1}),\sigma(z),\iota/2s)$$
$$\leq O(\Delta(z),\Delta(\omega_n),\iota/2s) \circ O(\Delta(\omega_{n-1}),\Delta(z),\iota/2sk) \leq 0 \circ 0 = 0 \text{ as } n \to +\infty.$$

By Definition 2.1, it follows that $\Delta(z) = \sigma(z)$.

Case 2: Suppose that $\sigma(\zeta)$ is complete; then there exists an element $\upsilon \in \sigma(\zeta)$ such that $\lim_{n\to+\infty} \Delta(\omega_n) = \upsilon$. Since, $\sigma(\zeta) \in \Delta(\zeta)$, so there exists an element $z \in \zeta$ such that $\upsilon = \Delta(\zeta)$. The existence of a coincident point is obvious from case 1. Now, we examine the uniqueness of a coincident point of Δ and σ. Suppose υ_1 be another point of coincidence of Δ and σ. Then, $\upsilon_1 = \Delta(z_1) = \sigma(z_1)$ for some z_1 in ζ

$$1 \geq M(\upsilon, \upsilon_1, \iota) = M(\sigma(z), \sigma(z_1), \iota)$$

$$\geq M(\Delta(z), \Delta(z_1), \iota/k) = M(\upsilon, \upsilon_1, \iota/k)$$

$$\geq \cdots \geq M(\upsilon, \upsilon_1, \iota/k^n),$$

$$0 \leq N(\upsilon, \upsilon_1, \iota) = N(\sigma(z), \sigma(z_1), \iota)$$

$$\leq N(\Delta(z), \Delta(z_1), \iota/k) = N(\upsilon, \upsilon_1, \iota/k)$$

$$\leq \cdots \leq N(\upsilon, \upsilon_1, \iota/k^n)$$

and

$$0 \leq O(\upsilon, \upsilon_1, \iota) = O(\sigma(z), \sigma(z_1), \iota)$$

$$\leq O(\Delta(z), \Delta(z_1), \iota/k) = O(\upsilon, \upsilon_1, \iota/k)$$

$$\leq \cdots \leq O(\upsilon, \upsilon_1, \iota/k^n).$$

Thus, by Definition 2.1, $\lim_{n\to+\infty} M(\upsilon, \upsilon_1, \iota/k^n) = 1$, $\lim_{n\to+\infty} N(\upsilon, \upsilon_1, \iota/k^n) = 0$ and $\lim_{n\to+\infty} O(\upsilon, \upsilon_1, \iota/k^n) = 0$.

It follows that $1 \geq M(\upsilon, \upsilon_1, \iota) \geq 1$, $0 \leq N(\upsilon, \upsilon_1, \iota) \leq 0$ and $0 \leq O(\upsilon, \upsilon_1, \iota) \leq 0$, which implies that $\upsilon = \upsilon_1$, also by the Definition 2.1. $\lim_{n\to+\infty} M(\upsilon, \upsilon_1, \iota/k^n) = 1$, $\lim_{n\to+\infty} N(\upsilon, \upsilon_1, \iota/k^n) = 0$ and $\lim_{n\to+\infty} O(\upsilon, \upsilon_1, \iota/k^n) = 0$. It follows that $1 \geq M(\upsilon, \upsilon_1, \iota) \geq 1$, $0 \leq N(\upsilon, \upsilon_1, \iota) \leq 0$ and $0 \leq O(\upsilon, \upsilon_1, \iota) \leq 0$. Which implies that $\upsilon = \upsilon_1$.

Remark 3.1 If Δ or σ is a bijective, a unique coincident point must be exist.

Theorem 3.2 Suppose $(\zeta, M, N, O, *, \circ, s)$ be a complete NbMS and $\Delta, \sigma: \zeta \to \zeta$ are verifying the following circumstances:

(1) $\sigma(\zeta) \subseteq \Delta(\zeta)$,

(2) there exists $k, 0 \leq k < 1$, such that, for all $\omega, \upsilon \in \zeta$,

$$M(\sigma(\omega), \sigma(\upsilon), k\iota) \geq M(\Delta(\omega), \Delta(\upsilon), \iota),$$

$$N(\sigma(\omega), \sigma(\upsilon), k\iota) \leq N(\Delta(\omega), \Delta(\upsilon), \iota),$$

$$O(\sigma(\omega), \sigma(\upsilon), k\iota) \leq O(\Delta(\omega), \Delta(\upsilon), \iota),$$

(3) Δ and σ are w-compatible.

Then, Δ and σ have a unique-common FP in ζ.

Proof: By utilizing Theorem 3.1, there exists a unique coincidence point of Δ and σ in ζ. Therefore, we have, υ in ζ such that $\upsilon = \Delta(\sigma(z)) = \Delta(\upsilon)$.Let $\sigma = \Delta(\upsilon) = \sigma(\upsilon)$, then σ is a coincidence point of Δ and σ, therefore, the coincidence point is unique, this shows that

$$\sigma = \upsilon \Rightarrow \upsilon = \Delta(\upsilon) = \sigma(\upsilon).$$

Hence, υ is a unique common FP of Δ and σ.

Corollary 3.1 Suppose $(\zeta, M, N, O, *, \circ)$ be a complete NMS and Δ, σ: $\zeta \to \zeta$ be mappings verifying the below circumstances:

(1) $\sigma(\zeta) \subseteq \Delta(\zeta)$,

(2) there exists k, such that $0 \leq k < 1$, for all $\omega, \upsilon \in \zeta$,

$$M(\sigma(\omega), \sigma(\upsilon), k\iota) \geq M(\Delta(\omega), \Delta(\upsilon), \iota),$$
$$N(\sigma(\omega), \sigma(\upsilon), k\iota) \leq N(\Delta(\omega), \Delta(\upsilon), \iota)$$

and

$$O(\sigma(\omega), \sigma(\upsilon), k\iota) \leq O(\Delta(\omega), \Delta(\upsilon), \iota),$$

(3) Δ and σ are w-compatible.

Then, Δ and σ have a unique-common FP in ζ.

Proof: By taking $s = 1$ in Theorem 3.2, it is obvious.

Corollary 3.2 Suppose $(\zeta, M, *)$ be a complete fuzzy b-metric space and $\Delta, \sigma: \zeta \to \zeta$ are mappings verifying the below circumstances:

(1) $\sigma(\zeta) \subseteq \Delta(\zeta)$,

(2) there exists $k \in [0,1)$ such that, for all $\upsilon \in \zeta$,

$$M(\sigma(\omega), \sigma(\upsilon), k\iota) \geq M(\Delta(\omega), \Delta(\upsilon), \iota),$$

(3) Δ and σ are w-compatible.

Then, Δ and σ have a unique-common FP in ζ.

Proof: By taking $N = 0 = O$ (i.e., N and O are zero functions) in Theorem 3.2, it is obvious.

Corollary 3.3 Suppose $(\zeta, M, *)$ be a complete FMS and Δ, $\sigma: \zeta \to \zeta$ are mappings verifying the below circumstances:

(1) $\sigma(\zeta) \subseteq \Delta(\zeta)$,

(2) there exists $k \in [0,1)$ such that, for all $\upsilon \in \zeta$,

$$M(\sigma(\omega), \sigma(\upsilon), k\iota) \geq M(\Delta(\omega), \Delta(\upsilon), \iota),$$

(3) Δ and σ are w-compatible.

Then, Δ and σ have a unique-common FP in ζ.

Proof: By taking $N = 0 = O$ (i.e., N and O are zero functions) and $s = 1$ in Theorem 3.2, it is obvious.

Example 3.1 Suppose $\zeta = [0,1]$ and $\Delta: \zeta \to \zeta$ be a self mapping on ζ defined as $\Delta(\omega) = 3\omega$, for all $\omega \in \zeta$. Define $M, N, O: \zeta^2 \times [0, +\infty) \to [0,1]$ by

$$M(\omega, \upsilon, \iota) = \begin{cases} \dfrac{\iota}{\iota + |\omega - \upsilon|^2}, & \text{if } \iota > 0, \\ 0, & \text{if } \iota = 0, \end{cases}$$

$$N(\omega, \upsilon, \iota) = \begin{cases} \dfrac{|\omega - \upsilon|^2}{\iota + |\omega - \upsilon|^2}, & \text{if } \iota > 0, \\ 1, & \text{if } \iota = 0, \end{cases}$$

and

$$O(\omega, \upsilon, \iota) = \begin{cases} \dfrac{|\omega - \upsilon|^2}{\iota}, & \text{if } \iota > 0, \\ 1, & \text{if } \iota = 0. \end{cases}$$

It is clear that $(\zeta, M, N, O, *, \circ, s)$ is a complete NbMS but not a NMS, where $a * b = \min\{a, b\}, a \circ b = \max\{a, b\}$, and $for\ all a, b \in [0,1]$. Now, define $\sigma: \zeta \to \zeta$ as $\sigma(\omega) = 2\omega, for\ all \omega \in \zeta$. It is obvious that $\sigma(\zeta) \subseteq \Delta(\zeta)$ and Δ and σ are weakly compatible. Then

$$M(\sigma(\omega), \sigma(\upsilon), k\iota) = \frac{k\iota}{k\iota + |2\omega - 2\upsilon|^2}$$

$$= \frac{k\iota}{k\iota + 4|\omega - \upsilon|^2} \geq \frac{\iota}{\iota + 9|\omega - \upsilon|^2} = M(\Delta(\omega), \Delta(\upsilon), \iota),$$

$$N(\sigma(\omega), \sigma(\upsilon), k\iota) = \frac{|2\omega - 2\upsilon|^2}{k\iota + |2\omega - 2\upsilon|^2}.$$

$$= \frac{4|2\omega - 2\upsilon|^2}{k\iota + 4|\omega - \upsilon|^2} \leq \frac{9|\omega - \upsilon|^2}{\iota + 9|\omega - \upsilon|^2} = N(\Delta(\omega), \Delta(\upsilon), \iota)$$

and

$$O(\sigma(\omega), \sigma(\upsilon), k\iota) = \frac{|2\omega - 2\upsilon|^2}{k\iota}$$

$$= \frac{4|2\omega - 2\upsilon|^2}{k\iota +} \leq \frac{9|\omega - \upsilon|^2}{\iota} = O(\Delta(\omega), \Delta(\upsilon), \iota).$$

Thus, all the circumstances of Theorem 3.2 are fulfilled for $k = \left[0, \frac{1}{4}\right)$. That is, Δ and σ have a unique common FP $\mathbf{0}$. As we can see that the behavior of contractions in figure 4, figure 5 and figure 6. Also, it is easy to see in figure 7 that $\mathbf{0}$ is unique common FP.

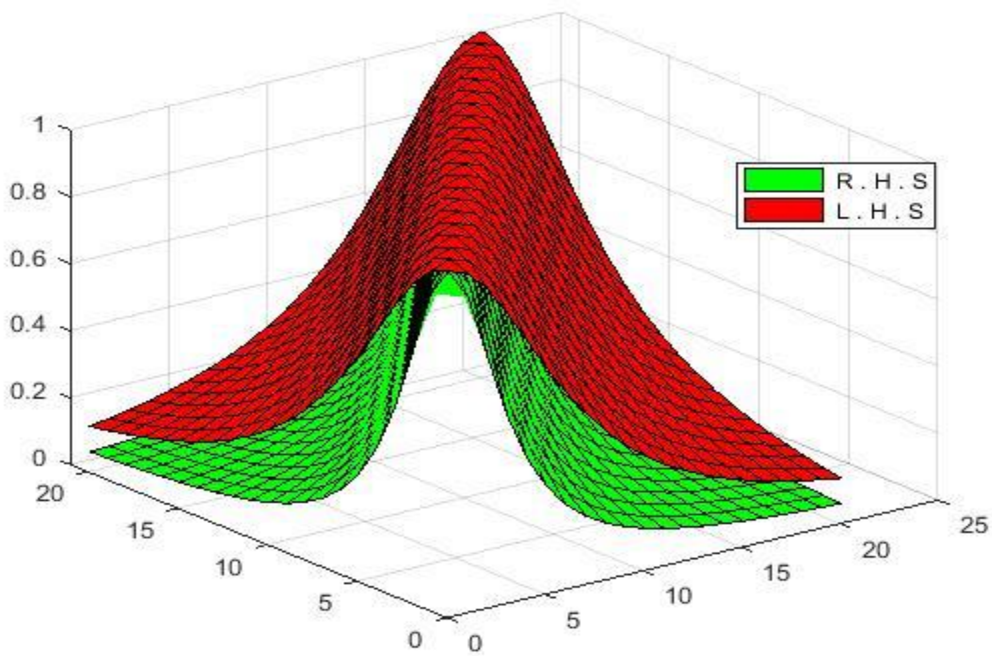

Figure 4 shows the graphical behavior of the contraction mapping $M(\sigma(\omega),\sigma(\upsilon),k\iota) \geq M(\Delta(\omega),\Delta(\upsilon),\iota)$ for $k = \frac{1}{10}$ and $\iota = 1$.

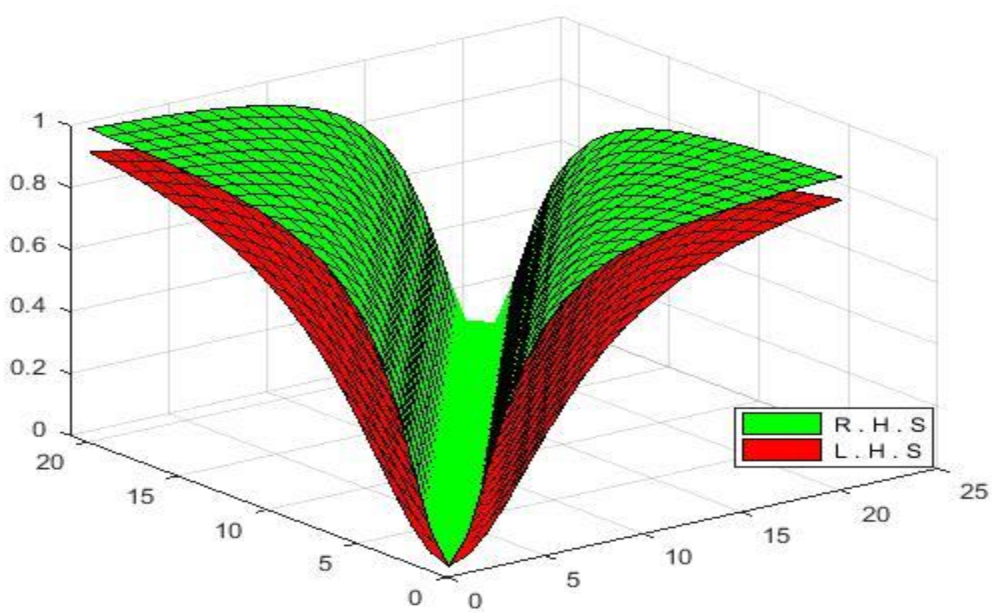

Figure 5 shows the graphical behavior of the contraction mapping $N(\sigma(\omega),\sigma(\upsilon),k\iota) \leq N(\Delta(\omega),\Delta(\upsilon),\iota)$ for $k = \frac{1}{10}$ and $\iota = 1$.

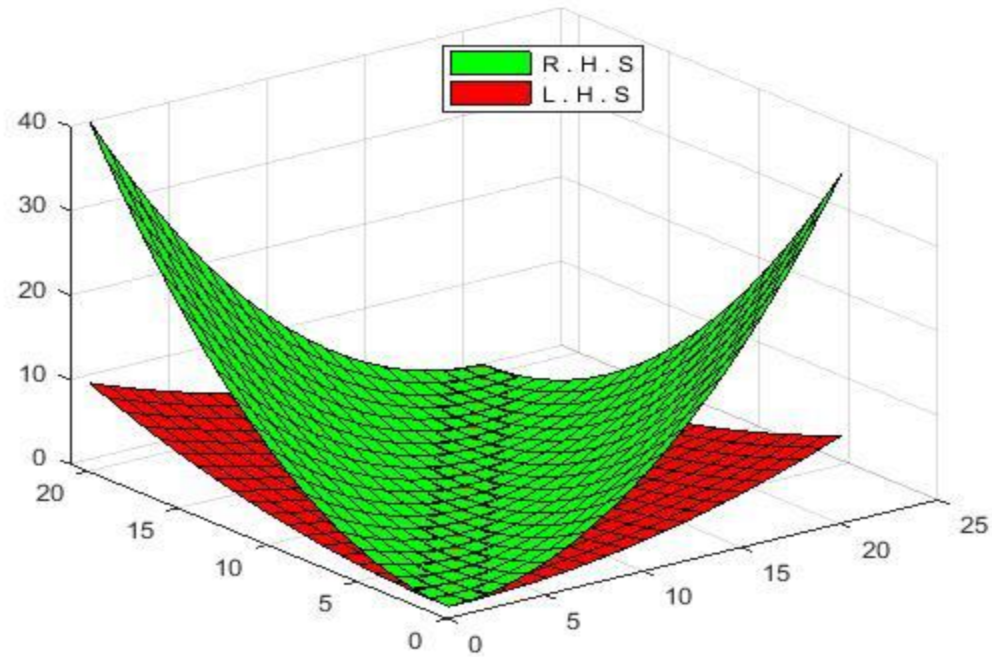

Figure 6 shows the graphical behavior of the contraction mapping $O(\sigma(\omega),\sigma(\upsilon),k\iota) \leq O(\Delta(\omega),\Delta(\upsilon),\iota)$ for $k = \frac{1}{10}$ and $\iota = 1$.

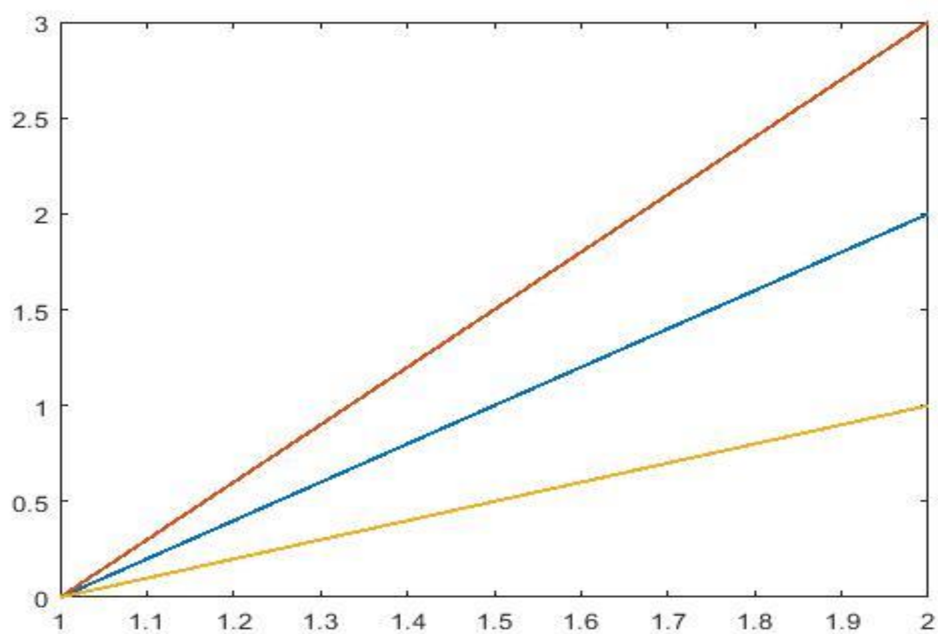

Figure 7 shows that 0 is a common FP, i.e., $0 = \Delta(0) = \sigma(0)$.

4. APPLICATION

Now, we will establish an application to show the validity of Theorem 3.1.

Theorem 4.1 Suppose continuous mappings $F, G: \mathbb{R} \times I \to \mathbb{R}$ and $Q: \mathbb{R} \to \mathbb{R}$ such that

$$G(\omega, \sigma) = F(\omega, \sigma) + Q(\omega),$$

where,

$$I = \{\sigma \in \mathbb{R}; a \leq \sigma \leq b, a, b \in \mathbb{R}\}.$$

Suppose $C(I)$ be the collection of all continuous functions defined from I into $\mathbb{R}$. Assume that, for each $\omega \in C(I)$, there exist $v \in C(I)$ such that $(Qv)(\sigma) = G(\omega(\sigma), \sigma)$ and $\{Q\omega: \omega \in C(I)\}$ is complete. If there exist $k \in [0,1]$ such that z, for all $\omega_1, \omega_2 \in C(I)$ and $\sigma \in I$, then the equation, $F(\omega, \sigma) = 0$, defines a continuous function ω in terms of σ.

Proof: Suppose $\zeta = Y = C(I)$. Define $M, N, O: \zeta^2 \times [0, +\infty) \to [0,1]$ as

$$M_\omega(\omega, v, \iota) = \begin{cases} \dfrac{\iota}{\iota + \max_{\sigma \in I} |\omega(\sigma) - v(\sigma)|}, & \text{if } \iota > 0, \\ 0, & \text{if } \iota = 0, \end{cases}$$

$$N_\omega(\omega, v, \iota) = \begin{cases} \dfrac{\max_{\sigma \in I} |\omega(\sigma) - v(\sigma)|}{\iota + \max_{\sigma \in I} |\omega(\sigma) - v(\sigma)|}, & \text{if } \iota > 0, \\ 1, & \text{if } \iota = 0, \end{cases}$$

and

$$O_\omega(\omega, v, \iota) = \begin{cases} \dfrac{\max_{\sigma \in I} |\omega(\sigma) - v(\sigma)|}{\iota}, & \text{if } \iota > 0, \\ 1, & \text{if } \iota = 0. \end{cases}$$

Define a mapping $\sigma: \zeta \to \zeta$ as follows: $\sigma(\omega(\sigma)) = G(\omega(\sigma), \sigma)$. Then, by assumption, $Q(\zeta) = \{Q\omega: \omega \in \zeta\}$ is complete. Let $\omega^* \in \sigma(\zeta)$; then, $\omega^* = \sigma\omega$ for $\omega \in \zeta$ and $\omega^*(\sigma) = \sigma\omega(\sigma) = G(\omega(\sigma), \sigma)$. By assumptions, there exists $v \in \zeta$ such that $\sigma\omega(\sigma) = G(\omega(\sigma), \sigma) = Qv(\sigma)$. Hence, $\sigma(\zeta) \subseteq Q(\zeta)$. Since

$$\begin{aligned} |(\sigma\, \omega)(\sigma) - (\sigma\, v)(\sigma)| &= |G(\omega(\sigma), \sigma) - G(v(\sigma), \sigma)| \\ &\leq \mathrm{k} |(Q\omega)(\sigma) - (Qv)(\sigma)| \\ &\leq k\big(\max_{\sigma \in I} \big|\big((Q\omega)(\sigma) - (Qv)(\sigma)\big)\big|\big). \end{aligned}$$

That is,

$$\begin{aligned} &\max_{\sigma \in I} |(\sigma\omega)(\sigma) - (\sigma v)(\sigma)| \leq k(\max_{\sigma \in I} |(Q\omega)(\sigma) - (Qv)(\sigma)|) \\ &\Rightarrow \frac{\max_{\sigma \in I} |(\sigma\omega)(\sigma) - (\sigma v)(\sigma)|}{k\iota} \leq \frac{(\max_{\sigma \in I} |(Q\omega)(\sigma) - (Qv)(\sigma)|)}{\iota} \\ &\Rightarrow \frac{k\iota}{\max_{\sigma \in I} |(\sigma\omega)(\sigma) - (\sigma v)(\sigma)|} \geq \frac{\iota}{(\max_{\sigma \in I} |(Q\omega)(\sigma) - (Qv)(\sigma)|)} \end{aligned}$$

$$\Rightarrow \frac{k\iota}{k\iota + \max_{\sigma\in I}|(\sigma\omega)(\sigma) - (\sigma\upsilon)(\sigma)|} \geq \frac{\iota}{\iota + (\max_{\sigma\in I}|(Q\omega)(\sigma) - (Q\upsilon)(\sigma)|)}$$
$$\Rightarrow M(\sigma\omega, \sigma\upsilon, k\iota) \geq M(Q\omega, Q\upsilon, \iota).$$

Also,

$$\frac{\max_{\sigma\in I}|(\sigma\omega)(\sigma) - (\sigma\upsilon)(\sigma)|}{k\iota} \leq \frac{(\max_{\sigma\in I}|(Q\omega)(\sigma) - (Q\upsilon)(\sigma)|)}{\iota}$$
$$\Rightarrow \frac{\max_{\sigma\in I}|(\sigma\omega)(\sigma) - (\sigma\upsilon)(\sigma)|}{k\iota + \max_{\sigma\in I}|(\sigma\omega)(\sigma) - (\sigma\upsilon)(\sigma)|} \leq \frac{(\max_{\sigma\in I}|(Q\omega)(\sigma) - (Q\upsilon)(\sigma)|)}{\iota + (\max_{\sigma\in I}|(Q\omega)(\sigma) - (Q\upsilon)(\sigma)|)}$$
$$\Rightarrow N(\sigma\omega, \sigma\upsilon, k\iota) \leq N(Q\omega, Q\upsilon, \iota).$$

and

$$\frac{\max_{\sigma\in I}|(\sigma\omega)(\sigma) - (\sigma\upsilon)(\sigma)|}{k\iota} \leq \frac{(\max_{\sigma\in I}|(Q\omega)(\sigma) - (Q\upsilon)(\sigma)|)}{\iota}$$
$$\Rightarrow O(\sigma\omega, \sigma\upsilon, k\iota) \leq O(Q\omega, Q\upsilon, \iota).$$

Therefore, all the circumstances of Theorem 3.1 are fulfilled to get a continuous function $z: I \to \mathbb{R}$ such that $\sigma z = Qz$. Then,

$$G(z(\sigma), \sigma) - Q\big(z(\sigma)\big) = 0,$$

where z will be a solution of the equation $F(z, \sigma) = 0$.

Example 4.1 If, we let an implicit form $F(\omega, \sigma) = 10\omega^5(\sigma - 1) + \sigma$, then by assumptions $G(\omega, \sigma) = 10\omega^5(\sigma - 1) + \sigma + 90\omega^5$ and $Q\big(\omega(\sigma)\big) = 90\omega^5$ in Theorem 4.1, we deduce the explicit representation as $\omega = \sqrt{[5]}\ \ \sigma/10\,(1 - \sigma)$.

Suppose the implicit equation,

$$\sigma + \sin(8\omega^5\sigma) - \omega^5 = 0,$$

in the space $C\left(\left[-\frac{1}{9}, \iota\frac{1}{9}\right]\right)$. Let

$$F(\omega, \sigma) = \sigma + \sin(8\omega^5\sigma) - \omega^5,$$
$$Q(\omega) = 5\omega^5 - 5,$$

where $F: \mathbb{R} \times C\left(\left[-\frac{1}{9}, \iota\frac{1}{9}\right]\right) \to \mathbb{R}$ and $Q: \mathbb{R} \to \mathbb{R}$. Let

$$G(\omega, \sigma) = \sigma + \sin(8\omega^5\sigma) + 4\omega^5 - 5.$$

Here, $Q(\omega) = 5\omega^5 - 5$ implies that $Q(\mathbb{R}) = \mathbb{R}$. Now,

$$|\sigma\omega_1 - \sigma\omega_2| = |G(\omega_1, \sigma) - G(\omega_2, \sigma)| = \left|\sigma + \sin\big(8\omega_1^5\sigma\big) + 4\omega_1^5 - 5 - \sigma - \sin\big(8\omega_2^5\sigma\big) - 4\omega_2^5 + 5\right|$$
$$\leq \sin\big(8\omega_1^5\sigma\big) - \sin\big(8\omega_2^5\sigma\big) + 4\omega_1^5 - 4\omega_2^5$$
$$\leq \left|\sin\big(8\omega_1^5\sigma\big) - \sin\big(8\omega_2^5\sigma\big)\right| + \left|4\omega_1^5 - 4\omega_2^5\right|$$
$$\leq 8|\sigma|\omega_1^5 - \omega_2^5\big|\omega_1^5 - \omega_2^5\big|$$

$$\leq \frac{44}{45}\left|5\omega_1^5 - 5 - \omega_2^5 + 5\right|.$$

Therefore, all the circumstances of Theorem 4.1 are fulfilled to apply Theorem 3.1, choose an initial guess $\omega_0(\sigma) = 0$,

$$\sigma(\omega_0(\sigma)) = G(\omega_0(\sigma), \sigma) = \sigma - 5 = Q\big(\omega_1(\sigma)\big) = 5\omega_1^5 - 5.$$

This shows that $\omega_1(\sigma) = \sqrt{[5]}\ \sigma/5$.

$$\sigma(\omega_1(\sigma)) = G(\omega_1(\sigma), \sigma) = \sigma + \sin\big(8\omega_1^5\sigma\big) + 5\omega_1^5 - 5$$

$$= \sigma + \sin\left(8\frac{\sigma^2}{5}\right) + 4\left(\frac{\sigma}{5}\right) - 5$$

$$Q(\omega) = \sigma + \sin\left(8\frac{\sigma^2}{5}\right) + 4\left(\frac{\sigma}{5}\right) - 5,$$

$$5\omega_2^5(\sigma) = \sigma + \sin\left(8\frac{\sigma^2}{5}\right) + 4\left(\frac{\sigma}{5}\right) - 5,$$

$$\Rightarrow \omega(\sigma) = \sqrt[5]{\frac{\sigma + \sin\left(8\frac{\sigma^2}{5}\right) + 4\left(\frac{\sigma}{5}\right) - 5}{5}}$$

$$\sigma\big(\omega_2(\sigma)\big) = G(w_2(\sigma), \sigma) = \sigma + \sin\big(8\omega_1^5\sigma\big) + 5\omega_1^5 - 5,$$

$$Q(\omega_3) = \sigma + \sin 8\left(\frac{\sigma + \sin 8\left(\frac{\sigma^2}{5}\right) + 9\left(\frac{\sigma^2}{5}\right) - 5}{5}\right) + 4\left(\frac{\sigma + \sin 8\left(\frac{\sigma^2}{5}\right) + 9\left(\frac{\sigma^2}{5}\right) - 5}{5}\right) - 5,$$

$$\Rightarrow \omega_3 = \sqrt[5]{\frac{\sigma + \sin 8\left(\sigma + \sin 8\left(\frac{\sigma^2}{5}\right) + 9\left(\frac{\sigma^2}{5}\right) - 5/5\right) + 4\left(\sigma + \sin 8\left(\frac{\sigma^2}{5}\right) + 9\left(\frac{\sigma}{5}\right) - 5\right)/5}{5}}$$

approximates the explicit form of $F(\omega, \sigma)$.

5. CONCLUSION

In this manuscript, we established the notion of NbMS that generalized the notions of fuzzy b-metric space, IFbMS and NMS. We provided numerous non-trivial examples and their graphical views via computational techniques. Also, we derived several coincident points and common fixed-point results for contraction mappings in the context of NbMS, as well, we presented a graphical view of defined contractions. At the end, we provided a novel application to support the validity of our main result.

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