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On the Exponential Stability of the Implicit Differential Systems in Hilbert Spaces

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Abstract. The aim of this research is to study the exponential stability of the stationary implicit system: $Ax'(t)+Bx(t)=0$, where A and B are bounded operators in Hilbert spaces. The achieved results are the generalization of Liapounov Theorem for the spectrum of the operator pencil $\lambda A+B$. We also establish the exponential stability conditions for the corresponding perturbed and quasi-linear implicit systems.

1. Introduction

Consider the general implicit differential system described by the following form:

$$Ax'(t)+Bx(t)=\theta(t,x(t)), \quad t\geq t_0, \; t_0\geq 0, \tag{1.1}$$

where A and B are bounded operators acting from the Hilbert space X into another Hilbert space Y, θ is an operator, usually non-linear from $[t_0,+\infty[\times X$ into Y.

Next, we assume that the system 1.1 has solutions.

The system 1.1 has been considered in various forms by many authors as A. Favini and A. Yagi [6],

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A.G. Rutkas [7], L.A. Vlasenko [8] and others.

In the present paper, we study not only the stationary implicit system

$$Ax'(t) + Bx(t) = 0, \quad t \geq t_0, \tag{1.2}$$

but also the quasi-linear system 1.1, with the initial condition $x(t_0) = x_0$.

In [2] the authors obtained results concerning the stability of the degenerate difference systems that is similar to 1.1.

We can find some practical examples for the above systems in [6–8].

We introduce at first, the basic bellow definition 1.1 of the exponential system. In section 2, we extend the famous general theorem of Liapounov [4] which plays an important role in our paper. In section 3, we establish some conditions of the exponential stability for the perturbed implicit systems and finally, we present our main results about the exponential stability for the solution of the quasi-linear implicit system 1.1.

Definition 1.1. *The system 1.1 is said to be exponential if there exist the constants M and α such that for all solutions $x(t)$, $t \geq t_0$, we have:*

$$\|x(t)\| \leq Me^{\alpha(t-t_0)}\|x_0\|. \tag{1.3}$$

If $\alpha < 0$, then the system 1.1 is said to be exponentially stable.

2. Stationary systems

Consider the implicit stationary system 1.2 above where A and B are bounded operators acting from X into Y.

Definition 2.1. *[7] The system 1.2 is well-posed, if it is determined (i.e, if $x(t_0) = x_0 = 0$ then, $x(t) = 0, \forall t \geq t_0$), and its evolution operator $S(t) : x_0 \mapsto x(t)$ is bounded for all $t \geq t_0$.*

In this work, we use the spectral theory of the operator pencil $\lambda A + B$.

Definition 2.2. *[3, 4] The complex number $\lambda \in \mathbb{C}$ is said to be a regular point of the pencil $\lambda A + B$, if the resolvant operator $R(\lambda) = (\lambda A + B)^{-1}$ exists and it is bounded. The set of all regular points is denoted by $\rho(A, B)$, and its complement $\sigma(A, B) = \mathbb{C} \backslash \rho(A, B)$ is called the spectrum of the pencil $\lambda A + B$.*

The set of all eigen-values of the pencil $\lambda A + B$ is denoted by

$$\sigma_p(A, B) = \{\lambda \in \mathbb{C} \backslash \exists v \neq 0, \quad (\lambda A + B)v = 0\}. \tag{2.1}$$

Proposition 2.1. *The system 1.2 is exponential if and only if it is well-posed.*

Proof. If the system 1.2 is exponential then, it admits a unique solution. In fact, if $x_0 = 0$, according to 1.3, we have $\|x(t)\| \leq 0$. Hence, $x(t) \equiv 0$ for all $t \geq t_0$, in one hand. On the other hand, we have

$$\|x(t)\| = \|S(t)x_0\| \leq Me^{\alpha(t-t_0)}\|x_0\|,$$

so, the operator $S(t)$ is bounded, therefore, the system 1.2 is well-posed.
Conversely, if the system 1.2 is well-posed then, according to [7], we have

$$\overline{\lim_{t\to\infty}} \frac{In\|S(t)\|}{t} = \omega < \infty.$$

Hence,

$$\forall \varepsilon > 0, \quad \exists N_\varepsilon / \|S(t)\| \leq e^{(\omega+\varepsilon)t}, \quad \forall t > N_\varepsilon.$$

Thus,

$$\|x(t)\| \leq \|S(t)x_0\| \leq e^{(\omega+\varepsilon)t}\|x_0\|, \quad \forall t > N_\varepsilon.$$

If we put

$$M_1 = \sup_{t\in[t_0,N_\varepsilon]} \|S(t)\|,$$

we obtain

$$\|x(t)\| = \|S(t)x_0\| \leq Me^{(\omega+\varepsilon)(t-t_0)}\|x_0\|,$$

with

$$M = \sup_{t\in[t_0,N_\varepsilon]} \{\frac{M_1}{e^{(\omega+\varepsilon)(t-t_0)}}, e^{(\omega+\varepsilon)t}\}.$$

Therefore, the system 1.2 is exponential. □

We can find some necessary and sufficient conditions for the system 1.2 to be well-posed in [7].

Proposition 2.2. *If the system 1.2 is exponential then, all the eigen-values of the pencil $\lambda A + B$ are in the half plane ($Re(\lambda) \leq \alpha$), where α is the constant appearing in 1.3.*
In particular, if the system 1.2 is exponentially stable then, all the eigen-values of the pencil $\lambda A + B$ belong to the left half plane, i.e:

$$\sigma_p(A, B) \subset \{\lambda : Re(\lambda) < 0\}.$$

Proof. Suppose that there exists an eigen-value $\lambda_0 \in \sigma_p(A, B)$ with $Re(\lambda_0) > \alpha$. Then, $(\lambda_0 A+B)v = 0$ and v is the corresponding eigen-vector.
Therefore, $y(t) = e^{\lambda_0(t-t_0)}v$ for $t \geq t_0$ is a solution of the system 1.2 such that $v = y(t_0) = y_0$. Moreover, we have:

$$\|y(t)\| = \|e^{\lambda_0(t-t_0)}y_0\| = e^{Re(\lambda_0)(t-t_0)}\|y_0\| > e^{\alpha(t-t_0)}\|y_0\|.$$

So, the solution $y(t)$ does not satisfy the condition 1.3 hence, the system 1.2 is not exponential. □

The general Liapounov theorem (see Theorem 2.4, [3]) can be also extended to the operator pencil $\lambda A + B$ for an arbitrary constant α as follows:

Theorem 2.1. *A necessary condition for the sepctrum $\sigma(A, B)$ of the pencil $\lambda A + B$ to lie inside the half-plane $Re(\lambda) < \alpha$, is that for any uniformly positive operator $G \gg 0$*[1]*, there exists an operator $W \gg 0$ such that:*

$$B^*WA + A^*WB + 2\alpha A^*WA = \frac{G}{\beta} \quad (\beta \neq 0), \tag{2.2}$$

and a sufficent condition is that $\alpha \mp i\beta \in \rho(A, B)$ and there exists an operator $W \gg 0$ such that:

$$B^*WA + A^*WB + 2\alpha A^*WA \gg 0. \tag{2.3}$$

Proposition 2.3. *If A and B are bounded operators in Hilbert spaces X, Y and there exists an operator $W \gg 0$ such that:*

$$F = B^*WA + A^*WB + 2\alpha A^*WA \gg 0,$$

then, there exists a real number $\beta \neq 0$ satisfies the property: $\alpha + i\beta \notin \sigma_p(A, B)$.

Proof. Suppose that, for all $\beta \neq 0$, we have $\alpha + i\beta \in \sigma_p(A, B)$, there exists $v \neq 0$ an eigen-vector verifies $[(\alpha + i\beta)A + B]v = 0$.
Now, we compute the inner product then, we get

$$\begin{aligned} < Fv, v > &= < B^*WAv + A^*WBv + 2\alpha A^*WAv, v >, \\ &= < WAv, Bv > + < WBv, Av > +2\alpha < WAv, Av >, \\ &= -(\alpha - i\beta) < WAv, Av > -(\alpha + i\beta) < WAv, Av > +2\alpha < WAv, Av >, \\ &= 0. \end{aligned}$$

So, we obtain a contradiction with our hypothesis $< Fv, v > \geq c\|v\|^2 > 0$, which proves the proposition. ☐

Proposition 2.4. *In finite dimentional spaces (i.e, $dim(X) = dim(Y) < \infty$), if*

$$\sigma(A, B) = \sigma_p(A, B) \subset \{\lambda : Re(\lambda) < \omega\},$$

then, the system 1.2 is exponential. Moreover, we have $\alpha \leq \omega$, where α is the constant appearing in 1.3.

Proof. Suppose that, the system 1.2 is not exponential. Using the method of elementary divisors (see for example F.R Gantmacher [5]) and noting that the pencil of matrices $\lambda A + B$ is regular (i.e, $det(\lambda A + B) \neq 0$) to prove our proposition. So,

$$\lambda A + B \sim \lambda \widetilde{A} + \widetilde{B} = \{N^{\mu_1}, N^{\mu_2}, ..., N^{\mu_s}; \Im + \lambda I\};$$

[1]it means that $G = G^*$ and that $< Gx, x >> c\|x\|$, $\forall c \in \mathbb{R}$ and for all x with $\|x\| = 1$.

where the first diagonal blocks correspond to the infinite elementary divisors.
Now, we put $x(t) = Qz(t)$ with $det(Q) \neq 0$. So, the system 1.2 is equivalent to the following system:

$$\begin{cases} Az'(t) + Bz(t) = 0, \\ \widetilde{A} = AQ, \quad \widetilde{B} = BQ, \quad \lambda\widetilde{A} + \widetilde{B} = (\lambda A + B)Q. \end{cases} \tag{2.4}$$

In accordance with the diagonal blocks, the system 1.2 can be written as follows:

$$\begin{cases} N^{\mu_k}\dfrac{dz_k}{dt} = 0, \qquad K = 1, 2, ..., s. \\ \dfrac{d\widetilde{z}_k}{dt} + \Im\widetilde{z} = 0, \quad \text{where} \quad z = (z_1, z_2, ..., z_s, \widetilde{z})^t. \end{cases} \tag{2.5}$$

Since, $\sigma(A, B) = \sigma(\widetilde{A}, \widetilde{B}) = \sigma(I, \Im) = \sigma(-\Im) \subset \{\lambda : Re(\lambda) < \omega\}$, then:

$$\|e^{-\Im t}\| \leq M_\omega e^{\omega t},$$

and

$$\|\widetilde{z}(t)\| = \|e^{-\Im(t-t_0)}\widetilde{z}(t_0)\| \leq M_\omega e^{\omega(t-t_0)}\|\widetilde{z}(t_0)\|.$$

So,

$$\|x(t)\| = \|Qz(t)\| \leq \|Q\|M_\omega e^{\omega(t-t_0)}\|z(t)\|.$$

Therefore, the system 1.2 is exponential for $\alpha \leq \omega$. We obtain a contradiction with our hypothesis ($< Fv, v > \geq c\|v\|^2$), which proves the proposition. □

Corollary 2.1. *If $dimX = dimY < \infty$, then the following conditions are equivalents:*

(1) *The system 1.2 is exponential.*
(2) $\sigma(A, B) = \sigma_p(A, B) \subset \{\lambda : Re(\lambda) < \alpha\}$.
(3) $\exists W \gg 0$ *such that* $B^*WA + A^*WB + 2\alpha A^*WA \gg 0$.

According to the proposition 2.2 and 2.3, we have (1) $\Longleftrightarrow$ (2) also, from Theorem 2.1 and Proposition 2.4, we obtain (2) $\Longleftrightarrow$ (3).
In particular, if $\alpha = 0$ then, we obtain the next result:

Corollary 2.2. *If the spaces X and Y have the same finite dimension then, the following assertions are equivalents:*

(1) *The system 1.2 is exponentially stable.*
(2) $\sigma(A, B) = \sigma_p(A, B) \subset \{\lambda : Re(\lambda) < 0\}$.
(3) $\exists W \gg 0$*such that* $B^*WA + A^*WB \gg 0$.

3. Perturbed sytems

We can use the method of variation of constants [7] to prove the following lemma:

Lemma 3.1. *Suppose that in the system 1.2, the operator $A_0 = A/D_0$ is invertible* [2] *with $D_0 = \{x_0\}$. If $\theta(s, x(s)) \in AD_0, \forall s \geq t_0$ and the function $S(t-s)A_0^{-1}\theta(s, x(s))$ is integrable(with respect to s), where $S(t)$ is the evolution operator of the system 1.2. Then, for all $x_0 \in AD_0$, the system 1.1 is equivalent to*

$$x(t) = S(t-s)x_0 + \int_{t_0}^{t} S(t-s)A_0^{-1}\theta(s, x(s))ds. \tag{3.1}$$

In the following, we use the lemma of Gronwall-Bellman:

Lemma 3.2. *[1] If:*

$$g(t) \leq c + \int_{t_0}^{t} g(s)h(s), \quad \forall t \geq t_0, \tag{3.2}$$

where h is a continuous positive real function and $c > 0$ is an arbitrary constant. Then,

$$g(t) \leq c.exp[\int_{t_0}^{t} h(s)ds]. \tag{3.3}$$

For the non stationary perturbation of the system 1.2 with:

$$\theta(t, x(t)) = -(B + B(t)),$$

we have:

Theorem 3.1. *Suppose that:*

1. *The system 1.2 is well-posed.*
2. *The operator A_0 is invertible.*
3. *The linear operators $B(t)$, $t \geq t_0$ which transforme D_0 into AD_0 such that*

$$\int_{t_0}^{\infty} \|A_0^{-1}B(t)\|dt < \infty. \tag{3.4}$$

Then, the perturbed system:

$$Ax'(t) + (B + B(t))x(t) = 0, \quad t \geq t_0 \tag{3.5}$$

is exponential with the same constant α as in 1.3.

Proof. According to the lemma 3.1 with $\psi(t, x(t)) \equiv -(B + B(t))x(t)$, the system 3.5 is equivalent to:

$$x(t) = S(t-t_0)x_0 - \int_{t_0}^{t} S(t-t_0)A_0^{-1}B(s)x(s)ds, \tag{3.6}$$

[2] In particular, if the system 1.2 is well posed then, the operator A_0 (i.e the restriction of A in D_0) is invertible.

where $S(t)$ is the evolution operator of the system 1.2.
Using the hypothesis (1) and the proposition 2.1, we obtain:

$$\|S(t-t_0)x_0\|) \leq Me^{\alpha(t-t_0)}\|x_0\|, \tag{3.7}$$

$$\|S(t-t_0)A_0^{-1}B(s)x(s)\| \leq Me^{\alpha(t-s)}\|A_0^{-1}B(s)x(s)\|, \tag{3.8}$$

From (2) and (3), we have $A_0^{-1}B(s)x(s) \in D_0$. According to 3.6, we obtain:

$$\|x(t)\| \leq Me^{\alpha(t-t_0)}\|x_0\| + M\int_{t_0}^{t} e^{\alpha(t-s)}\|A_0^{-1}B(s)\|\|x(s)\|ds, \tag{3.9}$$

or

$$e^{-\alpha(t-t_0)}\|x(t)\| \leq M\|x_0\| + M\int_{t_0}^{t} e^{\alpha(t_0-s)}\|A_0^{-1}B(s)\|\|x(s)\|ds. \tag{3.10}$$

Applying Lemma 3.2, where

$$g(t) = e^{-\alpha(t-t_0)}\|x(t)\|, \quad h(t) = M\|A_0^{-1}B(t)\|, \quad c = M\|x_0\|, \tag{3.11}$$

then,

$$e^{-\alpha(t-t_0)}\|x(t)\| \leq M\|x_0)\|.exp[M\int_{t_0}^{t}\|A_0^{-1}B(s)\|ds], \tag{3.12}$$

$$\leq M\|x_0\|.exp[M\int_{t_0}^{\infty}\|A_0^{-1}B(s)\|ds]. \tag{3.13}$$

Thus,

$$\|x(t)\| \leq M_1e^{\alpha(t-t_0)}\|x_0\|, \quad \text{where} \quad M_1 = M.exp[M\int_{t_0}^{\infty}\|A_0^{-1}B(s)\|ds] < \infty.$$

□

Corollary 3.1. *Under the condition of Theorem 3.1, if the unperturbed system 1.2 is exponentially stable then, the perturbed system 3.5 is also exponentially stable.*

Remark 3.1. *The theorem 3.1 is a generalization of Dini-Hukuhara's theorem [1], ($A \equiv I$, $B(t) \equiv -T(t)$, $\alpha = 0$).*

Example 3.1. *Consider the system 3.5 in finite dimensional spaces ($dimX = dimY = 2$), with the matrices:*

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad B(t) = e^{-t}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad t \geq t_0.$$

In our case we have:

$$D_0 = \{(a,b)/b = 0\}, \quad AD_0 = \{(a,b)/b = a\},$$

$$\lambda A + B = \begin{pmatrix} \lambda+1 & 0 \\ \lambda+1 & 1 \end{pmatrix}, \quad (\lambda A + B)^{-1} = \frac{1}{\lambda+1}\begin{pmatrix} 1 & 0 \\ -\lambda-1 & \lambda+1 \end{pmatrix}$$

It's clear that $B(t) : D_0 \mapsto AD_0$, $t \geq t_0$ and A_0 is invertible. Since,

$$\sigma(A,B) = \sigma_p(A,B) = \{-1\}.$$

Then, the system 1.2 is exponentially stable (see Corollary 2.2). According to Corollory 3.1, we conclude that the perturbed system 3.5 is also exponentially stable because:

$$\int_{t_0}^{\infty} \|A_0^{-1}B(t)\|dt \leq \|A_0^{-1}\| \int_{t_0}^{\infty} \|B(t)\|dt = \|A_0^{-1}\| \int_{t_0}^{\infty} e^{-t}dt = e^{-t_0}\|A_0^{-1}\| < \infty.$$

4. Quasi-linear systems

Using the same way for demonstration of Theorem 3.1, we can prove the following theorem:

Theorem 4.1. *Suppose that:*

(1) *The system 1.2 is exponential.*
(2) *The operator A_0 is invertible.*
(3) *The non-linear operator θ transforms $[t_0, \infty[\times D_0$ into AD_0 such that*

$$\|A_0^{-1}\theta(t, v)\| \leq \varphi(t).\|v\|, \quad \forall v \in D_0,$$

where φ is real positive function satisfies

$$\int_{t_0}^{\infty} \varphi(t)dt < \infty,$$

then, the quasi-linear system 1.1 is exponential with the same constant α as in 1.3.

Corollary 4.1. *If the linear system 1.2 is exponentially stable then, under the conditions of Theorem 4.1, the quasi-linear system 1.1 is also exponentially stable.*

Theorem 4.2. *Suppose that:*

(1) *The system 1.2 is exponential.*
(2) *The operator A_0 is invertible.*
(3) *The non-linear operator $\theta(t, .)$ transforms D_0 into AD_0 such that*

$$\|A_0^{-1}\theta(t, v)\| \leq \gamma\|Av\|, \quad \forall v \in D_0, \quad \forall t \geq t_0. \tag{4.1}$$

Then, the quasi-linear system 1.1 is also exponential with the constants:

$$\alpha_1 = \alpha + \gamma M, \quad M_1 = M. \tag{4.2}$$

This result represents the generalization of the famous Liapounov theorem on the stability by the first approximation for quasi-linear systems.

Corollary 4.2. *If under the conditions of Theorem 4.2, the constant γ is small enough ($\gamma < -\dfrac{\alpha}{M}$) then, the exponential stability of the linear system 1.2 implies the exponential stability of the corresponding quasi-linear system 1.1.*

The obtained results on the exponential stability can be used to obtain some conditions on the exponential stabilization of implicit controlled systems.

Finally, we note that similar results for the discrete implicit systems are obtained in the paper [2].

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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