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INTEGRAL BOUNDARY VALUE PROBLEMS FOR FRACTIONAL IMPULSIVE INTEGRO DIFFERENTIAL EQUATIONS IN BANACH SPACES

A. ANGURAJ[1], M. KASTHURI[2] AND P. KARTHIKEYAN [3,*]

ABSTRACT. We study in this paper,the existence of solutions for fractional integro differential equations with impulsive and integral conditions by using fixed point method. We establish the Sufficient conditions and unique solution for given problem. An Example is also explained to the main results.

1. INTRODUCTION

In the seventeenth century, Fractional calculus was originated and it has gained much attention in recent years by many researchers. Fractional differential equations appears in a large number of fields of science and engineering, thermodynamics, elasticity, wave propagation, electric railway systems, telecommunication lines and also in chemistry, analysing kinetical reaction problems (see [1, 5, 6, 13, 15, 16]).

Integral and anti-periodic boundary value conditions can be seen in models of a variety of physical, economic and biological processes, and they have been studied extensively in recent years (see [8, 9, 10, 11]) and related references therein for boundary value problems with integral boundary conditions [1, 2, 3, 6].

In [14], the authors have studied the impulsive problems for fractional differential equations with boundary value conditions. J.R. Wang et al. in [7] discussed the existence results for the boundary value problems for impulsive fractional differential equations. The authors in [17] proved the existence of solutions for multi-point nonlinear differential equations of fractional orders with integral boundary conditions without impulsive conditions.

Inspired by the above works, we consider the existence and uniqueness of solutions for impulsive fractional differential equations with integral boundary conditions

$$D^{\alpha}_{0+}u(t) = f(t, u(t), Bu(s)), \quad 1 < \alpha \leq 2, \qquad t \in J' = J\setminus\{t_1, ..., t_m\}, J := [0, T],\ T > 0, \tag{1.1}$$

$$u(t_k^+) = u(t_k^-) + y_k, \quad k = 1, 2, ..., m \qquad y_k \in X, \tag{1.2}$$

$$I^{2-\alpha}_{0+}u(t)|_{t=0} = 0, \quad D^{\alpha-2}_{0+}u(T) = \sum_{i=1}^{m} a_i I^{\alpha-1}_{0+}u(\xi_i), \tag{1.3}$$

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where $Bu(s) = \int_0^t k(t,s,u(s))ds$, $0 < \xi_i < T$, $T > 0$, $a_i \in X$, $m \geq 2$, D^{α}_{0+} and I^{α}_{0+} are the standard Riemann-Liouville fractional derivative and fractional integral respectively, $f : J \times J \times X \to X$, $k : J \times J \times X \to X$ are jointly continuous and t_k satisfy $0 = t_0 < t_1 < ... < t_m < t_{m+1} = T$, $u(t_k^+) = \lim_{\epsilon \to 0^+} u(t_k + \epsilon)$ and $u(t_k^-) = \lim_{\epsilon \to 0^-} u(t_k + \epsilon)$ represent the right and left limits of $u(t)$ at $t = t_k$.

In Section 2, we give definitions of fractional integral and derivative operators, lemma and some fixed point theorems. The main results discussed in section 3. Finally, in section 4, the example is also illustrated.

2. Preliminaries

Let E $= PC(J,X) = \{u : J \longrightarrow X : u \in C((t_k, t_{k+1}], X)\}$ $k = 0, ...m$,be a Banach space with norm $\|u\|_{PC} = sup_{t\in J}\|u(t)\|$. and there exist $u(t_k^+)$ and $u(t_k^-), k = 1, 2, ..., m$ with $u(t_k^+) = u(t_k^-)$, Set $J^{'} = [0,T] \setminus \{t_1, t_2, ...t_m\}$.

Theorem 2.1 ([12]). *(Schaefer's fixed point theorem) Let X be a Banach space. Assume that $T : X \to X$ is a completely continuous operator and the set $V = \{u \in X | u = \mu T u, 0 < \mu < 1\}$ is bounded. Then T has a fixed point in X.*

Theorem 2.2. *(PC-Type Ascoli-Arzela Theorem) Let X be a Banach space and $W \subset PC(J,X)$. If the following conditions are satisfied:*

- **(i):** *W is uniformly bounded subset of $PC(J,X)$*
- **(ii):** *W is equicontinuous in (t_k, t_{k+1}), $k = 0, 1, 2, ..., m$ where $t_0 = 0$, $t_{m+1} = T$;*
- **(iii):** *$W(t) = \{u(t)|u \in W,\ t \in J\setminus\{t_1, ..., t_m\}\}, W(t_k^+ = \{u(t_k^+)|u \in W\}$ and $W(t_k^- = \{u(t_k^-)|u \in W\}$ is a relatively compact subsets of X.*

Then W is a relatively compact subsets of $PC(J,X)$.

Definition 2.3. The fractional integral of order $\alpha > 0$ of a function $y : (0,\infty) \to R$ is given by

$$I^{\alpha}_{0+}y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds,$$

provided the right side is pointwise defined on $(0,\infty)$, where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.4. The fractional derivative of order $\alpha > 0$ of a function $y : (0,\infty) \to R$ is given by

$$D^{\alpha}_{0+}y(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$, provided the right side is pointwise defined on $(0,\infty)$.

Lemma 2.5. *Let $\alpha > 0$ and $u \in C(0,1) \cap L^1(0,1)$.Then fractional differential equation $D^{\alpha}_{0+}u(t) = 0$ has*

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{\alpha-N}, \quad c_i \in \mathbb{R},\ N = [\alpha] + 1,$$

as unique solution.

Lemma 2.6. *Assume that $u \in C(0,1) \cap L^1(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L^1(0,1)$. Then*

$$I^{\alpha}_{0+} D^{\alpha}_{0+} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{\alpha-N},$$

for some $c_i \in \mathbb{R}, i = 1, 2, \ldots, N$, where N is the smallest integer grater than or equal to α.

Lemma 2.7 ([8]). *Let $\alpha > 0$, $n = [\alpha] + 1$. Assume that $u \in L^1(0,1)$ with a fractional integration of order $n - \alpha$ that belongs to $AC^n[0,1]$. Then the equality*

$$(I_{0+}^{\alpha} D_{0+}^{\alpha} u)(t) = u(t) - \sum_{i=1}^{n} \frac{((I_{0+}^{n-\alpha} u)(t))^{n-i}|_{t=0}}{\Gamma(\alpha - i + 1)} t^{\alpha - i}$$

holds almost everywhere on $[0,1]$.

Lemma 2.8 ([8]).

(i) *Let $k \in N, \alpha > 0$. If $D_{a+}^{\alpha} y(t)$ and $(D_{a+}^{\alpha+k} y)(t)$ exist, then*

$$(D^k D_{a+}^{\alpha}) y(t) = (D_{a+}^{\alpha+k} y)(t);$$

(ii) *If $\alpha > 0, \beta > 0, \alpha + \beta > 1$, then*

$$(I_{a+}^{\alpha} I_{a+}^{\alpha}) y(t) = (I_{a+}^{\alpha+\beta} y)(t)$$

satisfies at any point on $[a,b]$ for $y \in L_p(a,b)$ and $1 \le p \le \infty$;

(iii) *Let $\alpha > 0$ and $y \in C[a,b]$. Then $(D_{a+}^{\alpha} I_{a+}^{\alpha}) y(t) = y(t)$ holds on $[a,b]$;*

(iv) *Note that for $\lambda > -1, \lambda \neq \alpha - 1, \alpha - 2, \ldots, \alpha - n$, we have*

$$D^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda - \alpha + 1)} t^{\lambda - \alpha},$$
$$D^{\alpha} t^{\alpha - i} = 0, i = 1, 2, \ldots, n$$

Lemma 2.9. *For any $y(t) \in PC(J, X)$, the linear impulsive fractional boundary-value problem*

$$\begin{aligned} D_{0+}^{\alpha} u(t) = y(t), \quad 1 < \alpha \le 2,\ t \in [0,T], \\ u(t_k^+) = u(t_k^-) + y_k, \quad k = 1, 2, ..., m \quad y_k \in X \\ I_{0+}^{2-\alpha} u(t)|_{t=0} = 0, \quad D_{0+}^{\alpha-2} u(T) = \sum_{i=1}^{m} a_i I_{0+}^{\alpha-1} u(\xi_i), \end{aligned} \tag{2.1}$$

has unique solution

$$u(t) = \begin{cases} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds \\ \quad + \frac{t^{\alpha-1}}{\Gamma(\alpha)(T-A)} \Big[\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)} \int_0^{\xi_i} (\xi_i - s)^{2\alpha-2} y(s) ds - \int_0^T (T-s) y(s) ds \Big], \ for\ t \in [0, t_1) \\ y_1 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds \\ \quad + \frac{t^{\alpha-1}}{\Gamma(\alpha)(T-A)} \Big[\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)} \int_0^{\xi_i} (\xi_i - s)^{2\alpha-2} y(s) ds - \int_0^T (T-s) y(s) ds \Big], \ for\ t \in (t_1, t_2) \\ y_1 + y_2 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds \\ \quad + \frac{t^{\alpha-1}}{\Gamma(\alpha)(T-A)} \Big[\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)} \int_0^{\xi_i} (\xi_i - s)^{2\alpha-2} y(s) ds - \int_0^T (T-s) y(s) ds \Big], \ for\ t \in (t_2, t_3) \\ \vdots \\ \sum_{i=0}^{m} y_i + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds \\ \quad + \frac{t^{\alpha-1}}{\Gamma(\alpha)(T-A)} \Big[\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)} \int_0^{\xi_i} (\xi_i - s)^{2\alpha-2} y(s) ds - \int_0^T (T-s) y(s) ds \Big], \ for\ t \in (t_m, T] \end{cases}$$

where $A = \sum_{i=1}^m a_i \xi_i^{2\alpha-2} / \Gamma(2\alpha - 1)$ and $T \neq A$.

Step:1 For $t \in [0, t_1]$ we have

By Lemma 2.6. the solution of (2.1) can be written as

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds.$$

From $I_{0+}^{2-\alpha} u(t)|_{t=0} = 0$, and by Lemmas 2.7 and 2.8, we know that $c_2 = 0$, and

$$D_{0+}^{\alpha-2} u(t) = c_1 t \Gamma(\alpha) + I_{0+}^2 y(t),$$

$$I_{0+}^{\alpha-1} u(t) = c_1 \frac{\Gamma(\alpha)}{\Gamma(2\alpha-1)} t^{2\alpha-2} + I_{0+}^{\alpha-1} I_{0+}^{\alpha} y(t),$$

from $D_{0+}^{\alpha-2} u(T) = \sum_{i=1}^m a_i I_{0+}^{\alpha-1} u(\xi_i)$, we have

$$c_1 = \frac{1}{\Gamma(\alpha)(T-A)} \Big[\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)} \int_0^{\xi_i} (\xi_i - s)^{2\alpha-2} y(s) ds - \int_0^T (T-s) y(s) ds \Big],$$

where $A = \sum_{i=1}^m a_i \xi_i^{2\alpha-2} / \Gamma(2\alpha-1)$ and $T \neq A$, so

$$\begin{aligned} u(t) &= \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds \\ &\quad + \frac{t^{\alpha-1}}{\Gamma(\alpha)(T-A)} \Big[\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)} \int_0^{\xi_i} (\xi_i - s)^{2\alpha-2} y(s) ds - \int_0^T (T-s) y(s) ds \Big]. \end{aligned}$$

Step:2 If $t \in (t_1, t_2]$,with $u(t_1^+ = u(t_1^-) + y_1$ then we have

$$\begin{aligned} u(t) &= c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + u(t_1^+) - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} y(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds, \\ &= c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + u(t_1^-) + y_1 - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} y(s) ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds, \\ &= c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + y_1 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds. \end{aligned}$$

Then,

$$\begin{aligned} u(t) &= y_1 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds \\ &\quad + \frac{t^{\alpha-1}}{\Gamma(\alpha)(T-A)} \Big[\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)} \int_0^{\xi_i} (\xi_i - s)^{2\alpha-2} y(s) ds - \int_0^T (T-s) y(s) ds \Big]. \end{aligned}$$

Preceding in this way,

Step:3 For $t \in (t_m, T]$, we have

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \sum_{i=1}^m y_i + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds.$$

Then,

$$u(t) = \sum_{i=1}^m y_i + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s) ds$$

$$+\frac{t^{\alpha-1}}{\Gamma(\alpha)(T-A)}\Big[\frac{\sum_{i=1}^{m}a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}y(s)ds-\int_0^T(T-s)y(s)ds\Big].$$

The proof is complete.

□

3. Main Results

In this section, we prove the existence and uniqueness results of the problem (1.1)-(1.3) by using the following assumptions:

(H1) There exist positive functions L , such that

$$|f(t,x,u)-f(t,y,v)|\leq L[|x-y|+|u-v|],\quad \forall t\in[0,T],\quad x,y,u,v\in X,$$

(H2) The function L satisfies

$$2L\leq\Big[\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{T^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^{m}a_i\xi_i^{2\alpha-1}}{\Gamma(2\alpha)}-\frac{T^2}{2}\Big)\Big]^{-1}+\sum_{i=1}^{m}y_i.$$

(H3) There exists a positive constant L_1 such that

$$|f(t,u,v)|\leq L_1\quad for\quad t\in[0,T],\ u,v\in X.$$

Theorem 3.1. *Assume that* (H1), (H2)*are satisfied, then the problem* (1.1)–(1.3) *has a unique solution.*

Proof:

Choose

$$r\geq 2M_1\Big[\frac{T^{\alpha}}{\Gamma(\alpha+1)}+\frac{T^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^{m}a_i\xi_i^{2\alpha-1}}{\Gamma(2\alpha)}-\frac{T^2}{2}\Big)\Big]+\sum_{i=1}^{m}y_i$$

Then we show that $\theta Br\subset Br$, where $Br=\{u\in E:\|u\|\leq r\}$. Let us set $\sup_{t\in[0,T]}|f(t,s,0)|=M_1$,

Step :1 For $t\in[0,t_1]$, we have

$$\begin{aligned}
&\|(\theta u)(t)\|\\
&=\Big|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s),Bu(s))ds\\
&\quad+\frac{t^{\alpha-1}}{\Gamma(\alpha)(T-A)}\Big(\frac{\sum_{i=1}^{m}a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}f(s,u(s),Bu(s))ds\\
&\quad-\int_0^T(T-s)f(s,u(s),Bu(s))ds\Big)\Big|\\
&\leq\Big[\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s),Bu(s))|ds\\
&\quad+\frac{t^{\alpha-1}}{\Gamma(\alpha)(T-A)}\Big(\frac{\sum_{i=1}^{m}a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}|f(s,u(s),Bu(s))|ds\\
&\quad-\int_0^T(T-s)|f(s,u(s),Bu(s))|ds\Big)\Big]\\
&\leq\Big[\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(|f(s,u(s),Bu(s)-f(\sigma,s,0)|+|f(\sigma,s,0)|)ds
\end{aligned}$$

$$+\frac{t^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}(|f(s,u(s),Bu(s)-f(\sigma,s,0)|+|f(\sigma,s,0)|)ds$$
$$-\int_0^T(T-s)(|f(s,u(s),Bu(s)-f(\sigma,s,0)|+|f(\sigma,s,0)|)ds\Big)\Big]$$
$$\le\Big[(2Lr+M_1)\Big(\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds$$
$$+\frac{t^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}ds-\int_0^T(T-s)ds\Big)\Big)\Big]$$
$$\le(2Lr+M_1)\Big[\frac{T^\alpha}{\Gamma(\alpha+1)}+\frac{T^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i\xi_i^{2\alpha-1}}{\Gamma(2\alpha)}-\frac{T^2}{2}\Big)\Big]\le r$$

Taking the maximum over the interval $[0,t_1]$, we obtain $\|\theta(u)(t)\|\le r$.

In view of (H1), for every $t\in[0,t_1]$, we have

$$\|(\theta x)(t)-(\theta y)(t)\|$$
$$=\Big|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(f(t,x)-f(t,y)ds$$
$$+\frac{t^{\alpha-1}}{\Gamma(\alpha)(T-A)}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}(f(t,x,u)-f(t,y,v))ds$$
$$-\int_0^T(T-s)(f(t,x,u)-f(t,y,v))ds\Big)\Big|$$
$$\le\Big[\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|(f(t,x,u)-f(t,y,v))|ds$$
$$+\frac{t^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}|(f(t,x,u)-f(t,y,v))|ds$$
$$-\int_0^T(T-s)|(f(t,x,u)-f(t,y,v))|ds\Big)\Big]$$
$$\le\Big[L[\|x-y\|+\|u-v\|]\Big(\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds$$
$$+\frac{t^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}ds-\int_0^T(T-s)ds\Big)\Big)\Big]$$
$$\le L[\|x-y\|+\|u-v\|]\Big[\frac{T^\alpha}{\Gamma(\alpha+1)}+\frac{T^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i\xi_i^{2\alpha-1}}{\Gamma(2\alpha)}-\frac{T^2}{2}\Big)\Big]$$
$$=A[\|x-y\|+\|u-v\|],$$

where

$$A=L\Big[\frac{T^\alpha}{\Gamma(\alpha+1)}+\frac{T^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i\xi_i^{2\alpha-1}}{\Gamma(2\alpha)}-\frac{T^2}{2}\Big)\Big],$$

which depends only on the parameters involved in the problem. As $A<1$, θ is contraction mapping for the interval $t\in[0,t_1]$.

Step :2 For $t\in(t_1,t_2]$, we have

$$\|(\theta u)(t)\|$$
$$=\Big|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s),Bu(s))ds$$

$$
\begin{aligned}
&+\frac{t^{\alpha-1}}{\Gamma(\alpha)(T-A)}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}f(s,u(s),Bu(s))ds \\
&-\int_0^T(T-s)f(s,u(s),Bu(s))ds\Big)+y_1\Big| \\
\le&\Big[(2Lr+M_1)\Big(\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds \\
&+\frac{t^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}ds-\int_0^T(T-s)ds\Big)\Big)+y_1\Big] \\
\le&(2Lr+M_1)\Big[\frac{T^\alpha}{\Gamma(\alpha+1)}+\frac{T^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i\xi_i^{2\alpha-1}}{\Gamma(2\alpha)}-\frac{T^2}{2}\Big)\Big]+y_1\le r
\end{aligned}
$$

Taking the maximum over the interval $(t_1,t_2]$, we obtain $\|\theta(u)(t)\|\le r$.

In view of (H1), for every $t\in(t_1,t_2]$, we have

$$
\begin{aligned}
&\|(\theta x)(t)-(\theta y)(t)\| \\
&=\Big|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(f(t,x)-f(t,y)ds \\
&\quad+\frac{t^{\alpha-1}}{\Gamma(\alpha)(T-A)}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}(f(t,x,u)-f(t,y,v))ds \\
&\quad-\int_0^T(T-s)(f(t,x,u)-f(t,y,v))ds\Big)+y_1\Big| \\
&\le\Big[L[\|x-y\|+\|u-v\|]\Big(\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds \\
&\quad+\frac{t^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}ds-\int_0^T(T-s)ds\Big)\Big)+y_1\Big] \\
&\le L[\|x-y\|+\|u-v\|]\Big[\frac{T^\alpha}{\Gamma(\alpha+1)}+\frac{T^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i\xi_i^{2\alpha-1}}{\Gamma(2\alpha)}-\frac{T^2}{2}\Big)\Big]+y_1 \\
&=A[\|x-y\|+\|u-v\|],
\end{aligned}
$$

where

$$
A=L\Big[\frac{T^\alpha}{\Gamma(\alpha+1)}+\frac{T^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i\xi_i^{2\alpha-1}}{\Gamma(2\alpha)}-\frac{T^2}{2}\Big)\Big]+y_1,
$$

As $A<1$, θ is therefore a contraction in the interval $t\in(t_1,t_2]$.

Preceding in this way, we got

Step:3 For $t\in(t_m,T]$, we have

$$
\begin{aligned}
&\|(\theta u)(t)\| \\
&=\Big|\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s),Bu(s))ds \\
&\quad+\frac{t^{\alpha-1}}{\Gamma(\alpha)(T-A)}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}f(s,u(s),Bu(s))ds
\end{aligned}
$$

$$
\begin{aligned}
&- \int_0^T (T-s) f(s,u(s),Bu(s))ds\Big) + \sum_{i=1}^m y_i \Big| \\
&\le \Big[(2Lr+M_1)\Big(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds \\
&\quad + \frac{t^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}ds - \int_0^T (T-s)ds\Big)\Big) + \sum_{i=1}^m y_i\Big] \\
&\le (2Lr+M_1)\Big[\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i\xi_i^{2\alpha-1}}{\Gamma(2\alpha)} - \frac{T^2}{2}\Big)\Big] + \sum_{i=1}^m y_i \le r
\end{aligned}
$$

Taking the maximum over the interval $(t_m, T]$, we obtain $\|\theta(u)(t)\| \le r$.

In view of (H1), for every $t \in (t_m, T]$, we have

$$
\begin{aligned}
&\|(\theta x)(t) - (\theta y)(t)\| \\
&= \Big|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}(f(t,x) - f(t,y)ds \\
&\quad + \frac{t^{\alpha-1}}{\Gamma(\alpha)(T-A)}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}(f(t,x,u)-f(t,y,v))ds \\
&\quad - \int_0^T (T-s)(f(t,x,u)-f(t,y,v))ds\Big) + \sum_{i=1}^m y_i\Big| \\
&\le \Big[L[\|x-y\| + \|u-v\|]\Big(\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds \\
&\quad + \frac{t^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}ds - \int_0^T (T-s)ds\Big)\Big) + \sum_{i=1}^m y_i\Big] \\
&\le L[\|x-y\| + \|u-v\|]\Big[\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i\xi_i^{2\alpha-1}}{\Gamma(2\alpha)} - \frac{T^2}{2}\Big)\Big] + \sum_{i=1}^m y_i \\
&= A[\|x-y\| + \|u-v\|],
\end{aligned}
$$

where

$$
A = L\Big[\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i\xi_i^{2\alpha-1}}{\Gamma(2\alpha)} - \frac{T^2}{2}\Big)\Big] + \sum_{i=1}^m y_i,
$$

which depends only on the parameters involved in the problem. Then by Banach fixed point theorem, the operator θ has fixed point in the interval $t \in (t_m, T]$.

□

Theorem 3.2. *Assume that (H1)-(H3) are satisfied. Then* (1.1)-(1.3) *has at least one solution.*

Proof:

We define an operator $P : PC(E) \to PC(E)$, as

$$(Pu)(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(t,u(s),v(s))ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)(T-A)}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}f(t,u(s),v(s))ds - \int_0^T (T-s)f(t,u(s),v(s))ds\Big) + \sum_{i=1}^m y_i. \tag{3.1}$$

To show that the operator P is completely continuous. Clearly, continuity of the operator P follows from the continuity of f. Let $\Omega \subset E$ be bounded. Then, $\forall u, v \in \Omega$ together with (H3) we obtain

$$\begin{aligned}(Pu)(t) &\le \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(t,u(s),v(s))|ds \\ &\quad + \frac{t^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}|f(t,u(s),v(s))|ds \\ &\quad - \int_0^T (T-s)|f(t,u(s),v(s))|ds\Big) + \sum_{i=1}^m y_i \\ &\le L_1\Big[\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds \\ &\quad + \frac{t^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}ds - \int_0^T (T-s)ds\Big)\Big] + \sum_{i=1}^m y_i \\ &\le L_1\Big[\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i\xi^{2\alpha-1}}{\Gamma(2\alpha)} - \frac{T^2}{2}\Big)\Big] + \sum_{i=1}^m y_i,\end{aligned}$$

which implies

$$\|Pu\| \le L_1\Big[\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i\xi^{2\alpha-1}}{\Gamma(2\alpha)} - \frac{T^2}{2}\Big)\Big] + \sum_{i=1}^m y_i < \infty.$$

Hence, $P(\Omega)$ is uniformly bounded.

For any $s_1, s_2 \in [0, t_1], u \in \Omega$, we have

$$\begin{aligned}&|(Pu)(s_1) - (Pu)(s_2)| \\ &= \Big|\int_0^{s_1} \frac{(s_1-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s),v(s))ds \\ &\quad + \frac{s_1^{\alpha-1}}{\Gamma(\alpha)(T-A)}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}f(s,u(s),v(s))ds \\ &\quad - \int_0^T (T-s)^f(s,u(s))ds\Big) - \int_0^{s_2}\frac{(s_2-s)^{\alpha-1}}{\Gamma(\alpha)}f(s,u(s),v(s))ds - \frac{t_2^{\alpha-1}}{\Gamma(\alpha)(T-A)} \\ &\quad \times\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}f(s,u(s),v(s))ds - \int_0^T (T-s)f(s,u(s),v(s))ds\Big)\Big| \\ &\le L_1\Big|\int_0^{s_1}\frac{(s_1-s)^{\alpha-1}-(s_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds\end{aligned}$$

$$
\begin{aligned}
&+\frac{t_1^{\alpha-1}-s_2^{\alpha-1}}{\Gamma(\alpha)(T-A)}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}ds\\
&-\int_0^T(T-s)ds\Big)-\int_{s_1}^{s_2}\frac{(s_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds\Big|\\
\le{}& L_1\Big[\Big|\int_0^{s_1}\frac{(s_1-s)^{\alpha-1}-(s_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds-\int_{s_1}^{s_2}\frac{(s_2-s)^{\alpha-1}}{\Gamma(\alpha)}ds\Big|\\
&+\Big|\frac{s_1^{\alpha-1}-s_2^{\alpha-1}}{\Gamma(\alpha)(T-A)}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}ds-\int_0^T(T-s)ds\Big)\Big|\Big]\\
\to{}& 0 \quad \text{as } s_1\to s_2.
\end{aligned}
$$

Thus, by the PC-type Arzela-Ascoli theorem, $P(\Omega)$ is equicontinuous. Consequently, the operator P is compact.

Next, we consider the set $S=\{u\in E: u=\mu Pu, 0<\mu<1\}$, and show that it is bounded. Let $u\in S$; then $u=\mu Pu, 0<\mu<1$. For any $t\in[0,T]$, we have

$$
\begin{aligned}
u(t)={}&\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f(t,u(s),v(s))ds\\
&+\frac{t^{\alpha-1}}{\Gamma(\alpha)(T-A)}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}f(t,u(s),v(s))ds\\
&-\int_0^T(T-s)f(t,u(s),v(s))ds\Big)+\sum_{i=1}^m y_i,
\end{aligned}
$$

and

$$
\begin{aligned}
|u(t)|={}&\mu|Pu|\\
\le{}&\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(t,u(s),v(s))|ds\\
&+\frac{t^{\alpha-1}}{\Gamma(\alpha)(T-A)}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}|f(t,u(s),v(s))|ds\\
&-\int_0^T(T-s)|f(t,u(s),v(s))|ds\Big)+\sum_{i=1}^m|y_i|\\
\le{}& L_1\Big[\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds\\
&+\frac{t^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i}{\Gamma(2\alpha-1)}\int_0^{\xi_i}(\xi_i-s)^{2\alpha-2}ds-\int_0^T(T-s)ds\Big)\Big]+\sum_{i=1}^m|y_i|\\
\le{}&\max_{t\in[0,T]}\Big\{L_1\Big[\frac{|t^\alpha|}{\Gamma(\alpha+1)}+\frac{|t^{\alpha-1}|}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^m a_i\xi^{2\alpha-1}}{\Gamma(2\alpha)}-\frac{T^2}{2}\Big)\Big]+\sum_{i=1}^m|y_i|\Big\}=M^*.
\end{aligned}
$$

Thus, $\|u\|\le M^*$. So, the set S is bounded. Thus, by the conclusion of Theorem 3.1, the operator P has at least one fixed point, which implies that (1.1)-(1.2) has at least one solution. □

4. Example

Consider the impulsive fractional integro differential equation

$$ {}^{c}D^{\frac{3}{2}}u(t) = \frac{e^t|u(t)|}{(9+e^t)(1+|u(t)|)} + \int_0^t \frac{e^{-(s-t)}}{10}|u(s)|ds, \tag{4.1}$$

$$ t \in J = [0,2], t \neq \frac{1}{2}, \tag{4.2}$$

$$ y(\frac{1}{2}^{+}) = \frac{|u(\frac{1}{2}^{-})|}{3+|u(\frac{1}{2}^{-})|}, \tag{4.3}$$

$$ I_{0+}^{2-\alpha}u(t)|_{t=0} = 0, \quad D_{0+}^{\alpha-2}u(T) = \sum_{i=1}^{m} a_i I_{0+}^{\alpha-1}u(\xi_i), \tag{4.4}$$

where $f(t,u,Bu) = \frac{e^t u}{(9+e^t)(1+u)} + Bu(t), a_1 = 2, a_2 = 3, \xi_1 = 1/2, \xi_2 = 1/3, T = 2$ we have $A = \sum_{i=1}^{m} a_i \xi_i^{2\alpha-2}/\Gamma(2\alpha-1) = 1 \neq T = 1$. Clearly, $L = 1/10$ as

$$|f(t,x,u) - f(t,y,v)| \leq \frac{1}{10}[|x-y| + |u-v|].$$

Further,

$$L\Big[\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T^{\alpha-1}}{\Gamma(\alpha)|T-A|}\Big(\frac{\sum_{i=1}^{m} a_i\xi_i^{2\alpha-1}}{\Gamma(2\alpha)} - \frac{T^2}{2}\Big)\Big] + \sum_{i=1}^{m} y_i \approx 0.01058612753 < 1.$$

Thus, all the assumptions of Theorem 3.1 are satisfied and hence the problem (4.1)-(4.4) has unique solution.

References

[1] A. Anguraj, P. Karthikeyan, and G. M. NGuérékata; *Nonlocal Cauchy problem for some fractional abstract integrodifferential equations in Banach space*, Communications in Mathematical Analysis , vol.55, no. 6, pp. 1?, 2009.

[2] A. Anguraj, P. Karthikeyan and J.J. Trujillo; *Existence of Solutions to Fractional Mixed Integrodifferential Equations with Nonlocal Initial Condition*, Advances in Difference Equations,Volume 2011, Article ID 690653,12pages, doi:10.1155/2011/690653

[3] B. Ahmad, J. J. Nieto; *Existence Results for Nonlinear Boundary Value Problems of Fractional Integrodifferential Equations with Integral Boundary Conditions*, Bound. Value Probl.(2009) Art. ID 708576, 11 pp..

[4] B. Ahmad, A. Alsaedi; *Existence of approximate solutions of the forced Duffing equation with discontinuous type integral boundary conditions*, Nonlinear Analysis, 10 (2009) 358-367.

[5] C. Bai; *Positive solutions for nonlinear fractional differential equations with coefficient that changes sign* Nonlinear Analysis: Theory, Methods and Applications, 64 (2006) 677-685.

[6] Z. Hu, W. Liu; *Solvability for fractional order boundary value problem at resonance*, Boundary value problem, 20(2011)1-10.

[7] J. R Wang, Y. Z. and M. Feckan; *On recent developments in the theory of boundary value problems for impulsive fractional differential equations*, Computers and mathematics with Applications, 64(2012) 3008-3020.

[8] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo; *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[9] V. Lakshmikantham, S. Leela, J. Vasundhara Devi; *Theory of Fractional Dynamic Systems*, Cambridge Academic Publishers, Cambridge, 2009.

[10] J. Sabatier, O. P. Agrawal, J. A. T. Machado (Eds.); *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, Dordrecht, 2007.

[11] S. G. Samko, A. A. Kilbas, O. I. Marichev; *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, New York, NY, USA, 1993.
[12] D. R. Smart; *Fixed Point Theorems*, Cambridge University Press, 1980.
[13] X. Su; *Boundary value problem for a coupled system of nonlinear fractional differential equations*, Applied Mathematics Letters, 22 (2009) 64-69.
[14] T.L. Guo and W. Jiang, *Impulsive problems for fractional differential equations with boundary value conditions*, Computers and mathematics with Applications, 64(2012) 3281-3291.
[15] G. Wang, W. Liu; *The existence of solutions for a fractional 2m-point boundary value problems*, Journal of Applied Mathematics.
[16] G. Wang, W. Liu; *Existence results for a coupled system of nonlinear fractional 2m-point boundary value problems at resonance,* Advances in difference equations,doi:10.1186/1687-1847-2011-44.
[17] G.Wang, W. Liu, C. Ren, *Existence Of Solutions For Multi-Point Nonlinear Differential Equations Of Fractional Orders With Integral Boundary Conditions* , Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 54, pp. 1?0.

[1]Department of Mathematics, PSG College of Arts and Science, Coimbatore, TN, India

[2]Department of Mathematics, P.K.R. Arts College for Women, Gobichettipalayam, TN, India

[3]Department of Mathematics, KSR College of Arts and Science, Tiruchengode, TN, India

*Corresponding author