Int. J. Anal. Appl. (2023), 21:56

International Journal of Analysis and Applications

# Frictional Contact Problem With Wear for Thermo-Viscoelastic Materials With Damage

**Safa Gherian[1], Abdelaziz Azeb Ahmed[1], Fares Yazid[2,*], Fatima Siham Djeradi[2]**

[1] *Laboratory of Operator Theory and PDE: Foundations and Applications, Faculty of Exact Sciences, University of El Oued, El Oued 39000, Algeria*

[2] *Laboratory of Pure and Applied Mathematics, Amar Telidji University of Laghouat, Laghouat 03000, Algeria*

[*] *Corresponding author:* f.yazid@lagh-univ.dz

Abstract. We consider a mathematical model which describes a dynamic frictional contact problem for thermo-viscoelastic materials with long memory and damage. The contact is modeled by the normal compliance condition and wear between surfaces are taken into account. We establish a variational formulation for the model and prove the existence and uniqueness of the weak solution. The proof is based on arguments of hyperbolic nonlinear differential equations, parabolic variational inequalities and Banach fixed point.

## 1. Introduction

Contact problems involving deformable bodies arise naturally in many situations and industrial processes as well as in everyday life and play an important role in mechanical and structural systems. That is way have been widely studied in the last years. The aim of this paper is to model and establish the variational analysis of a frictional contact problem for a dynamic thermo-viscoelastic body with wear and damage. The contact is modelled with normal compliance and wear. Thermoviscoelastic contact problems by taking into account the evolution of the temperature parameter could be found in [5, 6, 13]. General models for thermoelastic frictional contact, derived from thermodynamical principles, have been obtained in [14, 23, 24]. Quasistatic contact problems with normal compliance and friction have been considered in [1] and [16]. Dynamic problems with normal compliance were

Received: Mar. 8, 2023.

2020 *Mathematics Subject Classification.* 74M15, 74M10, 74F15, 49J40.

*Key words and phrases.* thermoviscoelastic; damage; wear; dynamic contact problem.

https://doi.org/10.28924/2291-8639-21-2023-56



ISSN: 2291-8639

first considered in [17]. The existence of weak solutions to dynamic thermoelastic contact problems with frictional heat generation have been proven in [2] and when wear is taken into account in [3, 9]. Recently contact problems with wear were studied in [6, 8, 10, 18, 19]. The damage of the material caused by growth, temperature and various other external factors. The evolution of the microscopic-cracks responsible for the damage is determined by a parabolic inclusion with a constitutive function describing the source of damage in the system which results from tension or compression. Using the subdifferential of indicator function of the interval $[0, 1]$ guarantees that the damage function $\varsigma$, which measures the decrease in the load bearing capacity of the material, varies between 0 and 1; when $\varsigma = 1$ the material has its full capacity; wvhen $\varsigma = 0$ it is completely damaged, and if $\varsigma = 1$, the material is partially damaged. Because of the importance of the subject, three-dimensional problems which include this approach to material damage have been investigated recently in [7, 12, 13]
This paper is organized as follows. In Section 2, we present the notation and some preliminaries. In section 3 we present the original model and list the assumptions on the problem's data and we derive the variational formulation. In section 4 we present our main result stated in Theorem 4.1 and its proof which is based on arguments of time-dependent variational inequalities, parabolic inequalities, differential equations and fixed point.

## 2. Notations and Preliminaries

First we will introduce some notations and preliminaries that we will use later. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d (d = 1, 2, 3)$. Let " : " and " $\cdot$ " represent the inner product on $\mathbb{S}^d$ and $\mathbb{R}^d$, respectively, and $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{S}^d$ and $\mathbb{R}^d$. Thus, for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$, $\mathbf{u} \cdot \mathbf{v} = u_i v_i$, $\|\mathbf{v}\| = (\mathbf{v} \cdot \mathbf{v})^{\frac{1}{2}}$ and for all $\sigma, \zeta \in \mathbb{S}^d$, $\sigma : \zeta = \sigma_{ij}\zeta_{ij}$, $\|\zeta\| = (\zeta : \zeta)^{\frac{1}{2}}$, $1 \leq i, j \leq d$. Also, an index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g. $u_{i;j} = \frac{\partial ui}{\partial x_j}$. We denote by $t$ the time variable and a dot superscript represents the time derivative with respect to the time variable t, e.g. $\dot{\mathbf{u}} = \frac{\partial \mathrm{u}}{\partial t}, \ddot{\mathbf{u}} = \frac{\partial^2 \mathrm{u}}{\partial t^2}$.
In what follows, we use the standard notation for Lebesgue and Sobolev spaces associated to $\Omega$ and $\Gamma$ introduce the spaces

$$H = \{\mathbf{u} = (u_i) : u_i \in L^2(\Omega)\},$$
$$\mathcal{H} = \{\sigma = (\sigma_{ij}) : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\},$$
$$H_1 = \{\mathbf{u} = (u_i) : \varepsilon(\mathbf{u}) \in H\},$$
$$\mathcal{H}_1 = \{\sigma \in \mathcal{H} : \mathrm{Div}œ \in \mathrm{H}\}.$$

Here $\varepsilon$ and Div are the deformation and divergence operators, respectively, defined by

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \ \ \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \ \ \mathrm{Div}\,\sigma = (\sigma_{ij,j}).$$

The spaces $H$, $\mathcal{H}$, $H_1$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$
\begin{aligned}
&(\mathbf{u},\mathbf{v})_H = \int_\Omega \mathbf{u}.\mathbf{v}dx \ \ \forall \mathbf{u},\mathbf{v} \in H, \quad (\zeta,\sigma)_{\mathcal{H}} = \int_\Omega \zeta : \sigma dx \ \ \forall \zeta,\sigma \in \mathcal{H},\\
&(\mathbf{u},\mathbf{v})_{H_1} = (\mathbf{u},\mathbf{v})_H + (\varepsilon(\mathbf{u}),\varepsilon(\mathbf{v}))_{\mathcal{H}} \ \ \forall \mathbf{u},\mathbf{v} \in H_1,\\
&(\zeta,\sigma)_{\mathcal{H}_1} = (\zeta,\sigma)_{\mathcal{H}} + (\mathrm{Div}ı, \mathrm{Div}œ)_{\mathrm{H}} \ \ \forall ı, œ \in \mathcal{H}_1.
\end{aligned}
$$

The associated norms on the spaces $H$, $\mathcal{H}$, $H_1$ and $\mathcal{H}_1$ are denoted by $\|\cdot\|_H$, $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{H_1}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively. For every element $\mathbf{v} \in H_1$ we also use the notation $\mathbf{v}$ for the trace of $\mathbf{v}$ on $\Gamma$ and we denote by $v_\nu$ and $\mathbf{v}_\tau$ the normal and the tangential components of $\mathbf{v}$ on $\Gamma$ given by $v_\nu = \mathbf{v}\cdot\nu$, $\mathbf{v}_\tau = \mathbf{v} - v_\nu \nu$. We also denote by $\sigma_\nu$ and $\sigma_\tau$ the normal and the tangential traces of a function $\sigma \in \mathcal{H}_1$, we recall that when $\sigma$ is a regular function then $\sigma_\nu = \sigma\nu\cdot\nu$, $\sigma_\tau = \sigma\nu - \sigma_\nu\nu$, and the following Green's formula holds

$$
(\sigma,\varepsilon(\mathbf{v}))_{\mathcal{H}} + (\mathrm{Div}œ, \mathbf{v})_{\mathrm{H}} = \int_\Gamma œ\,^\circ\,.\,^\circ\,\mathrm{da} \ \ \forall \mathrm{v} \in \mathrm{H}_1. \tag{2.1}
$$

For a real Banach space $(X, \|\cdot\|_X)$ we use the usual notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$ where $k \in \mathbb{N}$ and $1 \le p \le \infty$; we also denote by $C(0,T;X)$ and $C^1([0,T];X)$ the spaces of continuous and continuously differentiable functions on $[0,T]$ with values in $X$, with the respective norms

$$
\begin{aligned}
\|x\|_{C([0,T];X)} &= \max_{t\in[0,T]} \|x(t)\|_X,\\
\|x\|_{C^1([0,T];X)} &= \max_{t\in[0,T]} \|x(t)\|_X + \max_{t\in[0,T]} \|\dot{x}(t)\|_X.
\end{aligned}
$$

We end this section by giving an existence, uniqueness and regularity result concerning evolution problems, taken from [4, p.268].

**Theorem 2.1.** *Let $V$ and $H$ be two real Hilbert spaces such that $V \subseteq H$ and the inclusion mapping of $V$ into $H$ is continuous and densely defined. We suppose that $V$ is endowed with the norm $\|\cdot\|$ induced by the inner product $(\cdot,\cdot\rangle)$ and $H$ is endowed with the norm $|\cdot|$. We denote by $V'$ the dual space of $V$, by $\langle\cdot,\cdot\rangle_{V'\times V}$ the duality pairing between an element of $V$ and an element of $V'$, and $H$ is identified with its own dual $H'$. We assume that $M$ is a maximal monotone set in $V'\times V$ and $A$ is a linear, continuous and symmetric operator from $V$ to $V'$ satisfying the following coerciveness condition:*

$$
\langle A\mathbf{v},\mathbf{v}\rangle_{V'\times V} + \lambda|\mathbf{v}|^2 \ge \omega\|v\|^2 \ \forall \mathbf{v} \in V \tag{2.2}
$$

*where $\lambda \in \mathbb{R}$ and $\omega > 0$. Let $\mathbf{f} \in W^{1,1}(0,T;H)$ be given in $W^{1,1}(0,T;H)$ and $\mathbf{u}_0, \mathbf{v}_0$ be given with*

$$
\mathbf{u}_0 \in V, \quad \mathbf{v}_0 \in D(M), \quad \{A\mathbf{u}_0 + M\mathbf{v}_0\} \cap H \neq \emptyset. \tag{2.3}
$$

*Then there exists a unique solution* $\mathbf{u}$ *to the following problem:*

$$\begin{cases} \dfrac{d^2\mathbf{u}}{dt^2} + A\mathbf{u} + M(\dfrac{d\mathbf{u}}{dt}) \ni \mathbf{f}(t) \ a.e\ on\ (0,T), \\ \mathbf{u}(0) = \mathbf{u}_0,\ \dfrac{d\mathbf{u}}{dt}(0) = \mathbf{v}_0. \end{cases}$$

*which satisfies*

$$\mathbf{u} \in W^{1,\infty}(0,T;V) \cap W^{2,\infty}(0,T;H).$$

## 3. Mechanical and variational formulations

The physical setting is the following. A thermo-viscoelastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ with outer surface $\Gamma = \partial\Omega$, assumed to be sufficiently smooth and decomposed into three disjoint measurable parts $\Gamma_1, \Gamma_2$, and $\Gamma_3$, such that $meas(\Gamma_1) > 0$. Let us denote by $[0,T]$, $T > 0$ the time interval of interest. The body is clamped on $\Gamma_1$, so the displacement field vanishes there. A surface traction of density $\mathbf{f}_2$ act on $\Gamma_2$. Moreover, the body is submitted to the action of body forces of density $\mathbf{f}_0$ and a heat source of constant strength $q$.
The body could come in sliding frictional contact with a moving obstacle made of a hard perfectly rigid material, and assume that the contact surface of the body $\Gamma_3$ is covered by a layer of soft material. This layer is deformable and the foundation may penetrate it, and could deteriorate over time as a result of frictional contact with the foundation.

**Problem P.** Find a displacement field $\mathbf{u} : \Omega \times [0,T] \longrightarrow \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0,T] \longrightarrow \mathbb{S}^d$, a damage field $\varsigma : \Omega \times [0,T] \longrightarrow \mathbb{R}$, a temperature field $\theta : \Omega \times [0,T] \longrightarrow \mathbb{R}$ and a wear field $w : \Gamma_3 \times [0,T] \longrightarrow \mathbb{R}$ such that

$$\sigma = \mathcal{A}\varepsilon(\dot{\mathbf{u}}) + \mathcal{G}\varepsilon(u) + \int_0^t \mathcal{B}(t-s, \varepsilon(\mathbf{u}(s)), \varsigma(s), \theta(s))ds \quad \text{in } \Omega \times (0,T), \tag{3.1}$$

$$\dot{\varsigma} - \Delta\varsigma + \partial\Psi_k(\varsigma) \ni \mathcal{S}(\varepsilon(\mathbf{u}), \varsigma, \theta) \quad \text{in } \Omega \times (0,T), \tag{3.2}$$

$$\dot{\theta} - \mu_0\Delta\theta = \Phi(\varepsilon(\mathbf{u}), \varsigma, \theta) + q \quad \text{in } \Omega \times (0,T), \tag{3.3}$$

$$\operatorname{Div}\sigma + \mathbf{f}_0 = \rho\ddot{\mathbf{u}} \quad \text{in } \Omega \times (0,T), \tag{3.4}$$

$$\mathbf{u} = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{3.5}$$

$$\sigma\nu = \mathbf{f}_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{3.6}$$

$$\begin{cases} -\sigma_\nu = p_\nu(u_\nu - w), \|\sigma_\tau\| \le p_\tau(u_\nu - w), \\ \|\sigma_\tau\| < p_\tau(u_\nu - w) \Longrightarrow \dot{\mathbf{u}}_\tau = \mathbf{v}^*, \\ \|\sigma_\tau\| = p_\tau(u_\nu - w) \Longrightarrow \dot{\mathbf{u}}_\tau = \mathbf{v}^* - \lambda\sigma_\tau,\ \lambda > 0, \\ \dot{w} = k_\omega\|\mathbf{v}^*\|p_\nu(u_\nu - w), \end{cases} \quad \text{on } \Gamma_3 \times (0,T), \tag{3.7}$$

$$\mu_0 \frac{\partial \theta}{\partial \nu} + \mu_1 \theta = 0 \quad \text{on } \Gamma \times (0, T), \tag{3.8}$$

$$\frac{\partial \varsigma}{\partial \nu} = 0 \quad \text{on } \Gamma \times (0, T), \tag{3.9}$$

$$w(0) = 0 \quad \text{on } \Gamma_3, \tag{3.10}$$

$$\mathbf{u}(0) = \mathbf{u}_0,\ \dot{\mathbf{u}}(0) = \dot{\mathbf{u}}_0,\ \theta(0) = \theta_0,\ \varsigma(0) = \varsigma_0 \quad \text{on } \Omega. \tag{3.11}$$

We now describe problem (3.1)−(3.11). Equation (3.1) represents the thermo-viscoelastic constitutive with long memory and damage, $\mathcal{A}$ and $\mathcal{G}$ denote the linear viscosity operator and the elastic operator, respectively and $\mathcal{B}$ is the relaxation tensor depending on the damage $\varsigma$ and the temperature $\theta$. Equation (3.2) describes the evolution of the damage field, governed by the source damage function $\mathcal{S}$ and $\partial\Psi_K$ is the subdifferential of indicator function of the set of admissible damage functions. Equation (3.3) represents the evolution of the temperature field $\theta$ where $\Phi$ is a nonlinear constitutive function which represents the heat generated by the work of internal forces, $q$ represents the density of volume heat sources and $\mu_0$ is a strictly positive constant. Equation (3.4) represents the equilibrium equation for the stress displacement fields. Equations (3.5) and (3.6) are the displacement and traction boundary conditions, respectively. Equation (3.7) describes the condition with normal compliance, wear and the Coulomb's friction law. The wear function $w$ which measures the wear accumulated of the surface. The evolution of the wear of the contacting surface is governed by the differential form of Archard's law (see, eg., [2, 21, 23, 24]), where $\mathbf{v}^*$ is a constant vector which represents the displacement of the foundation, $k_w > 0$ is a wear coefficient, $p_\nu$ and $p_\tau$ are prescribed functions of the normal compliance and friction bound, respectively. (3.8) is a Fourier boundary condition for the temperature $\theta$ where $\mu_1 > 0$ and it represents a conduction coefficient of $\Gamma$. (3.9) is a homogeneous Niemann boundary condition for the damage $\varsigma$, where $\frac{\partial \varsigma}{\partial \nu}$ is the normal derivative of $\varsigma$. (3.10) represents the initial condition for the wear function, which shows that at the initial moment the foundation is new. Finally the functions $\mathbf{u}_0$, $\dot{\mathbf{u}}_0$, $\theta_0$ and $\varsigma_0$ in (3.11) are the initial data.
We now turn to the variational formulation of Problem $P$.
We introduce the following space for the temperature field denoted by

$$E = H^1(\Omega).$$

The following Friedrichs-Poincaré inequality holds on $E$ is

$$\|\nabla \vartheta\|_{L^2(\Omega)^d} \geq C_F \|\vartheta\|_E, \qquad \forall\, \vartheta \in E. \tag{3.12}$$

$L^2(\Omega)$ is identified with its dual and with a subspace of the dual $E'$ of $E$, i.e., $E \subset L^2(\Omega) \subset E'$, and we say that the inclusions above define a Gelfand triple. We use the notation $\langle .,. \rangle_{E' \times E}$ to represent the duality pairing between $E'$ and $E$.

$$\langle \omega, \vartheta \rangle_{E' \times E} = (\omega, \vartheta)_{L^2(\Omega)}, \quad \forall \omega, \vartheta \in L^2(\Omega). \tag{3.13}$$

We define the admissible space

$$V = \{\mathbf{v} \in (H^1(\Omega))^d : \ \mathbf{v} = 0 \text{ on } \Gamma_1\},$$

also, the admissible damage set

$$K = \{\varsigma \in H^1(\Omega) : \ 0 \leq \varsigma \leq 1 \text{ a.e} \quad \text{in } \Omega\}.$$

Since $meas(\Gamma_1) > 0$, Korn's inequality holds and there exists a constant $C_0 > 0$, that depends only on $\Omega$ and $\Gamma_1$, such that

$$C_0 \|\mathbf{v}\,\|_H \leq \|\varepsilon(\mathbf{v})\,\|_{\mathcal{H}}, \quad \forall\, \mathbf{v} \ \in \ V.$$

On the space $V$, we consider the inner product and the associated norm given by

$$(\mathbf{u}, \mathbf{v})_V = (\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}}, \quad \|\mathbf{u}\|_V = \|\varepsilon(\mathbf{u})\|_{\mathcal{H}}, \quad \forall \mathbf{u}, \mathbf{v} \ \in V. \tag{3.14}$$

It follows that $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent norms on $V$ and therefore $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem and (3.14), there exists a constant $C_1 > 0$, depending only on $\Omega$, $\Gamma_1$ and $\Gamma_3$, such that

$$\|\mathbf{v}\|_{L^2(\Gamma_3)^d} \leq C_1 \|\mathbf{v}\|_V, \ \forall\, \mathbf{v} \ \in V. \tag{3.15}$$

In the study of the mechanical problem (3.1)-(3.11), we still need to assume that the operators $\mathcal{A}, \mathcal{G}, \mathcal{B}$ and the functions $\mathcal{S}, \Phi, p_r$ (for $r = \nu,\ \tau$) satisfy the following conditions

$$\begin{cases} (1) & \mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d; \\ (2) & \text{There exists } m_{\mathcal{A}} > 0 \text{ such that} \\ & (\mathcal{A}(x, \varepsilon_1) - \mathcal{A}(x, \varepsilon_2)).(\varepsilon_1 - \varepsilon_2) \geq m_{\mathcal{A}} \|\varepsilon_1 - \varepsilon_2\|^2, \\ & \ \text{for any } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d \text{ a.e } x \in \Omega; \\ (3) & \text{There exists } M_1, M_2 > 0 \text{ such that} \\ & \|(\mathcal{A}(x, \varepsilon)\| \leq M_1 \|\varepsilon\| + M_2, \ \text{ for any } \varepsilon \in \mathbb{S}^d \text{ a.e } x \in \Omega; \\ (4) & \text{The mapping } x \mapsto \mathcal{A}(x, \varepsilon) \text{ is continuous on } \mathbb{S}^d, \quad \text{a.e. } x \in \Omega; \\ (5) & \text{The mapping } x \mapsto \mathcal{A}(x, \varepsilon) \text{is Lebesgue measurable on } \Omega \\ & \ \text{for any } \varepsilon \in \mathbb{S}^d. \end{cases} \tag{3.16}$$

$$\begin{cases} (1) & \mathcal{G} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d; \\ (2) & \text{There exists } m_{\mathcal{G}} > 0 \text{ such that} \\ & \mathcal{G}(x, \varepsilon) \cdot \varepsilon \geq m_{\mathcal{G}} \|\varepsilon\|^2, \ \text{ for any } \varepsilon \in \mathbb{S}^d \text{ a.e } x \in \Omega; \\ (3) & \mathcal{G}(x, \varepsilon_1) \cdot \varepsilon_2 = \varepsilon_1 \cdot \mathcal{G}(x, \varepsilon_2) \text{ for all } \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \ \text{ a.e. } x \in \Omega; \\ (4) & \mathcal{G}_{ijkl} \in L^\infty(\Omega), \forall i, j, k, l = 1, .., d \end{cases} \tag{3.17}$$

$$\begin{cases} (1) & \mathcal{B} : \Omega \times [0, T] \times \mathbb{S}^d \times \mathbb{R} \times \mathbb{R} \to \mathbb{S}^d; \\ (2) & \text{There exists } L_{\mathcal{B}} > 0 \text{ such that} \\ & \|\mathcal{B}(x, t, \varepsilon_1, \zeta_1, \theta_1) - \mathcal{B}(x, \varepsilon_2, \zeta_2, \theta_2)\| \le L_{\mathcal{B}}(\|\varepsilon_1 - \varepsilon_2\| + |\zeta_1 - \zeta_2| + |\theta_1 - \theta_2|) \\ & \text{for all } t \in [0, T], \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \zeta_1, \zeta_2, \theta_1, \theta_2 \in \mathbb{R}, \text{ a.e. } x \in \Omega; \\ (3) & \text{The mapping } x \mapsto \mathcal{B}(x, t, \varepsilon, \zeta, \theta) \text{ is Lebesgue measurable on } \Omega \\ & \text{ for any } t \in [0, T], \varepsilon \in \mathbb{S}^d, \zeta, \theta \in \mathbb{R}; \\ (4) & \text{The mapping } t \mapsto \mathcal{B}(x, t, \varepsilon, \zeta, \theta) \text{ is continuous measurable on } [0, T] \\ & \text{ for any } \varepsilon \in \mathbb{S}^d, \zeta \in, \theta \in \mathbb{R}, \text{ a.e. } x \in \Omega; \\ (5) & \text{The mapping } x \mapsto \mathcal{B}(x, t, \mathbf{0}, 0, 0) \in \mathcal{H}. \end{cases} \tag{3.18}$$

$$\begin{cases} (1) & \mathcal{S} : \Omega \times \mathbb{S}^d \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}; \\ (2) & \text{There exists } L_{\mathcal{S}} > 0 \text{ such that} \\ & |\mathcal{S}(x, \varepsilon_1, \zeta_1, \theta_1) - \mathcal{S}(x, \varepsilon_2, \zeta_2, \theta_2)| \le L_{\mathcal{S}}(\|\varepsilon_1 - \varepsilon_2\| + |\zeta_1 - \zeta_2| + |\theta_1 - \theta_2|) \\ & \forall x \in \Omega, \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \ \forall\ \zeta_1, \zeta_2, \theta_1, \theta_2 \ \in \ \mathbb{R}; \\ (3) & \text{ The function } x \longrightarrow \mathcal{S}(x, \varepsilon, \zeta, \theta) \text{ is Lebesgue measurable on } \Omega \\ & \qquad \forall\ \varepsilon \ \in \mathbb{S}^d, \forall \zeta \in \mathbb{R} \\ (4) & \text{ The function } x \longrightarrow \mathcal{S}(x, \mathbf{0}, 0, 0) \ \in \ L^2(\Omega). \end{cases} \tag{3.19}$$

$$\begin{cases} (1) & \Phi : \Omega \times \mathbb{S}^d \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}; \\ (2) & \text{There exists } L_{\Phi} > 0 \text{ such that} \\ & |\Phi(x, \varepsilon_1, \varsigma_1, \theta_1) - \Phi(x, \varepsilon_2, \varsigma_2, \theta_2)| \le L_{\Phi}(\|\varepsilon_1 - \varepsilon_2\| + |\varsigma_1 - \varsigma_2| + |\theta_1 - \theta_2|) \\ & \forall x \in \Omega, \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d\ \forall\ \varsigma_1, \varsigma_2, \theta_1, \theta_2 \ \in \ \mathbb{R}; \\ (3) & \text{ The mapping} x \longrightarrow \Phi(x, \varepsilon, \ \forall \varsigma, \theta) \text{ is Lebesgue mesurable on } \Omega \\ & \forall\ \varepsilon \in \mathbb{S}^d, \varsigma, \theta \ \in \mathbb{R}; \\ (4) & \text{ The function } x \longrightarrow \Phi(x, \mathbf{0}, 0, 0) \in L^2(\Omega). \end{cases} \tag{3.20}$$

$$\begin{cases} (1) & p_r : \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}_+, (r = \nu, \tau); \\ (2) & \text{There exists } L_r > 0 \text{ such that } \|p_r(x, a_1) - p_r(x, a_2)\| \le L_r |a_1 - a_2| \\ & \forall x \in \Omega, \forall a_1, a_2 \in \mathbb{R}, a.e.\ x \in \Gamma_3; \\ (3) & p_r(x, a) \le 0 \quad \forall a \le 0, \ a.e.\ x \in \Gamma_3; \\ (4) & \text{The mapping } x \longrightarrow p_r(x, a) \text{ is Lebesgue measurable on } \Gamma_3 \quad \forall a \in \mathbb{R}. \end{cases} \tag{3.21}$$

We suppose that the mass density satisfies, for $\rho^* > 0$

$$\rho \in L^\infty(\Omega), \ \rho(x) \ge \rho^*, \quad \text{a.e.} x \in \Omega. \tag{3.22}$$

We also suppose the mechanical and heat forces satisfy

$$q \in L^2(0, T; L^2(\Omega)), \ \mathbf{f}_0 \in W^{1,1}(0, T; H), \ \mathbf{f}_2 \in W^{1,1}(0, T; L^2(\Gamma_2)^d). \tag{3.23}$$

Next, we define the elements $f(t) \in V$ by

$$(\mathbf{f}(t), \mathbf{v})_V = \int_\Omega \mathbf{f}_0 \cdot \mathbf{v} dx + \int_{\Gamma_2} \mathbf{f}_2 \cdot \mathbf{v} da, \quad \forall \mathbf{v} \in V.$$

Let us define $j : V \times V \times L^2(\Gamma_3) \to \mathbb{R}$ be the functional

$$j(\mathbf{u}, \mathbf{v}, w) = \int_{\Gamma_3} p_\nu(u_\nu - w) v_\nu da + \int_{\Gamma_3} p_\tau(u_\nu - w) \|\mathbf{v}_\tau\| da, \tag{3.24}$$

the functional $j$ satisfies

$$\begin{aligned} &\text{for all } \mathbf{u} \in V \text{and } w \in L^2(\Gamma_3) \\ &j : \mathbf{v} \to j(\mathbf{u}, \mathbf{v}, w) \text{ is proper, convex and lower semicontinuous on } V. \end{aligned} \tag{3.25}$$

Taking into account assumptions (3.21) combined with (3.15), we get

$$\begin{aligned} &j(\mathbf{u}_1, \mathbf{v}_2, w) - j(\mathbf{u}_1, \mathbf{v}_1, w) + j(\mathbf{u}_2, \mathbf{v}_1, w) - j(\mathbf{u}_2, \mathbf{v}_2, w) \\ &\leq C_1^2 (L_\nu + L_\tau) \|\mathbf{u}_1 - \mathbf{u}_2\|_V \|\mathbf{v}_1 - \mathbf{v}_2\|_V, \\ &\quad \forall \mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in V, \ \forall w \in L^2(\Gamma_3). \end{aligned} \tag{3.26}$$

We note that condition (3.23) implies

$$\mathbf{f} \in W^{1,1}(0, T; V) \tag{3.27}$$

We suppose that the initial data satisfy

$$\theta_0 \in L^2(\Omega), \ \varsigma_0 \in K, \ w_0 \in L^2(\Gamma_3), \ \mathbf{u}_0 \in V, \ \dot{\mathbf{u}}_0 \in D(\partial_2 j), \tag{3.28}$$

where $\partial_2 j$ denotes the partial subdifferential with respect to the second argument of the operator $j$ and $D(\partial_2 j)$ represent its domain.

There exists $\mathbf{g} \in H$ such that

$$\begin{aligned} &\big(\mathcal{A}\varepsilon(\dot{\mathbf{u}}_0) + \mathcal{B}\varepsilon(\mathbf{u}_0), \varepsilon(\mathbf{v}) - \varepsilon(\dot{\mathbf{u}}_0)\big)_{\mathcal{H}} + j(\mathbf{u}_0, \mathbf{v}, 0) - j(\mathbf{u}_0, \dot{\mathbf{u}}_0, 0) \\ &\geq (\mathbf{g}, \mathbf{v} - \dot{\mathbf{u}}_0), \quad \forall \mathbf{v} \in V. \end{aligned} \tag{3.29}$$

We introduce the following continuous functionals $A : V \to V'$ and $B : V \to V'$ defined by $\forall \mathbf{u} \in V$, $\forall \mathbf{v} \in V$

$$\langle A\mathbf{u}, \mathbf{v} \rangle_{V' \times V} = (\mathcal{G}\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}}, \tag{3.30}$$

$$\langle B\mathbf{u}, \mathbf{v} \rangle_{V' \times V} = (\mathcal{A}\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}}. \tag{3.31}$$

We define the bilinear forms $a : E \times E \to \mathbb{R}$ and $b : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ defined by

$$a(\xi, \kappa) = \mu_0 \int_\Omega \nabla \xi \cdot \nabla \kappa dx + \mu_1 \int_\Gamma \xi \kappa da, \ \ \forall \ \xi, \ \kappa \in E, \tag{3.32}$$

$$b(\varsigma, \zeta) = \int_\Omega \nabla \varsigma \cdot \nabla \zeta dx, \ \ \forall \ \varsigma, \ \zeta \in H^1(\Omega). \tag{3.33}$$

We will use a modified inner product on the Hilbert space $H$ given by

$$((\mathbf{u},\mathbf{v}))_H = (\rho\mathbf{u},\mathbf{v})_H, \quad \forall \mathbf{u},\mathbf{v}\in H, \tag{3.34}$$

that is, it is weighted with $\rho$, and we let $|||\cdot|||_H$ be the associated norm, i.e.,

$$|||\mathbf{v}|||_H = (\rho\mathbf{v},\mathbf{v})_H^{\frac{1}{2}}, \quad \forall \mathbf{v}\in H. \tag{3.35}$$

It follows from assumptions (3.22) that $|||\cdot|||_H$ and $\|\cdot\|_H$ are equivalent norms on $H$, and also the inclusion mapping of $(V,\|\cdot\|_H)$ into $(H,|||\cdot|||_H)$ is continuous and dense. We denote by $V'$ the dual space of $V$, and by identifying $H$ with its own dual, write

$$V\subset H = H\subset V'.$$

We use the notation $\langle\cdot,\cdot\rangle_{V'\times V}$ to represent the duality pairing between $V'$ and $V$ and recall that

$$\langle\mathbf{u},\mathbf{v}\rangle_{V'\times V} = ((\mathbf{u},\mathbf{v}))_H, \quad \forall \mathbf{u},\mathbf{v}\in H. \tag{3.36}$$

Using the above notation and a standard procedure based on integrals by parts, we have the following variational formulation of the problem thermo-mechanical (3.1)-(3.11).

**Problem PV.** Find a displacement field $\mathbf{u}:\Omega\times[0,T]\longrightarrow V$, a damage field $\varsigma:\Omega\times[0,T]\longrightarrow L^2(\Omega)$, a temperature field $\theta:\Omega\times[0,T]\longrightarrow E$ and a wear field $w:\Gamma_3\times[0,T]\longrightarrow L^2(\Gamma_3)$ such that

$$\begin{aligned}&(\mathcal{A}\varepsilon(\dot{\mathbf{u}})+\mathcal{G}\varepsilon(\mathbf{u})+\int_0^t\mathcal{B}\big(\varepsilon(\mathbf{u}(s)),\varsigma(s),\theta(s)\big)ds,\varepsilon(\mathbf{v})-\varepsilon(\dot{\mathbf{u}}))_{\mathcal{H}}\\&+((\ddot{\mathbf{u}},\mathbf{v}-\dot{\mathbf{u}}))_H+j(\mathbf{u},\mathbf{v},w)-j(\mathbf{u},\dot{\mathbf{u}},w)\geq(\mathbf{f}(t),\mathbf{v}-\dot{\mathbf{u}})_V,\ \ \forall\,\mathbf{v}\ \in\ V,\end{aligned} \tag{3.37}$$

$$\begin{aligned}&(\dot{\varsigma},\beta-\varsigma(t))_{L^2(\Omega)}+b(\varsigma(t),\beta-\varsigma(t))\\&\geq(\mathcal{S}(\varepsilon(\mathbf{u}(t)),\varsigma(t),\theta(t)),\beta-\varsigma(t))_{L^2(\Omega)},\ \ \forall\,\beta\in K,\end{aligned} \tag{3.38}$$

$$\begin{aligned}&\langle\dot{\theta}(t),\alpha\rangle_{E'\times E}+a(\theta(t),\alpha)=\langle\Phi(\varsigma(t),\varepsilon(\mathbf{u}(t)),\theta(t)),\alpha\rangle_{E'\times E}\\&+(q(t),\alpha)_{L^2(\Omega)},\ \ \forall\,\alpha\in E,\quad \text{a.e}\ \ t\in(0,T),\end{aligned} \tag{3.39}$$

$$\dot{w}=k_\omega\|\mathbf{v}^*\|p_\nu(u_\nu-w), \qquad \text{a.e.}\quad t\in(0,T). \tag{3.40}$$

To study Problem (3.37)-(3.40), we need the following smallness assumption

$$L_\nu+L_\tau<\frac{2m_{\mathcal{A}}}{C_1^2+C_1}, \tag{3.41}$$

where $m_{\mathcal{A}}$, $C_1$ and $L_r,(r=\nu,\tau)$ are given in (3.16), (3.15) and (3.17), respectively.
Our main existence and uniqueness result is stated and proved in the next section.

## 4. Existence and uniqueness result

**Theorem 4.1.** *Let the assumptions (3.16)-(3.29) and (3.41). Then there exists an unique solution* $(\mathbf{u}, \varsigma, \theta, w)$ *to problem* $PV$*. Moreover, the solution satisfies*

$$\mathbf{u} \in W^{1,\infty}(0,T;V) \cap W^{2,\infty}(0,T;H), \tag{4.1}$$

$$\varsigma \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)), \tag{4.2}$$

$$\theta \in C(0,T;L^2(\Omega)) \cap L^2(0,T;E), \;\; \dot{\theta} \in L^1(0,T;E') \tag{4.3}$$

$$w \in C^1(0,T;L^2(\Gamma_3)). \tag{4.4}$$

The functions $\mathbf{u}, \sigma, \varsigma, \theta$ and $w$ which satisfy (3.1) and (3.37)-(3.40) are called a weak solution of the contact problem $P$.

We conclude that, under the assumptions (3.16)-(3.29) and (3.41), the mechanical problem (3.1)-(3.11) has a unique weak solution satisfying (4.1)-(4.4). The regularity of the weak solution is given by (4.1)-(4.4) and, in term of stress,

$$\sigma \in W^{1,\infty}(0,T;\mathcal{H}_1). \tag{4.5}$$

Indeed, it follows from (3.1), (3.16)(3), (3.17)(3), (3.17)(4), (3.18)(2) and the regularity (4.1)-(4.3) implies $\sigma \in W^{1,\infty}(0,T;\mathcal{H})$. Let $t \in [0,T]$ and we choose as a test function $\mathbf{v} = \dot{\mathbf{u}}(t) \pm \mathbf{z}$ where $\mathbf{z} \in \mathcal{D}(\Omega)^d$ in (3.37) and using (2.1), (3.34) and (3.36) to obtain

$$\operatorname{Div} \sigma + \mathbf{f}_0 = \rho \ddot{\mathbf{u}} \;\; \text{in } H.$$

It now follows from (3.23) and (4.1) that $\operatorname{Div} \sigma \in W^{1,\infty}(0,T;H)$ which show (4.5).

The proof of Theorem 4.1 is carried out in several steps that we prove in what follows, everywhere in this section we suppose that assumptions of Theorem 4.1 hold, and we consider that $C$ is a generic positive constant which may depend on the problem's data but it is independent on time, and whose value may change from place to place.

Let $w \in C(0,T;L^2(\Gamma_3))$ and $\eta \in L^2(0,T;\mathcal{H})$. In the first step we consider the following variational problem

**Problem $PV_{w\eta}$.** Find $\mathbf{u}_{w\eta} : [0,T] \to V$ such that

$$\begin{aligned}
&((\ddot{\mathbf{u}}_{w\eta}(t), \mathbf{v} - \dot{\mathbf{u}}_{w\eta}(t)))_H + (A\mathbf{u}_{w\eta}(t), \mathbf{v} - \dot{\mathbf{u}}_{w\eta}(t)) \\
&+ (B\dot{\mathbf{u}}_{w\eta}(t), \mathbf{v} - \dot{\mathbf{u}}_{w\eta}(t)) + j(\mathbf{u}_{w\eta}(t), \mathbf{v}, w(t)) \\
&- j(\mathbf{u}_{w\eta}(t), \dot{\mathbf{u}}_{w\eta}(t), w(t)) \geq (\mathbf{F}(t), \mathbf{v} - \dot{\mathbf{u}}_{w\eta}(t))_V, \;\; \forall\, \mathbf{v} \in V, \\
&\mathbf{u}_{w\eta}(0) = \mathbf{u}_0, \;\; \dot{\mathbf{u}}_{w\eta}(0) = \dot{\mathbf{u}}_0,
\end{aligned} \tag{4.6}$$

where $(F(t), \mathbf{v})_V = (f(t), \mathbf{v})_V - (\eta(t), \varepsilon(\mathbf{v}))_{\mathcal{H}}, \quad \forall \mathbf{v} \in V.$

In the study of Problem $PV_{w\eta}$ we have the following result.

**Lemma 4.1.** *The problem $PV_{w\eta}$ has a unique solution which satisfies $\mathbf{u}_{w\eta} \in W^{1,\infty}(0,T;V) \cap W^{2,\infty}(0,T;H)$*

*Proof.* By assumptions (3.30), (3.14), (3.17)(3), we see that the operator $A$ is linear, continuous, and symmetric from $V$ to $V'$ and satisfies the condition (2.2) with $\lambda = 0$ and $\omega = m_{\mathcal{G}}$.
After, we define the set-valued operator $\psi : V \to V'$ by

$$\psi = B + \partial_2 j. \tag{4.7}$$

From (3.16)(2), we deduce that the operator $B$ defined by (3.31), is monotone. Using (3.31) and (3.14), we have

$$\|B\mathbf{u} - B\mathbf{v}\|_{V'} \leq \|\mathcal{A}\varepsilon(\mathbf{u}) - \mathcal{A}\varepsilon(\mathbf{v})\|_{\mathcal{H}}, \ \forall \mathbf{u}, \mathbf{v} \in V,$$

keeping in mind (3.16)(2), (3.16)(3), (3.16)(4) and Krasnoselski's theorem (see [15, p.60]), we find that $B : V \to V'$ is a continuous operator. Using again (3.31) and (3.16)(2), we find that $B$ is bounded.
From (3.24) and (3.25), we deduce that $j$ is maximal monotone. Consequently, since $B$ is monotone, bounded and hemicontinuous from $V$ to $V'$, we conclude (see [4, p.39]) that $\psi = B + \partial_2 j$ is maximal monotone.
Moreover, the initial data $\mathbf{u}_0$ and $\dot{\mathbf{u}}_0$ satisfy (2.3) due to (3.28) and (3.29). Thus, all the requirements of Theorem 2.1, with $A$ defined by (3.30), $M = \psi$ given in (4.7) and $\mathbf{f} = \mathbf{F}$, are satisfied, it follows that there exists a unique solution $\mathbf{u}_{w\eta}$ to Problem $PV_{w\eta}$ satisfying the regularity expressed in (4.1). □

Let $\eta \in L^2(0,T;\mathcal{H})$. In the second step, we consider the operator $\chi : C(0,T;L^2(\Gamma_3)) \to C(0,T;L^2(\Gamma_3))$ defined by

$$\chi w(t) = k_w \|\mathbf{v}^*\| \int_0^t p_\nu(u_{w\eta\nu}(s) - w(s))ds \qquad \forall t \in [0,T]. \tag{4.8}$$

**Lemma 4.2.** *The operator $\chi$ has a unique fixed point $w^* \in C(0,T;L^2(\Gamma_3))$.*

*Proof.* Let $w_1, w_2 \in C(0,T;L^2(\Gamma_3))$ and denote by $u_i$, $i = 1,2$ the solutions to the problem $PV_{w\eta}$, for $w = w_i$ i.e. $\mathbf{u}_i = \mathbf{u}_{w_i\eta}$ and $\mathbf{v}_i = \dot{\mathbf{u}}_i = \mathbf{u}_{w_i\eta}$. From the definition (4.8) of $\chi$, we can write

$$|\chi w_1(t) - \chi w_2(t)|_{L^2(\Gamma_3)} \leq k_w \|\mathbf{v}^*\| \int_0^t |(p_\nu(u_{1\nu}(t) - w_1(t)) - p_\nu(u_{2\nu}(t) - w_2(t)))|ds$$

Using (3.21)(2) and (3.15), we get

$$\|\chi w_1(t) - \chi w_2(t)\|^2_{L^2(\Gamma_3)} \leq C\Big(\int_0^t \|w_1(s) - w_2(s)\|^2_{L^2(\Gamma_3)}ds + \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|^2_V\Big)ds \tag{4.9}$$

Using the relation (4.6), we find

$$\begin{aligned}
&((\dot{\mathbf{v}}_1(t) - \dot{\mathbf{v}}_2(t), \mathbf{v}_1(t) - \mathbf{v}_2(t)))_H + (A\mathbf{u}_1(t) - A\mathbf{u}_2(t), \mathbf{v}_1(t) - \mathbf{v}_2(t))_V \\
&+ (B\mathbf{v}_1(t) - B\mathbf{v}_2(t), \mathbf{v}_1(t) - \mathbf{v}_2(t))_V \\
&\leq j(\mathbf{u}_1(t), \mathbf{v}_2(t), w_1(t)) - j(\mathbf{u}_2(t), \mathbf{v}_2(t), w_2(t)) \\
&+ j(\mathbf{u}_2(t), \mathbf{v}_1(t), w_2(t)) - j(\mathbf{u}_1(t), \mathbf{v}_1, w_1(t)).
\end{aligned}$$

By virtue of (3.30), (3.31), (3.16)(2), (3.17)(2), (3.21) and (3.14)(2), this inequality becomes

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|||\mathbf{v}_1(t) - \mathbf{v}_2(t)|||_H^2 + \frac{m_{\mathcal{G}}}{2}\frac{d}{dt}\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 + m_{\mathcal{A}}\|\mathbf{v}_1(t) - \mathbf{v}_2(t)\|_V^2 \\
&\leq L_\nu \int_{\Gamma_3} (|u_{1\nu} - u_{2\nu}| + |w_1 - w_2|)|v_{2\nu}|da - L_\nu \int_{\Gamma_3} (|u_{1\nu} - u_{2\nu}| + |w_1 - w_2|)|v_{1\nu}|da \\
&+ L_\tau \int_{\Gamma_3} (|u_{1\nu} - u_{2\nu}| + |w_1 - w_2|)\|\mathbf{v}_{2\tau}\|da - L_\tau \int_{\Gamma_3} (|u_{1\nu} - u_{2\nu}| + |w_1 - w_2|)\|\mathbf{v}_{1\tau}\|da.
\end{aligned}$$

Integrating this inequality over the interval time variable $(0, t)$, using (3.15) and the inequality $2ab \leq a^2 + b^2$ leads to

$$\begin{aligned}
&|||\mathbf{v}_1(t) - \mathbf{v}_2(t)|||_H^2 + \frac{m_{\mathcal{G}}}{2}\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 + m_{\mathcal{A}} \int_0^t \|\mathbf{v}_1(s) - \mathbf{v}_2(s)\|_V^2 ds \\
&\leq (L_\nu + L_\tau)\frac{C_1}{2} \int_0^t \big(C_1\|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 + (C_1 + 1)\|\mathbf{v}_1(s) - \mathbf{v}_2(s)\|_V^2 \\
&+ \|w_1(s) - w_2(s)\|_{L^2(\Gamma_3)}^2\big) ds,
\end{aligned}$$

and keeping in mind (3.41), we obtain

$$\int_0^T \|\mathbf{v}_1(s) - \mathbf{v}_2(s)\|_V^2 ds \leq C \int_0^T \big(\|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V^2 + \|w_1(s) - w_2(s)\|_{L^2(\Gamma_3)}^2\big) ds. \tag{4.10}$$

On the other hand, since $\mathbf{u}_i(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}_i(s)\, ds$, we have

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 \leq \int_0^t \|\mathbf{v}_1(s) - \mathbf{v}_2(s)\|_V^2\, ds, \tag{4.11}$$

and using this inequality in (4.10) yields

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V^2 \leq C \int_0^T \Big(\|w_1(t) - w_2(t)\|_V^2 + \int_0^t \|\mathbf{v}_1(s) - \mathbf{v}_2(s)\|_V^2\, ds\Big).$$

It follows now from a Gronwall-type argument that

$$\|\mathbf{u}_1(t) - \mathbf{v}_2(t)\|_V^2\, ds \leq C \int_0^t \|w_1(s) - w_2(s)\|_V^2\ ds.$$

which implies for $s \leq t \leq T$

$$\begin{aligned}\int_0^t \|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|_V^2 ds \leq & C\int_0^t\int_0^s \|w_1(r)-w_2(r)\|_{L^2(\Gamma_3)}^2 drds \\ \leq & C\int_0^t\int_0^t \|w_1(r)-w_2(r)\|_{L^2(\Gamma_3)}^2 drds \\ \leq & C\int_0^t \|w_1(r)-w_2(r)\|_{L^2(\Gamma_3)}^2 dr \int_0^T ds.\end{aligned}$$

Then

$$\int_0^t \|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|_V^2 ds \leq CT\int_0^t \|w_1(s)-w_2(s)\|_{L^2(\Gamma_3)}^2 ds. \tag{4.12}$$

From (4.8) and (4.12), we deduce that

$$\|\chi w_1(s)-\chi w_2(s)\|_{L^2(\Gamma_3)}^2 \leq CT\int_0^t \|w_1(s)-w_2(s)\|_{L^2(\Gamma_3)}^2 ds. \tag{4.13}$$

Reiterating the last inequality $n$ times, we infer that

$$\|\chi^n w_1-\chi^n w_2\|_{C(0,T;L^2(\Gamma_3))} \leq \left(\frac{C^nT^{n+1}}{n!}\right)^{\frac{1}{2}} \|w_1-w_2\|_{C(0,T;L^2(\Gamma_3))}.$$

Thus, for $n$ sufficiently large, a power $\chi^n$ of $\chi$ is a contraction in the Banach space $C(0,T;L^2(\Gamma_3))$. Which implies that the operator $\chi$ has a unique fixed point $w^* \in C(0,T;L^2(\Gamma_3))$. □

In the third step, let $\gamma \in L^2(0,T;L^2(\Omega))$ be given and consider the following variational problem for the damage field.

**Problem** $PV_\gamma$**.** Find a damage field $\varsigma_\gamma : [0,T] \to H^1(\Omega)$ such that

$$\begin{aligned}\varsigma_\gamma \in K, \quad & (\dot{\varsigma}_\gamma, \beta-\varsigma_\gamma(t))_{L^2(\Omega)} + b(\varsigma_\gamma(t),\beta-\varsigma_\gamma(t)) \\ & \geq (\gamma(t),\beta-\varsigma_\gamma(t))_{L^2(\Omega)} \quad \forall \beta \in K, \\ & \varsigma_\gamma(0)=\varsigma_0.\end{aligned} \tag{4.14}$$

**Lemma 4.3.** *The problem $PV_\gamma$ has a unique solution $\varsigma_\gamma$ satisfying*

$$\varsigma_\gamma \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \tag{4.15}$$

*Proof.* Using (3.33) and (3.28), after some algebraic computations and from a classical existence and uniqueness result of parabolic equations (see for example the reference [22, p.60]), we find that the problem $PV_\gamma$ has a unique solution $\varsigma_\gamma \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$. □

In the forth step, let $\varphi \in L^2(0,T;E')$, we consider the following variational problem

**Problem** $PV_\varphi$**.** Find a temperature $\theta_\varphi : \Omega \times (0, T) \to \mathbb{R}$ such that

$$\begin{aligned} &\langle \dot{\theta_\varphi}(t), \alpha \rangle_{E' \times E} + a(\theta_\varphi(t), \alpha) = \langle \varphi(t) + q(t), \alpha \rangle_{E' \times E} \\ &\forall\, \alpha \in E \ a.e\ t \in (0, T), \qquad (4.16) \\ &\theta_\varphi(0) = \theta_0. \end{aligned}$$

**Lemma 4.4.** *The problem $PV_\varphi$ has a unique solution $\theta_\varphi$ satisfies the regularity (4.3).*

*Proof.* From the Friedrichs-poincaré inequality, we can find that there exists a constant $\mu_2 > 0$ such that

$$\int_\Omega \|\nabla(\xi)\|^2 dx + \frac{\mu_1}{\mu_0} \int_\Gamma \|\xi\|^2 da \geq \mu_2 \int_\Omega \|\xi\|^2 dx.$$

Thus, we obtain

$$a(\xi, \xi) \geq \mu_3 \|\xi\|_E^2, \qquad (4.17)$$

where $\mu_3 = \frac{\mu_0 min(1,\mu_2)}{2}$, which implies that $a$ is elliptic on $E$. Consequently, based on a classical arguments of functional analysis concerning parabolic equations (see [4, p.140]), we conclude that the problem $PV_\varphi$ has a unique solution $\theta_\varphi$ which satisfies the regularity (4.3). □

Let us now consider the operator $\Lambda : L^2(0, T; \mathcal{H} \times L^2(\Omega) \times E) \to L^2(0, T; \mathcal{H} \times L^2(\Omega) \times E)$

$$\Lambda(\eta, \gamma, \varphi)(t) = (\Lambda_1(\eta, \gamma, \varphi)(t), \Lambda_2(\eta, \gamma, \varphi)(t), \Lambda_3(\eta, \gamma, \varphi)(t)), \qquad (4.18)$$

defined by

$$\Lambda_1(\eta, \gamma, \varphi)(t) = \int_0^t \mathcal{B}\big(\varepsilon(\mathbf{u}_{w^*\eta})(s), \varsigma_\gamma(s), \theta_\varphi(s)\big) ds, \qquad (4.19)$$

$$\Lambda_2(\eta, \gamma, \varphi)(t) = \mathcal{S}(\varepsilon(\mathbf{u}_{w^*\eta}(t)), \varsigma_\gamma(t), \theta_\varphi(t)), \qquad (4.20)$$

$$\Lambda_3(\eta, \gamma, \varphi)(t) = \Psi(\varepsilon(\mathbf{u}_{w^*\eta}(t)), \varsigma_\gamma(t), \theta_\varphi(t)). \qquad (4.21)$$

Here, $w^*$ be the fixed point of the operator $\chi$. For every $(\eta, \gamma, \varphi) \in L^2(0, T; \mathcal{H} \times L^2(\Omega) \times E^{'})$, $\mathbf{u}_{w^*\eta}$, $\varsigma_\gamma$ and $\theta_\varphi$ represent the displacement, the damage and the temperature obtained in Lemma 4.1, Lemma 4.3 and Lemma 4.4 respectively.

The last step in the proof of Theorem 4.1 is the next result.

**Lemma 4.5.** *The operator $\Lambda$ has a unique fixed point $(\eta^*, \gamma^*, \varphi^*) \in L^2(0, T; \mathcal{H} \times L^2(\Omega) \times E')$.*

*Proof.* Let $(\eta_1, \gamma_1, \varphi_1), (\eta_2, \gamma_2, \varphi_2) \in L^2(0, T; \mathcal{H} \times L^2(\Omega) \times E^{'})$.

We use the notation $\mathbf{u}_{w^*\eta_i} = \mathbf{u}_i$, $\dot{\mathbf{u}}_{w^*\eta_i} = \mathbf{v}_i$, $\varsigma_{\gamma_i} = \varsigma_i$ and $\theta_{\varphi_i} = \theta_i$ for $i = 1, 2$.

Using (4.19)-(4.21) and from (3.18)-(3.20) and (3.14), we get

$$\begin{aligned} &\|\Lambda(\eta_1, \gamma_1, \varphi_1)(t) - \Lambda(\eta_2, \gamma_2, \varphi_2)\|_{\mathcal{H} \times L^2(\Omega) \times E'} \\ &\leq L_{\mathcal{B}} \int_0^t \big(\|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V + \|\varsigma_1(s) - \varsigma_2(s)\|_{L^2(\Omega)} + \|\theta_1(s) - \theta_2(s)\|_E\big) ds \\ &+ (L_S + L_\Phi)\big(\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V + \|\varsigma_1(t) - \varsigma_2(t)\|_{L^2(\Omega)} + \|\theta_1(t) - \theta_2(t)\|_{L^2(\Omega)}\big) \end{aligned}$$

Employing Hölder's and Young's inequalities, we deduce that

$$\begin{aligned}
&\|\Lambda(\eta_1,\gamma_1,\varphi_1)(t)-\Lambda(\eta_2,\gamma_2,\varphi_2)\|^2_{\mathcal{H}\times L^2(\Omega)\times E'}\\
&\le C\int_0^t \big(\|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|^2_V+\|\varsigma_1(s)-\varsigma_2(s)\|^2_{L^2(\Omega)}\\
&+\|\theta_1(s)-\theta_2(s)\|^2_E\big)ds+C\big(\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|^2_V\\
&+\|\varsigma_1(t)-\varsigma_2(t)\|^2_{L^2(\Omega)}+\|\theta_1(t)-\theta_2(t)\|^2_{L^2(\Omega)}\big)
\end{aligned}\tag{4.22}$$

Using the relation (4.6), we obtain

$$\begin{aligned}
&(\mathcal{A}\varepsilon(\mathbf{v}_1)-\mathcal{A}\varepsilon(\mathbf{v}_2),\varepsilon(\mathbf{v}_1-\mathbf{v}_2)_V+((\dot{\mathbf{v}}_1-\dot{\mathbf{v}}_2,\mathbf{v}_1-\mathbf{v}_2))_H\\
&\le(\mathcal{G}\varepsilon(\mathbf{u}_1)-\mathcal{G}\varepsilon(\mathbf{u}_2),\varepsilon(\mathbf{v}_1-\mathbf{v}_2))_{\mathcal{H}}+j(\mathbf{u}_1,\mathbf{v}_2,w)-j(\mathbf{u}_1,\mathbf{v}_1,w)+j(\mathbf{u}_2,\mathbf{v}_1,w)\\
&\quad-j(\mathbf{u}_2,\mathbf{v}_2,w)-(\eta_1-\eta_2,,\varepsilon(\mathbf{v}_1)-\varepsilon(\mathbf{v}_2))_{\mathcal{H}}
\end{aligned}$$

We use similar arguments that those used in the proof of the relation (4.12) to obtain that

$$\int_0^t\|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|^2_V ds\le C\int_0^t\|\eta_1(s)-\eta_2(s)\|^2_{\mathcal{H}}ds.\tag{4.23}$$

From (4.14), we get

$$(\dot{\varsigma}_1-\dot{\varsigma}_1,\varsigma_1-\varsigma_2)_{L^2(\Omega)}+b(\varsigma_1-\varsigma_2,\varsigma_1-\varsigma_2)\le(\varsigma_1-\varsigma_2,\varsigma_1-\varsigma_2)_{L^2(\Omega)}\quad \text{a.e. } t\in(0,T).$$

Integrating the previous inequality with respect to time, using the initial conditions $\varsigma_1(0)=\varsigma_1(0)=\varsigma_0$ and the inequality $b(\varsigma_1-\varsigma_2,\varsigma_1-\varsigma_2)\ge 0$, we find

$$\frac{1}{2}\|\varsigma_1(t)-\varsigma_2(t)\|^2_{L^2(\Omega)}\le\int_0^t(\varsigma_1(s)-\varsigma_2(s),\varsigma_1(s)-\varsigma_2(s))_{L^2(\Omega)}ds,$$

which implies that

$$\|\varsigma_1(t)-\varsigma_2(t)\|^2_{L^2(\Omega)}\le\int_0^t\big(\|\gamma_1(s)-\gamma_2(s)\|^2_{L^2(\Omega)}+\|\varsigma_1(s)-\varsigma_2(s)\|^2_{L^2(\Omega)}\big)ds,$$

this inequality combined with Gronwall's inequality leads to

$$\|\varsigma_1(t)-\varsigma_2(t)\|^2_{L^2(\Omega)}\le C\int_0^t\|\gamma_1(s)-\gamma_2(s)\|^2_{L^2(\Omega)}ds,\quad\forall t\in[0,T].\tag{4.24}$$

In order words from (4.16), it follows

$$(\dot{\theta}_1-\dot{\theta}_1,\theta_1-\theta_2)_{E'\times E}+a(\theta_1-\theta_2,\theta_1-\theta_2)=(\varphi_1-\varphi_2,\theta_1-\theta_2)_{E'\times E}\quad\text{a.e. } t\in(0,T).$$

We integrate the previous equality, using (3.13), the initial conditions $\theta_1(0)=\theta_1(0)=\theta_0$ and as $a$ is $E$-elliptic, we get

$$\begin{aligned}
&\|\theta_1(t)-\theta_2(t)\|^2_{L^2(\Omega)}+\mu_3\int_0^t\|\theta_1(t)-\theta_2(t)\|^2_E\\
&\le\int_0^t\|\varphi_1(s)-\varphi_2(s)\|_{E'}\|\theta_1(s)-\theta_2(s)\|_E\quad\text{a.e. } t\in(0,T),
\end{aligned}$$

employing Young's and Hölder's inequalities, we get

$$\begin{aligned} &\|\theta_1(t)-\theta_2(t)\|^2_{L^2(\Omega)}+\int_0^t\|\theta_1(s)-\theta_2(s)\|^2_E ds\\ &\leq C\int_0^t\|\varphi_1(s)-\varphi_2(s)\|^2_{E'}ds \quad \text{a.e. } t\in(0,T). \end{aligned}\tag{4.25}$$

Now, we combine (4.23), (4.24) and (4.25), we find that

$$\|\Lambda(\eta_1,\gamma_1,\varphi_1)(t)-\Lambda(\eta_2,\gamma_2,\varphi_2)\|^2_{\mathcal{H}\times L^2(\Omega)\times E'}\leq C\int_0^T\|(\eta_1,\gamma_1,\varphi_1)(t)-(\eta_2,\gamma_2,\varphi_2)\|^2_{\mathcal{H}\times L^2(\Omega)\times E'}.$$

Reiterating this inequality $n$ times we are led to

$$\begin{aligned} &\|\Lambda(\eta_1,\gamma_1,\varphi_1)(t)-\Lambda(\eta_2,\gamma_2,\varphi_2)(t)\|^2_{\mathcal{H}\times L^2(\Omega)\times E'}\\ &\leq\ C^n\underbrace{\int_0^t\int_0^s\dots\int_0^m}_{n\text{ integrals}}\int_0^T\|(\eta_1,\gamma_1,\varphi_1)(r)-(\eta_2,\gamma_2,\varphi_2)(r)\|^2_{\mathcal{H}\times L^2(\Omega)\times E'}\,dr\dots ds, \end{aligned}$$

which implies that

$$\begin{aligned} &\|\Lambda^n(\eta_1,\gamma_1,\varphi_1)-\Lambda^n(\eta_2,\gamma_2,\varphi_2)\|^2_{L^2(\mathcal{H}\times L^2(\Omega)\times E')}\\ &\leq\frac{C^n\,T^n}{n!}\|(\eta_1,\gamma_1,\varphi_1)-(\eta_2,\gamma_2,\varphi_2)\|^2_{L^2(\mathcal{H}\times L^2(\Omega)\times E')}. \end{aligned}\tag{4.26}$$

Since $\lim\limits_{n\to\infty}\frac{C^n\,T^n}{n!}=0$, it follows that there exists a positive integer $n$ such that $\frac{C^n\,T^n}{n!}<1$ and, therefore, (4.26) shows that the operator $\Lambda^n$ is a contraction on the Banach space $L^2(\mathcal{H}\times L^2(\Omega)\times E^{'})$ and, so, there exists a unique fixed point $(\eta^*,\gamma^*,\varphi^*)\in L^2(0,T;\mathcal{H}\times L^2(\Omega)\times E^{'})$ such that $\Lambda(\eta^*,\gamma^*,\varphi^*)=(\eta^*,\gamma^*,\varphi^*)$. □

We have now all the ingredient to prove Theorem 4.1 which we complete now.

**Existence.** Let $w^*$ be the fixed point of the operator $\chi$ given by (4.8) and $(\eta^*,\gamma^*,\varphi^*)$ be the fixed point of the operator $\Lambda$ given by (4.18)-(4.21) and denote

$$\mathbf{u}_*=\mathbf{u}_{w^*\eta^*},\quad \varsigma_*=\varsigma_{\gamma^*},\quad \theta_*=\theta_{\varphi^*}.\tag{4.27}$$

It follows from (4.19)-(4.21) that

$$\begin{aligned} &\eta^*(t)=\int_0^t\mathcal{B}\big(\varepsilon(\mathbf{u}_*(s)),\varsigma_*(s),\theta_*(s)\big)ds,\\ &\gamma^*(t)=\mathcal{S}(\varepsilon(\mathbf{u}_*(t)),\varsigma_*(t),\theta_*(t)),\\ &\varphi^*(t)=\Psi(\varepsilon(\mathbf{u}_*(t)),\varsigma_*(t),\theta_*(t)), \end{aligned}$$

and, therefore, (4.6), (4.14), (4.16) and (4.8) imply that $(\mathbf{u}_*,\varsigma_*,\theta_*,w^*)$ is a solution of problem $PV$.

**Uniqueness.** The uniqueness of the solution follows from the uniqueness of the fixed point of the operators $\chi$ and $\Lambda$ defined by (4.8) and (4.18)-(4.21) respectively.

**Conflicts of Interest:** The authors declare that there are no conflicts of interest regarding the publication of this paper.

## References


[1] L.E. Andersson, A Quasistatic Frictional Problem With Normal Compliance, Nonlinear Anal.: Theory Methods Appl. 16 (1991), 347–369. https://doi.org/10.1016/0362-546X(91)90035-Y.

[2] K.T. Andrews, M. Shillor, S. Wright, A. Klarbring, A Dynamic Thermoviscoelastic Contact Problem With Friction and Wear, Int. J. Eng. Sci. 35 (1997), 1291–1309. https://doi.org/10.1016/S0020-7225(97)87426-5.

[3] K.T. Andrews, K.L. Kuttler, M. Shillor, On the Dynamic Behaviour of a Thermoviscoelastic Body in Frictional Contact With a Rigid Obstacle, Eur. J. Appl. Math. 8 (1997), 417–436. https://doi.org/10.1017/S0956792597003173.

[4] V. Barbu, Nonlinear Semigroups and Differential Equations in Banach Spaces, Springer, Dordrecht, 1976.

[5] M. Bouallala, E.H. Essoufi, Analysis Results for Dynamic Contact Problem With Friction in Thermo-Viscoelasticity, Methods Funct. Anal. Topol. 26 (2020), 317–326. https://doi.org/10.31392/mfat-npu26_4.2020.03.

[6] I. Boukaroura, S. Djabi, Analysis of a Quasistatic Contact Problem With Wear and Damage for Thermo-Viscoelastic Materials, Malaya J. Mat. 06 (2018), 299–309. https://doi.org/10.26637/mjm0602/0001.

[7] O. Chau, A. Petrov, A. Heibig, M.M. Marques, A Frictional Dynamic Thermal Contact Problem with Normal Compliance and Damage, in: T.M. Rassias, P.M. Pardalos (Eds.), Nonlinear Analysis and Global Optimization, Springer International Publishing, Cham, 2021: pp. 71–107. https://doi.org/10.1007/978-3-030-61732-5_4.

[8] J. Chen, W. Han, M. Sofonea, Numerical Analysis of a Quasistatic Problem of Sliding Frictional Contact With Wear, Methods Appl. Anal. 7 (2000), 687–704. https://doi.org/10.4310/MAA.2000.v7.n4.a5.

[9] C. Ciulcu, T.V. Hoarau-Mante, M. Sofonea, Viscoelastic Sliding Contact Problems With Wear, Math. Computer Model. 36 (2002), 861–874. https://doi.org/10.1016/S0895-7177(02)00233-9.

[10] L. Chouchane, L. Selmani, A History-Dependent Frictional Contact Problem With Wear for Thermoviscoelastic Materials, Math. Model. Anal. 24 (2019), 351–371. https://doi.org/10.3846/mma.2019.022.

[11] A. Djabi, A. Merouani, Bilateral Contact Problem With Friction and Wear for an Electro Elastic-Viscoplastic Materials With Damage, Taiwan. J. Math. 19 (2015), 1161-1182. https://doi.org/10.11650/tjm.19.2015.5453.

[12] J.R. Fernández, M. Sofonea, Variational and Numerical Analysis of the Signorini's Contact Problem in Viscoplasticity With Damage, J. Appl. Math. 2003 (2003), 87–114. https://doi.org/10.1155/S1110757X03202023.

[13] L. Gasiński, A. Ochal, Dynamic Thermoviscoelastic Problem With Friction and Damage, Nonlinear Anal.: Real World Appl. 21 (2015), 63–75. https://doi.org/10.1016/j.nonrwa.2014.06.004.

[14] L. Johansson, A. Klarbring, Thermoelastic Frictional Contact Problems: Modelling, Finite Element Approximation and Numerical Realization, Computer Methods Appl. Mech. Eng. 105 (1993), 181–210. https://doi.org/10.1016/0045-7825(93)90122-E.

[15] O. Kavian, Introduction à la Théorie des Points Critiques et Applications aux Équations Elliptiques, Springer, Berlin, 1993.

[16] A. Klarbring, A. Mikelić, M. Shillor, A Global Existence Result for the Quasistatic Frictional Contact Problem with Normal Compliance, in: G. Del Piero, F. Maceri (Eds.), Unilateral Problems in Structural Analysis IV, Birkhäuser Basel, Basel, 1991: pp. 85–111. https://doi.org/10.1007/978-3-0348-7303-1_8.

[17] J.A.C. Martins, J.T. Oden, Existence and Uniqueness Results for Dynamic Contact Problems With Nonlinear Normal and Friction Interface Laws, Nonlinear Anal.: Theory Methods Appl. 11 (1987), 407–428. https://doi.org/10.1016/0362-546X(87)90055-1.

[18] M.S. Mesai Aoun, M. Selmani, A.A. Ahmed, Variational Analysis of a Frictional Contact Problem With Wear and Damage, Math. Model. Anal. 26 (2021), 170–187. https://doi.org/10.3846/mma.2021.11942.

[19] K. Rimi, T.H. Ammar, Variational Analysis of an Electro-Elasto-Viscoplastic Contact Problem With Friction and Wear, Commun. Math. Appl. 12 (2021), 145–159. https://doi.org/10.26713/cma.v12i1.1461.

[20] M. Selmani, Frictional Contact Problem With Wear for Electro-Viscoelastic Materials With Long Memory, Bull. Belg. Math. Soc. Simon Stevin. 20 (2013), 461-479. https://doi.org/10.36045/bbms/1378314510.

[21] M. Shillor, M. Sofonea, J.J. Telega, Quasistatic Viscoelastic Contact With Friction and Wear Diffusion, Quart. Appl. Math. 62 (2004), 379–399. https://doi.org/10.1090/qam/2054605.

[22] M. Sofonea, W. Han, M. Shillor, Analysis and Approximation of Contact Problems With Adhesion or Damage. Pure and Applied Mathematics 276. Chapman-Hall/CRC Press, New York 2006. http://doi.org/10.1201/9781420034837.

[23] N. Stromberg, Continuum Thermodynamics of Contact, Friction and Wear. Ph.D. Thesis, Linkoping University, Sweeden, 1995.

[24] N. Strömberg, L. Johansson, A. Klarbring, Derivation and Analysis of a Generalized Standard Model for Contact, Friction and Wear, Int. J. Solids Struct. 33 (1996), 1817–1836. https://doi.org/10.1016/0020-7683(95)00140-9.