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L^{∞}-Convergence Analysis of a Finite Element Linear Schwarz Alternating Method for a Class of Semi-Linear Elliptic PDEs

Qais Al Farei, Messaoud Boulbrachene*

Department of Mathematics, Sultan Qaboos University, P.O. Box 36, Muscat 123, Oman

**Corresponding author:* boulbrac@squ.edu.om

Abstract. In this paper, we prove uniform convergence of the standard finite element method for a Schwarz alternating procedure for a class of semi-linear elliptic partial differential equations, in the context of linear iterations and non-matching grids. More precisely, making use of the subsolution-based concept, we prove that finite element Schwarz iterations converge, in the maximum norm, to the true solution of the PDE. We also give numerical results to validate the theory. This work introduces a new approach and generalizes the one in [14] as it encompasses a larger class of problems.

1. INTRODUCTION

The Schwarz alternating method can be used to solve elliptic boundary value problems on domains which consist of two or more overlapping subdomains. The solution is approximated by an infinite sequence of functions which results from solving a sequence of elliptic boundary value problems in each of the subdomains. The literature in this area is huge and one can refer to [2], [3] and to proceedings of the annual International Symposium on Domain Decomposition for Partial Differential Equations, starting from [1].

The mathematical analysis of Schwarz alternating method for nonlinear elliptic boundary value problems has been extensively studied in the last three decades (c.f., e.g., [2], [3], [5], [6] and the references therein).

On the numerical analysis side and, more specifically, non-matching grid discretizations, to the best of our knowledge, only few works are known in the literature regarding the convergence and error estimates analysis for discrete Schwarz procedures (c.f. [7], [8], [9], [10], [12], [15]).

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The main motivation in using non-matching grid discretizations resides in their flexibility as they can be applied to solve many practical problems which cannot be handled by global discretizations because of the complexity of the domain's geometry. They allow the choice of different mesh sizes and different orders of approximate polynomials in different subdomains according to the different properties of the solution and different requirements of the practical problems.

In the present paper, we are interested in a non-matching grid finite element approximation method for the class of PDEs

$$\begin{cases} -\Delta u = f(x,u) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$, $d = 2,3$ is a bounded domain with boundary $\partial\Omega$, Δ is the Laplace operator, $f(.)$ is a smooth nonlinearity, and g is a regular function defined on $\partial\Omega$.

To be more specific, let $\Omega = \Omega_1 \cup \Omega_2$ such that $\Omega_1 \cap \Omega_2 \neq \emptyset$, $\gamma_i = \partial\Omega_i \cap \Omega_j$, $\Gamma_i = \partial\Omega_i \cap \partial\Omega$ and $\partial\Omega_i; i = 1,2$, the boundary of Ω_i. Let also $c(x)$ be a positive smooth function. Then following the work of S.H.Lui [6], given initial smooth guesses u_1^0 and u_2^0, we approximate the solution of problem (1.1) by Schwarz sequences $\left(u_i^n\right)$ such that $u_1^n \in C^2\left(\bar{\Omega}_1\right)$, $n \geq 1$ solves the linear subproblem

$$\begin{cases} -\Delta u_1^n + c u_1^n = f(u_1^{n-1}) + c u_1^{n-1} & \text{in } \Omega_1 \\ u_1^n = u_2^{n-1} & \text{on } \gamma_1 \\ u_1^n = g & \text{on } \Gamma_1 \end{cases} \tag{1.2}$$

on Ω_1,and $u_2^n \in C^2\left(\bar{\Omega}_2\right)$, $n \geq 1$ solves the linear subproblem

$$\begin{cases} -\Delta u_2^n + c u_2^n = f(u_2^{n-1}) + c u_2^{n-1} & \text{in } \Omega_2 \\ u_2^n = u_1^n & \text{on } \gamma_2 \\ u_2^n = g & \text{on } \Gamma_2 \end{cases} \tag{1.3}$$

on Ω_2.

In this paper, motivated by the uniform convergence result [6],

$$\lim_{n\to\infty} \|u_i^n - u\|_{L^\infty(\Omega_i)} = 0,\ i = 1,2,$$

we prove that the corresponding finite elements Schwarz sequences $(u_{1h_1}^n)$ and $(u_{2h_2}^n)$, generated in the context of non-matching grids, converge, in the maximum norm, to the exact solution of problem (1.1). That is,

$$\lim_{n\to\infty} \left\|u - u_{ih_i}^n\right\|_{L^\infty(\Omega_i)} = 0,\ i = 1,2,$$

where, h_i is the mesh-size on Ω_i, and $u_i = u/_{\Omega_i}$.

To that end, by means of the concept of subsolutions, we establish a fundamental lemma which consists of estimating the error, at each iteration, between the continuous and the discrete Schwarz iterations, on each subdomain.

This work introduces a new approach and uses weaker assumptions on the nonlinearity than the one developed in [14] to derive the convergence result.

The layout of the paper is as follows. In section 2, we recall some standard results related to linear elliptic boundary problems. In section 3, we recall the existence of a solution for the nonlinear PDE, and define both the continuous and discrete variational formulations of subproblems (1.2) and (1.3). In section 4, we prove the main results of this paper. Finally, in section 5, we give some numerical results to validate the theory.

2. PRELIMINARIES

We begin by recalling some definitions and classical results related to linear elliptic equations.

2.1. **Linear Elliptic equations.** We introduce the bilinear form

$$a(\xi, v) = \int_{\Omega} (\nabla \xi . \nabla v + c \xi v) dx \quad \forall v \in H^1(\Omega), \tag{2.1}$$

the linear form

$$(f, v) = \int_{\Omega} f(x).v(x) dx \quad \forall v \in H^1(\Omega), \tag{2.2}$$

where the right hand side

$$f \quad \text{is a regular function}, \tag{2.3}$$

and the space

$$\mathbb{V}^{(g)} = \{v \in H^1(\Omega) \text{ such that } v = g \text{ on } \partial\Omega\}, \tag{2.4}$$

where g is a regular function defined on $\partial\Omega$. Note that $\mathring{\mathbb{V}} = H_0^1(\Omega)$.

We consider the linear elliptic equation: Find $\xi \in \mathbb{V}^{(g)}$ such that

$$a(\xi, v) = (f, v), \forall v \in \mathring{\mathbb{V}}(\Omega) \tag{2.5}$$

Lemma 2.1. *[6] (Weak maximum principle) Let $w \in H^1(\Omega) \cap C(\bar{\Omega})$ satisfy $a(w, \phi) \geq 0$ $\forall$ non-negative $\phi \in \mathring{\mathbb{V}}$, and $w \geq 0$ on $\partial\Omega$. Then $w \geq 0$ on $\bar{\Omega}$.*

Definition 2.1. *A function $\check{\xi} \in H^1(\Omega)$ is a subsolution of (2.5) if*

$$\begin{cases} a(\check{\xi}, v) \leq (f, v) \ \forall v \geq 0, v \in \mathring{\mathbb{V}}(\Omega) \\ \check{\xi} \leq g \end{cases} \tag{2.6}$$

Definition 2.2. *A function $\hat{\xi} \in H^1(\Omega)$ is a supersolution of (2.5) if*

$$\begin{cases} a(\hat{\xi}, v) \geq (f, v) \ \forall v \geq 0, v \in \mathring{\mathbb{V}}(\Omega) \\ \hat{\xi} \geq g \end{cases} \tag{2.7}$$

Lemma 2.2. *The solution ξ of (2.5) is the least upper bound of the set of subsolutions.*

Proof. (2.6) can be re-written as

$$a(-\check{\xi}, v) \geq (-f, v) \quad \forall v \geq 0, v \in \mathring{\mathbb{V}}.$$

Subtracting this result from (2.5) yields

$$a(\xi - \check{\xi}, v) \geq 0 \quad \forall v \geq 0, v \in \mathring{\mathbb{V}}.$$

Since $\xi - \check{\xi} \geq 0$ on $\partial\Omega$, it follows from lemma 2.1 that $\check{\xi} \leq \xi$ on $\bar{\Omega}$, which completes the proof. □

The proposition below establishes a continuous Lipschitz property of the solution with respect to the data.

Notation 2.1. *Let (f, g) and $(\tilde{f}, \tilde{g})$ be a pair of data, and $\xi = \partial(f, g)$ and $\tilde{\xi} = \partial(\tilde{f}, \tilde{g})$ be the corresponding solutions to (2.5).*

Proposition 2.1. *[9] Let β be a positive constant such that $c/\beta \geq 1$. Let also lemma 2.1 hold. Then,*

$$\left\|\xi - \tilde{\xi}\right\|_{L^\infty(\Omega)} \leq \max\left\{\frac{1}{\beta}\left\|f - \tilde{f}\right\|_{L^\infty(\Omega)}; \|g - \tilde{g}\|_{L^\infty(\partial\Omega)}\right\} \tag{2.8}$$

Let $\mathbb{V}_h$ be the space of finite elements consisting of continuous piece-wise linear functions, ϕ_s, $s = 1, 2, ..., m(h)$ be the basis functions of $\mathbb{V}_h$. Let also $\mathring{\mathbb{V}}_h$ be the subspace of $\mathbb{V}_h$ defined by

$$\mathring{\mathbb{V}}_h = \{v \in \mathbb{V}_h \text{ such that } v = 0 \text{ on } \partial\Omega\} \tag{2.9}$$

The discrete counterpart of (2.5) consists of finding $\xi_h \in \mathbb{V}_h^{(g)}$ such that

$$a(\xi_h, v) = (f, v) \quad \forall v \in \mathring{\mathbb{V}}_h \tag{2.10}$$

where

$$\mathbb{V}_h^{(g)} = \{v \in \mathbb{V}_h \text{ such that } v = \pi_h g \text{ on } \partial\Omega\ \}, \tag{2.11}$$

and π_h is the linear Lagrange interpolation operator on $\partial\Omega$.

The discrete version of lemma 2.1 stays true provided a discrete maximum principle **(d.m.p)** holds (the matrix resulting from the finite element discretization is an M-matrix). See [16].

Lemma 2.3. *Let $w_h \in \mathbb{V}_h$ satisfy $a(w_h, \phi_s) \geq 0\ \forall s = 1, 2, ..., m(h)$ and $w_h \geq 0$ on $\partial\Omega$. Then, under the d.m.p, we have $w_h \geq 0$ on $\bar{\Omega}$.*

Definition 2.3. *A function $\check{\xi}_h \in \mathbb{V}_h$ is a subsolution of (2.10) if*

$$\begin{cases} a(\check{\xi}_h, \phi_s) \leq (f, \phi_s) & \forall \phi_s \geq 0, \forall s = 1, 2, ..., m(h) \\ \check{\xi}_h \leq \pi_h g \end{cases} \tag{2.12}$$

Definition 2.4. *A function $\hat{\xi}_h \in \mathbb{V}_h$ is a supersolution of (2.10) if*

$$\begin{cases} a(\hat{\xi}_h, \phi_s) \geq (f, \phi_s) \ \forall \phi_s \geq 0, \forall s = 1, 2, ..., m(h) \\ \hat{\xi}_h \geq \pi_h g \end{cases} \tag{2.13}$$

Lemma 2.4. *Let the d.m.p hold. Then the solution ξ_h of (2.10) is the least upper bound of the set of subsolutions.*

Proof. The proof is similar to that of lemma 2.2. Indeed, as $\phi_s \geq 0$ are non-negative, it suffices to make use of lemma 2.3. □

Now we give the finite element counterpart of proposition 2.1.

Notation 2.2. *Let (f, g) and $(\tilde{f}, \tilde{g})$ be a pair of data, with $\xi_h = \partial_h(f, g)$ and $\tilde{\xi}_h = \partial_h(\tilde{f}, \tilde{g})$ be the corresponding discrete solutions to (2.10).*

Proposition 2.2. *[9] Let β be a positive constant such that $c/\beta \geq 1$. Then, under the d.m.p and conditions of lemma 2.3, we have*

$$\left\|\xi_h - \tilde{\xi}_h\right\|_{L^\infty(\Omega)} \leq \max\left\{\frac{1}{\beta}\left\|f - \tilde{f}\right\|_{L^\infty(\Omega)}; \|g - \tilde{g}\|_{L^\infty(\partial\Omega)}\right\} \tag{2.14}$$

Finally, we recall a standard maximum norm error estimate [18].

Theorem 2.1. *[18] Under suitable regularity of the solution of problem (2.5), there exists a constant C independent of h such that*

$$\|\xi - \xi_h\|_{L^\infty(\Omega)} \leq Ch^2 |\ln h|$$

3. SCHWARZ METHOD FOR NONLINEAR PDEs

We first recall the following classical existence result due to Pao [4].

3.1. The Nonlinear PDE.

We shall consider the following nonlinear PDE: Find $u \in C^2(\Omega)$ such that

$$\begin{cases} -\Delta u = f(x, u) & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

For the sake of convenience, we will suppress the dependence of the space variable x.

Definition 3.1. *[4] A function $\breve{u} \in C^2(\Omega)$ is a subsolution of (3.1) if*

$$\begin{cases} -\Delta \breve{u} \leq f(\breve{u}) \ \text{in } \Omega \\ \breve{u} \leq g \quad \text{on } \partial\Omega \end{cases} \tag{3.2}$$

Definition 3.2. *[4] A function $\hat{u} \in C^2(\Omega)$ is a supersolution of (3.1) if*

$$\begin{cases} -\Delta \hat{u} \geq f(\hat{u}) \text{ in } \Omega \\ \hat{u} \geq g \qquad \text{on } \partial\Omega \end{cases} \tag{3.3}$$

Suppose that (3.1) has a subsolution $\breve{u}$ and a supersolution $\hat{u}$ such that $\breve{u} \leq \hat{u}$ on Ω. Define the sector

$$\mathcal{A} = \{u \in C^2(\bar{\Omega});\ \breve{u} \leq u \leq \hat{u} \text{ on } \bar{\Omega}\}. \tag{3.4}$$

Assume that

$$-c\,(u - v) \leq f(u) - f(v)\ \forall v \leq u \in \mathcal{A} \tag{3.5}$$

Then, thanks to Pao [4], (3.1) has a solution (not necessarily unique) in $\mathcal{A}$.

Theorem 3.1. *[6] Let $u_2^0 = \breve{u}$ on $\bar{\Omega}$; $i = 1, 2$, with $\breve{u} = 0$ on $\partial\Omega$. Define the linear Schwarz sequences generated by the subproblems (1.2) and (1.3).Then $u_i^n \to u$ in $C^2(\bar{\Omega}_i)$, where u is a solution of (3.1) in $\mathcal{A}$. Similarly, if $u_2^0 = \hat{u}$ on $\bar{\Omega}$ with $\hat{u} = 0$ on $\partial\Omega$ instead, then the same conclusion holds.*

3.2. **Continuous variational Schwarz subproblems.** The weak form of (1.2) and (1.3) read as follows: find $u_1^n \in H^1(\Omega)$ such that:

$$\begin{cases} a_1(u_1^n, v) = (F(u_1^{n-1}), v) & \forall v \in \mathring{\mathbb{V}}_1 \\ u_1^n = u_2^{n-1} & \text{on } \gamma_1 \\ u_1^n = g & \text{on } \Gamma_1, \end{cases} \tag{3.6}$$

and $u_2^n \in H^1(\Omega)$ such that

$$\begin{cases} a_2(u_2^n, v) = (F(u_2^{n-1}), v) & \forall v \in \mathring{\mathbb{V}}_2 \\ u_2^n = u_1^n & \text{on } \gamma_2 \\ u_2^n = g & \text{on } \Gamma_2 \end{cases} \tag{3.7}$$

respectively, where

$$a_i(u_i, v) = \int_{\Omega_i} (\nabla u_i \nabla v + c u_i v) dx, \tag{3.8}$$

and

$$(F(u_i), v) = \int_{\Omega_i} (f(u_i) + c u_i) v dx\ ;\ i = 1, 2. \tag{3.9}$$

3.3. **Finite element discretization.** Let $\mathcal{T}^{h_i}; i = 1, 2$ be a standard quasi-uniform regular finite element triangulation on Ω_i; h_i being its mesh size. We introduce the finite element spaces $\mathbb{V}_{h_i}$ and $\mathring{\mathbb{V}}_{h_i}$ as follows:

$$\mathbb{V}_{h_i} = \{v \in C^0(\bar{\Omega}_i) : v/_K \in P_1 \ \forall K \in \mathcal{T}^{h_i}\}, \tag{3.10}$$

and

$$\mathring{\mathbb{V}}_{h_i} = \{v \in \mathbb{V}_{h_i} : v = 0 \text{ on } \Gamma_i\}, \tag{3.11}$$

where P_1 denotes the space of linear polynomials on $K \in \mathcal{T}^{h_i}$, with degree ≤ 1. The two meshes are also assumed to be overlaping and non-matching in the sense that they are mutually independent on the overlap region.

The discrete maximum principle (d.m.p). We assume that the meshing on each subdomain satisfies the discrete maximum principle. In other words, the matrices resulting from the discretization of (3.6) and (3.7) are M-matrices.

3.4. **Discrete variational Schwarz subproblems.** Let $u^0_{ih_i}$ be the discrete analog of u^0_i, i.e.; $u^0_{ih_i} = r_{h_i}(u^0_i)$, where r_{h_i} denotes the finite element Lagrange interpolation operator on Ω_i. Now, we define the discrete Schwarz sequences $(u^n_{1h_1})$ such that $u^n_{1h_1} \in \mathbb{V}_{h_1}$ solves

$$\begin{cases} a_1(u^n_{1h_1}, v) = (F(u^{n-1}_{1h_1}), v) & \forall v \in \mathring{\mathbb{V}}_{h_1} \\ u^n_{1h_1} = \pi_{h_1}(u^{n-1}_{2h_2}) & \text{on } \gamma_1 \\ u^n_{1h_1} = \pi_h g & \text{on } \Gamma_1, \end{cases} \tag{3.12}$$

and $\left(u^n_{2h_2}\right)$ such that $u^n_{2h_2} \in \mathbb{V}_{h_2}$ solves

$$\begin{cases} a_2(u^n_{2h_2}, v) = (F(u^{n-1}_{2h_2}), v) & \forall v \in \mathring{\mathbb{V}}_{h_2} \\ u^n_{2h_2} = \pi_{h_2}(u^n_{1h_1}) & \text{on } \gamma_2 \\ u^n_{2h_2} = \pi_h g & \text{on } \Gamma_2, \end{cases} \tag{3.13}$$

where π_{h_i} denotes the Lagrange interpolation operator on γ_i. Below, we construct a finite element discretization of subproblems (3.12) and (3.13), as in Figure 1, using a quasi-uniform regular finite element triangulation on both subdomains as stated before.

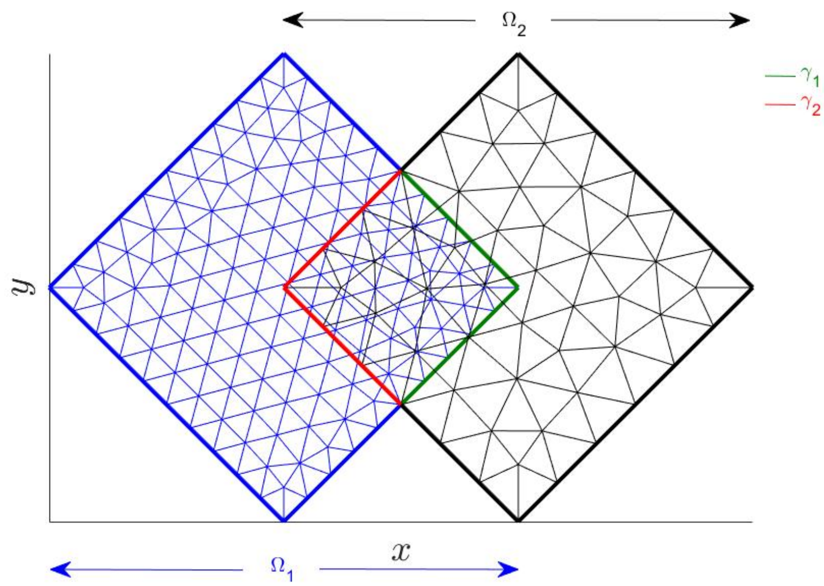


Figure 1. A sample of two overlapping nonmatching grids.

4. L^∞- CONVERGENCE ANALYSIS

This section is devoted to the proof of the main results of the present paper. We first introduce two continuous and two discrete auxiliary Schwarz sequences and prove a fundamental lemma.

4.1. **Continuous auxiliary Schwarz subproblems.** For $\tilde{u}_i^0 = u_i^0; i = 1, 2$, we define the continuous auxiliary Schwarz sequence $(\tilde{u}_1^n)$ such that $\tilde{u}_1^n \in \mathbb{V}_1$ solves

$$\begin{cases} a_1(\tilde{u}_1^n, v) = (F(u_{1h_1}^{n-1}), v) & \forall v \in \mathring{\mathbb{V}}_1 \\ \tilde{u}_1^n = \pi_{h_1}(u_{2h_2}^{n-1}) & \text{on } \gamma_1 \\ \tilde{u}_1^n = \pi_h g & \text{on } \Gamma_1 \end{cases} \tag{4.1}$$

and $\left(\tilde{u}_2^n\right)$ such that $\tilde{u}_2^n \in \mathbb{V}_2$ solves

$$\begin{cases} a_2(\tilde{u}_2^n, v) = (F(u_{2h_2}^{n-1}), v) & \forall v \in \mathring{\mathbb{V}}_2 \\ \tilde{u}_2^n = \pi_{h_2}(u_{1h_1}^n) & \text{on } \gamma_2 \\ \tilde{u}_2^n = \pi_h g & \text{on } \Gamma_2 \end{cases} \tag{4.2}$$

where $u_{1h_1}^n$ and $u_{2h_2}^n$ are the Schwarz iterates defined in (3.12) and (3.13), respectively.

4.2. **Discrete Auxiliary Schwarz subproblems.** Likewise, for $\tilde{u}^0_{ih_i} = u^0_{ih_i}; i = 1, 2$, we define the discrete auxiliary Schwarz sequences $(\tilde{u}^n_{1h_1})$ such that $\tilde{u}^n_{1h_1} \in \mathbb{V}_{h_1}$ solves

$$\begin{cases} a_1(\tilde{u}^n_{1h_1}, v) = (F(u^{n-1}_1), v) & \forall v \in \mathring{\mathbb{V}}_{h_1} \\ \tilde{u}^n_{1h_1} = \pi_{h_1}(u^{n-1}_2) & \text{on } \gamma_1 \\ \tilde{u}^n_{1h_1} = \pi_h g & \text{on } \Gamma_1 \end{cases} \tag{4.3}$$

and $\left(\tilde{u}^n_{2h_2}\right)$ such that $\tilde{u}^n_{2h_2} \in \mathbb{V}_{h_2}$ solves

$$\begin{cases} a_2(\tilde{u}^n_{2h_2}, v) = (F(u^{n-1}_2), v) & \forall v \in \mathring{\mathbb{V}}_{h_2} \\ \tilde{u}^n_{2h_2} = \pi_{h_2}(u^n_1) & \text{on } \gamma_2 \\ \tilde{u}^n_{2h_2} = \pi_h g & \text{on } \Gamma_2 \end{cases} \tag{4.4}$$

where u^n_1 and u^n_2 are the Schwarz iterates defined in (3.6) and (3.7), respectively.

Notation 4.1. *From now onward, we shall adopt the following notations:*

$$C \text{ is ageneric constant independent of } h \text{ and } n,$$

$$\|.\|_1 = \|.\|_{L^\infty(\Omega_1)} \;\; ; \; |.|_1 = \|.\|_{L^\infty(\gamma_1)} \, ,$$

$$\|.\|_2 = \|.\|_{L^\infty(\Omega_2)} \;\; ; \; |.|_2 = \|.\|_{L^\infty(\gamma_2)} \, ,$$

$$\pi_{h_1} = \pi_{h_2} = \pi_h,$$

and

$$h = \max_{i=1,2} h_i.$$

Lemma 4.1. *Assume that*

$$\max\left\{\|\tilde{u}^n_i\|_{W^{2,p}(\Omega_i)} \, , \; \|u^n_i\|_{W^{2,p}(\Omega_i)}\right\} \leq C.$$

Then, we have

$$\left\|\tilde{u}^n_i - u^n_{ih_i}\right\|_{L^\infty(\Omega_i)} \leq Ch^2 \left|\ln h\right| , \tag{4.5}$$

$$\left\|u^n_i - \tilde{u}^n_{ih_i}\right\|_{L^\infty(\Omega_i)} \leq Ch^2 \left|\ln h\right| . \tag{4.6}$$

where C is a constant independent of both $h_i; i = 1, 2$ and n.

Proof. It is clear that $u^n_{ih_i}$ and $\tilde{u}^n_{ih_i}$ are the discrete counterparts of $\tilde{u}^n_i$ and u^n_i, respectively. So, as the latter are both uniformly bounded in $W^{2,p}(\Omega_i)$, the desired error estimates follows from Theorem 2.1. □

4.3. **The main results.** The following lemma plays a crucial role in deriving the main result of this paper.

Lemma 4.2. *Assume that $f(.)$ is a Lipschitz continuous function, i.e., there is a constant $k > 0$ such that*

$$|f(x) - f(y)| \leq k\,|x - y| \quad \forall x, y \in \mathbb{R}. \tag{4.7}$$

Then,

$$\|u_1^n - u_{1h}^n\|_1 \leq (2n)Ch^2\,|\ln h| \tag{4.8}$$

and

$$\|u_2^n - u_{2h}^n\|_2 \leq (2n+1)Ch^2\,|\ln h|\,. \tag{4.9}$$

Remark 4.1. *Note that the assumption $k/\beta \leq 1$ used in [14] is no longer needed in this paper.*

Proof. The proof will be carried out by induction. Also, for the sake of simplicity, we shall ignore the boundary condition on $\Gamma_i; i = 1, 2$.

Indeed, on Ω_1, problem (4.1) for $n = 1$ reads as follows

$$\begin{cases} a_1(\tilde{u}_1^1, v) = (F(u_{1h}^0), v) & \forall v \in \mathring{\mathbb{V}}_1 \\ \tilde{u}_1^1 = \pi_h(u_{2h}^0) & \text{on } \gamma_1. \end{cases} \tag{4.10}$$

As $\tilde{u}_1^1$ is also a subsolution for (4.10), we have

$$\begin{cases} a_1(\tilde{u}_1^1, v) \leq (F(u_{1h}^0), v) & \forall v \in \mathring{\mathbb{V}}_1, v \geq 0 \\ \tilde{u}_1^1 \leq \pi_h(u_{2h}^0) & \text{on } \gamma_1. \end{cases}$$

But

$$\begin{cases} a_1(\tilde{u}_1^1, v) \leq (F(u_{1h}^0) - F(u_1^0) + F(u_1^0), v) & \forall v \in \mathring{\mathbb{V}}_1, v \geq 0 \\ \tilde{u}_1^1 \leq \pi_h(u_{2h}^0) - \pi_h(u_2^0) + \pi_h(u_2^0) & \text{on } \gamma_1, \end{cases}$$

then, since $F(.)$ is Lipschitz continuous and $\gamma_1 \subset \Omega_2$, this implies

$$\begin{cases} a_1(\tilde{u}_1^1, v) \leq (C\left\|u_1^0 - u_{1h}^0\right\|_1 + F(u_1^0), v) & \forall v \in \mathring{\mathbb{V}}_1, v \geq 0 \\ \tilde{u}_1^1 \leq \left\|u_2^0 - u_{2h}^0\right\|_2 + \pi_h(u_2^0) & \text{on } \gamma_1. \end{cases}$$

Then, making use of standard uniform estimate, we have

$$\left\|u_i^0 - r_h(u_i^0)\right\|_i \leq Ch^2\,|\ln h|\,; i = 1, 2. \tag{4.11}$$

Hence

$$\begin{cases} a_1(\tilde{u}_1^1, v) \leq (F(u_1^0) + Ch^2\,|\ln h|\,, v) & \forall v \in \mathring{\mathbb{V}}_1, v \geq 0 \\ \\ \tilde{u}_1^1 \leq \pi_h(u_2^0) + Ch^2\,|\ln h| & \text{on } \gamma_1 \end{cases}$$

Let $\tilde{U}_1^1$ be the solution of the problem with source term $F(u_1^0)+Ch^2|\ln h|$ and boundary data $\pi_h(u_2^0)+Ch^2|\ln h|$. That is,

$$\tilde{U}_1^1 = \partial(F(u_1^0) + Ch^2|\ln h| \ ,\ \pi_h(u_2^0) + Ch^2|\ln h|)$$

Then, as

$$u_1^1 = \partial(F(u_1^0)\ ,\ u_2^0),$$

making use of proposition 2.1, yields

$$\begin{aligned}\left\|\tilde{U}_1^1 - u_1^1\right\|_1 &\leq \max\left\{Ch^2|\ln h|\,;Ch^2|\ln h|\right\}\\ &\leq Ch^2|\ln h|\,.\end{aligned}$$

Hence, due to lemma 2.2, we have

$$\tilde{u}_1^1 \leq \tilde{U}_1^1 \leq u_1^1 + Ch^2|\ln h|\,.$$

Putting

$$\alpha_1^1 = \tilde{u}_1^1 - Ch^2|\ln h|\,,$$

we get

$$\alpha_1^1 \leq u_1^1. \tag{4.12}$$

and using (4.5), for $n=1$, we also get

$$\left\|\tilde{u}_1^1 - u_{1h}^1\right\|_1 \leq Ch^2|\ln h|\,.$$

Thus,

$$\begin{aligned}\left\|\alpha_1^1 - u_{1h}^1\right\|_1 &= \left\|\tilde{u}_1^1 - Ch^2|\ln h| - u_{1h}^1\right\|_1 \\ &\leq Ch^2|\ln h| + Ch^2|\ln h|\\ &\leq 2Ch^2|\ln h|\,.\end{aligned} \tag{4.13}$$

Now, consider problem (4.3) for $n=1$:

$$\begin{cases} a_1(\tilde{u}_{1h}^1, v) = (F(u_1^0), v) & \forall v \in \mathring{\mathbb{V}}_{1h} \\ \tilde{u}_{1h}^1 = \pi_h(u_2^0) & \text{on } \gamma_1. \end{cases} \tag{4.14}$$

As $\tilde{u}_{1h}^1$ is also a subsolution for (4.14), we have

$$\begin{cases} a_1(\tilde{u}_{1h}^1, \phi_s) \leq (F(u_1^0), \phi_s) & \forall \phi_s \geq 0, \forall s \\ \tilde{u}_{1h}^1 \leq \pi_h(u_2^0) & \text{on } \gamma_1, \end{cases}$$

which implies

$$\begin{cases} a_1(\tilde{u}_{1h}^1, \phi_s) \leq (F(u_1^0) - F(u_{1h}^0) + F(u_{1h}^0), \phi_s) & \forall \phi_s \geq 0, \forall s \\ \tilde{u}_{1h}^1 \leq \pi_h(u_2^0) - \pi_h(u_{2h}^0) + \pi_h(u_{2h}^0) & \text{on } \gamma_1. \end{cases}$$

Since $F(.)$ and π_h are Lipschitz, we get

$$\begin{cases} a_1(\tilde{u}_{1h}^1, \phi_s) \leq (C \left\| u_1^0 - u_{1h}^0 \right\|_1 + F(u_{1h}^0), \phi_s) & \forall \phi_s \geq 0, \forall s \\ \tilde{u}_{1h}^1 \leq \left\| \ u_2^0 - u_{2h}^0 \right\|_2 + \pi_h(u_{2h}^0) & \text{on } \gamma_1. \end{cases}$$

Hence, using (4.11), yields

$$\begin{cases} a_1(\tilde{u}_{1h}^1, \phi_s) \leq (F(u_{1h}^0) + Ch^2 \left| \ln h \right|, \phi_s) & \forall \phi_s \geq 0, \forall s \\ \tilde{u}_{1h}^1 \leq \pi_h(u_{2h}^0) + Ch^2 \left| \ln h \right| & \text{on } \gamma_1. \end{cases}$$

Let $\tilde{U}_{1h}^1$ be the solution of the problem with source term $F(u_{1h}^0) + Ch^2 \left| \ln h \right|$ and boundary data $\pi_h(u_{2h}^0) + Ch^2 \left| \ln h \right|$, that is,

$$\tilde{U}_{1h}^1 = \partial_h(F(u_{1h}^0) + Ch^2 \left| \ln h \right| \ , \ \pi_h(u_{2h}^0) + Ch^2 \left| \ln h \right|)$$

Then, as

$$u_{1h}^1 = \partial_h(F(u_{1h}^0) \ , \ \pi_h(u_{2h}^0)),$$

making use of proposition 2.2, yields

$$\begin{aligned} \left\| \tilde{U}_{1h}^1 - u_{1h}^1 \right\|_1 &\leq \max \left\{ Ch^2 \left| \ln h \right| ; Ch^2 \left| \ln h \right| \right\} \\ &\leq Ch^2 \left| \ln h \right|, \end{aligned}$$

and due to lemma 2.4, we have

$$\tilde{u}_{1h}^1 \leq \tilde{U}_{1h}^1 \leq u_{1h}^1 + Ch^2 \left| \ln h \right|.$$

Now, putting

$$\alpha_{1h}^1 = \tilde{u}_{1h}^1 - Ch^2 \left| \ln h \right|,$$

it follows that

$$\alpha_{1h}^1 \leq u_{1h}^1. \tag{4.15}$$

And making use of (4.6) for $n = 1$, we get

$$\left\| u_1^1 - \tilde{u}_{1h}^1 \right\|_1 \leq Ch^2 \left| \ln h \right|.$$

Thus,

$$\begin{aligned} \left\| \alpha_{1h}^1 - u_1^1 \right\|_1 &= \left\| \tilde{u}_{1h}^1 - Ch^2 \left| \ln h \right| - u_1^1 \right\|_1. \\ &\leq Ch^2 \left| \ln h \right| + Ch^2 \left| \ln h \right| \\ &\leq 2Ch^2 \left| \ln h \right| \end{aligned} \tag{4.16}$$

Now, combining (4.12), (4.13), (4.15) and (4.16), we get

$$\begin{aligned} u_1^1 &\le \alpha_{1h}^1 + 2Ch^2 \left|\ln h\right| \\ &\le u_{1h}^1 + 2Ch^2 \left|\ln h\right| \\ &\le \alpha_1^1 + 4Ch^2 \left|\ln h\right| \\ &\le u_1^1 + 4Ch^2 \left|\ln h\right| . \end{aligned}$$

That is,

$$\left\| u_1^1 - u_{1h}^1 \right\|_1 \le 2Ch^2 \left|\ln h\right| . \tag{4.17}$$

Similarly on Ω_2, for $n = 1$ in (4.2), we have

$$\begin{cases} a_2(\tilde{u}_2^1, v) = (F(u_{2h}^0), v) & \forall v \in \mathring{\mathbb{V}}_2 \\ \tilde{u}_2^1 = \pi_h(u_{1h}^1) & \text{on } \gamma_2. \end{cases} \tag{4.18}$$

The solution $\tilde{u}_2^1$ is also a subsolution for (4.18). That is,

$$\begin{cases} a_2(\tilde{u}_2^1, v) \le (F(u_{2h}^0), v) & \forall v \in \mathring{\mathbb{V}}_2, v \ge 0 \\ \tilde{u}_2^1 \le \pi_h(u_{1h}^1) & \text{on } \gamma_2, \end{cases}$$

or

$$\begin{cases} a_2(\tilde{u}_2^1, v) \le (F(u_{2h}^0, v) - F(u_2^0, v) + F(u_2^0), v) \; \forall v \in \mathring{\mathbb{V}}_2, v \ge 0 \\ \tilde{u}_2^1 \le \pi_h(u_{1h}^1) - \pi_h(u_1^1) + \pi_h(u_1^1) & \text{on } \gamma_2. \end{cases}$$

As $F(.)$ is Lipschitz continuous function and $\gamma_2 \subset \Omega_1$, this implies

$$\begin{cases} a_2(\tilde{u}_2^1, v) \le (\left\| u_2^0 - u_{2h}^0 \right\|_2 + F(u_2^0), v) & \forall v \in \mathring{\mathbb{V}}_2, v \ge 0 \\ \tilde{u}_2^1 \le \left\| u_1^1 - u_{1h}^1 \right\|_1 + \pi_h(u_1^1) & \text{on } \gamma_2 \end{cases}$$

Using (4.11) and the resulting estimate (4.17), we obtain

$$\begin{cases} a_2(\tilde{u}_2^1, v) \le (F(u_2^0) + Ch^2 \left|\ln h\right|, v) & \forall v \in \mathring{\mathbb{V}}_2, v \ge 0 \\ \tilde{u}_2^1 \le \pi_h(u_1^1) + 2Ch^2 \left|\ln h\right| & \text{on } \gamma_2. \end{cases}$$

Let $\tilde{U}_2^1$ be the solution of the equation with source term $F(u_2^0) + Ch^2 \left|\ln h\right|$ and boundary data $\pi_h(u_1^1) + 2Ch^2 \left|\ln h\right|$, that is,

$$\tilde{U}_2^1 = \partial(F(u_2^0) + Ch^2 \left|\ln h\right| \;,\; \pi_h(u_1^1) + 2Ch^2 \left|\ln h\right|).$$

Then, as

$$u_2^1 = \partial(F(u_2^0) \;,\; u_1^1),$$

making use of proposition 2.1, we get

$$\begin{aligned} \left\| \tilde{U}_2^1 - u_2^1 \right\|_2 &\le \max\left\{ Ch^2 \left|\ln h\right| ; 2Ch^2 \left|\ln h\right| \right\} \\ &\le 2Ch^2 \left|\ln h\right| . \end{aligned}$$

Also, due to lemma 2.2, we have

$$\tilde{u}_2^1 \le \tilde{U}_2^1 \le u_2^1 + 2Ch^2 \left|\ln h\right|$$

Now, putting

$$\alpha_2^1 = \tilde{u}_2^1 - 2Ch^2 \left|\ln h\right|,$$

yields

$$\alpha_2^1 \le u_2^1. \tag{4.19}$$

And due to (4.5) for $n = 1$, we have

$$\left\|\tilde{u}_2^1 - u_{2h}^1\right\|_2 \le Ch^2 \left|\ln h\right|.$$

Thus, it follows that

$$\begin{aligned} \left\|\alpha_2^1 - u_{2h}^1\right\|_2 &= \left\|\tilde{u}_2^1 - 2Ch^2 \left|\ln h\right| - u_{2h}^1\right\|_2 \\ &\le \left\|\tilde{u}_2^1 - u_{2h}^1\right\|_1 + 2Ch^2 \left|\ln h\right| \\ &\le 3Ch^2 \left|\ln h\right|. \end{aligned} \tag{4.20}$$

Again, on Ω_2, for $n = 1$ in (4.4), we have

$$\begin{cases} a_2(\tilde{u}_{2h}^1, v) = (F(u_2^0), v) & \forall v \in \mathring{\mathbb{V}}_{2h} \\ \tilde{u}_{2h}^1 = \pi_h(u_1^1) & \text{on } \gamma_2. \end{cases} \tag{4.21}$$

The solution $\tilde{u}_{2h}^1$ being also a subsolution, we have

$$\begin{cases} a_1(\tilde{u}_{2h}^1, \phi_s) \le (F(u_2^0), \phi_s) & \forall \phi_s \ge 0, \forall s \\ \tilde{u}_{2h}^1 \le \pi_h(u_1^1) & \text{on } \gamma_2. \end{cases}$$

Then, as $F(.)$ and π_h are Lipschitz, we get

$$\begin{cases} a_1(\tilde{u}_{2h}^1, \phi_s) \le (C\left\|u_2^0 - u_{2h}^0\right\|_2 + F(u_{2h}^0), \phi_s) & \forall \phi_s \ge 0, \forall s \\ \tilde{u}_{2h}^1 \le \left\|u_1^1 - u_{1h}^1\right\|_1 + \pi_h(u_{1h}^1) & \text{on } \gamma_2, \end{cases}$$

or

$$\begin{cases} a_1(\tilde{u}_{2h}^1, \phi_s) \le (F(u_{2h}^0) + Ch^2 \left|\ln h\right|, \phi_s) & \forall \phi_s \ge 0 \\ \tilde{u}_{2h}^1 \le \pi_h(u_{1h}^1) + 2Ch^2 \left|\ln h\right| & \text{on } \gamma_2. \end{cases}$$

Hence, $\tilde{u}_{2h}^1$ is a subsolution for the problem with source term $F(u_{2h}^0) + Ch^2 \left|\ln h\right|$ and boundary term $\pi_h(u_{1h}^1) + 2Ch^2 \left|\ln h\right|$. Let $\tilde{U}_{2h}^1$ be the solution of such a problem, that is,

$$\tilde{U}_{2h}^1 = \partial_h(F(u_{2h}^0) + Ch^2 \left|\ln h\right| \ , \ \pi_h(u_{1h}^1) + 2Ch^2 \left|\ln h\right|)$$

Then, we have

$$\tilde{u}_{2h}^1 \le \tilde{U}_{2h}^1.$$

As

$$u_{2h}^1 = \partial_h(F(u_{2h}^0)\ ,\ \pi_h(u_{1h}^1)),$$

making use of proposition 2.2, we get

$$\begin{aligned}\left\|\tilde{U}_{2h}^1 - u_{2h}^1\right\|_2 &\leq \max\left\{Ch^2\left|\ln h\right| ; 2Ch^2\left|\ln h\right|\right\}\\ &\leq 2Ch^2\left|\ln h\right|.\end{aligned}$$

So, due to lemma 2.4, we have

$$\tilde{u}_{2h}^1 \leq \tilde{U}_{2h}^1 \leq u_{2h}^1 + 2Ch^2\left|\ln h\right|$$

Now, putting

$$\alpha_{2h}^1 = \tilde{u}_{2h}^1 - 2Ch^2\left|\ln h\right|,$$

yields

$$\alpha_{2h}^1 \leq u_{2h}^1. \tag{4.22}$$

And making use of (4.6) for $n = 1$, we get

$$\left\|\alpha_{2h}^1 - u_2^1\right\|_2 = \left\|\tilde{u}_{2h}^1 - 2Ch^2\left|\ln h\right| - u_2^1\right\|_2 \tag{4.23}$$

$$\tag{4.24}$$

$$\leq 3Ch^2\left|\ln h\right|.$$

Now, combining statements (4.19), (4.20), (4.22) and (4.23), we obtain

$$\begin{aligned}u_2^1 &\leq \alpha_{2h}^1 + 3Ch^2\left|\ln h\right|\\ &\leq u_{2h}^1 + 3Ch^2\left|\ln h\right|\\ &\leq \alpha_2^1 + 6Ch^2\left|\ln h\right|\\ &\leq u_2^1 + 6Ch^2\left|\ln h\right|.\end{aligned}$$

That is,

$$\left\|u_2^1 - u_{2h}^1\right\|_2 \leq 3Ch^2\left|\ln h\right|. \tag{4.25}$$

Now, for $n = 2$ on Ω_1, (4.1) reads

$$\begin{cases} a_1(\tilde{u}_1^2, v) = (F(u_{1h}^1), v) & \forall v \in \mathring{\mathbb{V}}_1\\ \tilde{u}_1^2 = \pi_h(u_{2h}^1) & \text{on } \gamma_1.\end{cases} \tag{4.26}$$

As $\tilde{u}_1^2$ is also a subsolution, we have

$$\begin{cases} a_1(\tilde{u}_1^2, v) \leq (F(u_{1h}^1), v) & \forall v \in \mathring{\mathbb{V}}_1, v \geq 0\\ \tilde{u}_1^2 \leq \pi_h(u_{2h}^1) & \text{on } \gamma_1.\end{cases}$$

And, since $F(.)$ is Lipschitz, we get

$$\begin{cases} a_1(\tilde{u}_1^2, v) \leq (F(u_1^1) + 2Ch^2 \left|\ln h\right|, v) & \forall v \in \mathring{\mathbb{V}}_1 \\ \tilde{u}_1^2 \leq \pi_h(u_2^1) + 3Ch^2 \left|\ln h\right| & \text{on } \gamma_1. \end{cases}$$

So, $\tilde{u}_1^2$ is a subsolution for the problem with source term $F(u_1^1) + 2Ch^2 \left|\ln h\right|$ and boundary term $\pi_h(u_2^1) + 3Ch^2 \left|\ln h\right|$. Let $\tilde{U}_1^2$ be the solution of such a problem. That is,

$$\tilde{U}_1^2 = \partial(F(u_1^1) + 2Ch^2 \left|\ln h\right| \ , \ \pi_h(u_2^1) + 3Ch^2 \left|\ln h\right|).$$

Then, due to lemma 2.2, we have

$$\tilde{u}_1^2 \leq \tilde{U}_1^2.$$

Furthermore, as

$$u_1^2 = \partial(F(u_1^1) \ , \ u_2^1),$$

making use of proposition 2.1, we get

$$\begin{aligned} \left\|\tilde{U}_1^2 - u_1^2\right\|_1 &\leq \max\left\{2Ch^2 \left|\ln h\right| ; 3Ch^2 \left|\ln h\right|\right\} \\ &\leq 3Ch^2 \left|\ln h\right| . \end{aligned}$$

Hence,

$$\tilde{u}_1^2 \leq \tilde{U}_1^2 \leq u_1^2 + 3Ch^2 \left|\ln h\right| .$$

Putting

$$\alpha_1^2 = \tilde{u}_1^2 - 3Ch^2 \left|\ln h\right| ,$$

yields

$$\alpha_1^2 \leq u_1^2. \tag{4.27}$$

Making use of (4.5) for $n = 2$, we get

$$\begin{aligned} \left\|\alpha_1^2 - u_{1h}^2\right\|_1 &= \left\|\tilde{u}_1^2 - 3Ch^2 \left|\ln h\right| - u_{1h}^2\right\|_1 \\ &\leq Ch^2 \left|\ln h\right| + 3Ch^2 \left|\ln h\right| \\ &\leq 4Ch^2 \left|\ln h\right| . \end{aligned} \tag{4.28}$$

Now for $n = 2$ on Ω_1, (4.3) reads

$$\begin{cases} a_1(\tilde{u}_{1h}^2, v) = (F(u_1^1), v) & \forall v \in \mathring{\mathbb{V}}_{1h} \\ \tilde{u}_{1h}^2 = \pi_h(u_2^1) & \text{on } \gamma_1. \end{cases} \tag{4.29}$$

As $\tilde{u}_{1h}^2$ is also a subsolution, we have

$$\begin{cases} a_1(\tilde{u}_{1h}^2, \phi_s) \leq (F(u_1^1), \phi_s) & \forall \phi_s \geq 0, \forall s \\ \tilde{u}_{1h}^2 \leq \pi_h(u_2^1) & \text{on } \gamma_1. \end{cases}$$

Similarly, as above, this implies

$$\begin{cases} a_1(\tilde{u}_{1h}^2, \phi_s) \leq (F(u_{1h}^1) + 2Ch^2 \left|\ln h\right|, \phi_s) & \forall \phi_s \geq 0, \forall s \\ \tilde{u}_{1h}^2 \leq \pi_h(u_{2h}^1) + 3Ch^2 \left|\ln h\right| & \text{on } \gamma_1. \end{cases}$$

And, due to lemma 2.4,

$$\tilde{u}_{1h}^2 \leq \tilde{U}_{1h}^2 = \partial_h(F(u_{1h}^1) + 2Ch^2 \left|\ln h\right| \ , \ \pi_h(u_{2h}^1) + 3Ch^2 \left|\ln h\right|)$$

But

$$u_{1h}^2 = \partial_h(F(u_{1h}^1) \ , \ \pi_h(u_{2h}^1))$$

Then, using proposition 2.2, yields the estimate

$$\begin{aligned} \left\|\tilde{U}_{1h}^2 - u_{1h}^2\right\|_1 &\leq \max\left\{2Ch^2 \left|\ln h\right| ; 3Ch^2 \left|\ln h\right|\right\} \\ &\leq 3Ch^2 \left|\ln h\right| . \end{aligned}$$

Hence

$$\tilde{u}_{1h}^2 \leq \tilde{U}_{1h}^2 \leq u_{1h}^2 + 3Ch^2 \left|\ln h\right| .$$

Putting

$$\alpha_{1h}^2 = \tilde{u}_{1h}^2 - 3Ch^2 \left|\ln h\right| ,$$

it follows that

$$\alpha_{1h}^2 \leq u_{1h}^2, \tag{4.30}$$

and, (4.6) for $n = 2$, implies that

$$\begin{aligned} \left\|\alpha_{1h}^2 - u_1^2\right\|_1 &= \left\|\tilde{u}_{1h}^2 - 3Ch^2 \left|\ln h\right| - u_1^2\right\|_1 \\ &\leq 4Ch^2 \left|\ln h\right| . \end{aligned} \tag{4.31}$$

Combining (4.27), (4.28), (4.30) and (4.31), we obtain

$$\begin{aligned} u_1^2 &\leq \alpha_{1h}^2 + 4Ch^2 \left|\ln h\right| \\ &\leq u_{1h}^2 + 4Ch^2 \left|\ln h\right| \\ &\leq \alpha_1^2 + 8Ch^2 \left|\ln h\right| \\ &\leq u_1^2 + 8Ch^2 \left|\ln h\right| . \end{aligned}$$

Thus,

$$\left\|u_1^2 - u_{1h}^2\right\|_1 \leq 4Ch^2 \left|\ln h\right| . \tag{4.32}$$

Similarly, for $n = 2$ on Ω_2,(4.2) we have

$$\begin{cases} a_2(\tilde{u}_2^2, v) = (F(u_{2h}^1), v) & \forall v \in \mathring{\mathbb{V}}_2 \\ \tilde{u}_2^2 = \pi_h(u_{1h}^2) & \text{on } \gamma_2. \end{cases}$$

Using a similar argument as above, one can prove that $\tilde{u}_2^2$ is a subsolution for the problem with source term $F(u_2^1)+3Ch^2|\ln h|$ and boundary condition $\pi_h(u_1^2)+4Ch^2|\ln h|$. Let $\tilde{U}_2^2$ be the solution of such a problem, that is

$$\tilde{U}_2^2=\partial(F(u_2^1)+3Ch^2|\ln h|\ ,\ \pi_h(u_1^2)+4Ch^2|\ln h|),$$

Then, as

$$u_2^2=\partial(F(u_2^1)\ ,\ u_1^2),$$

making use of proposition 2.1, we get

$$\left\|\tilde{U}_2^2-u_2^2\right\|_2\leq 4Ch^2|\ln h|\,.$$

Putting

$$\alpha_2^2=\tilde{u}_2^2-4Ch^2|\ln h|\,,$$

we obtain

$$\alpha_2^2\leq u_2^2, \tag{4.33}$$

and, making use of (4.5) for $n=2$, we get

$$\begin{aligned}\left\|\alpha_2^2-u_{2h}^2\right\|_2&=\left\|\tilde{u}_2^2-4Ch^2|\ln h|-u_{2h}^2\right\|_1 \\ &\leq 5Ch^2|\ln h|\,.\end{aligned} \tag{4.34}$$

Likewise, for $n=2$ on Ω_2, we can also establish that

$$\alpha_{2h}^2\leq u_{2h}^2 \tag{4.35}$$

and

$$\left\|\alpha_{2h}^2-u_2^2\right\|_2\leq 5Ch^2|\ln h| \tag{4.36}$$

Hence, combining (4.33), (4.34), (4.35) and (4.36), we obtain

$$\left\|u_2^2-u_{2h}^2\right\|_2\leq 5Ch^2|\ln h|\,. \tag{4.37}$$

Now, let us assume that (4.8) and (4.9) hold. We need to prove it for the $(n+1)^{th}$ step. Indeed, consider the problem

$$\begin{cases} a_1(\tilde{u}_1^{n+1},v)=(F(u_{1h}^n),v) & \forall v\in\mathring{\mathbb{V}}_1 \\ \tilde{u}_1^{n+1}=\pi_h(u_{2h}^n) & \text{on } \gamma_1. \end{cases}$$

Then, we also have

$$\begin{cases} a_1(\tilde{u}_1^{n+1},v)\leq(F(u_{1h}^n),v) & \forall v\in\mathring{\mathbb{V}}_1, v\geq 0 \\ \tilde{u}_1^{n+1}\leq\pi_h(u_{2h}^n) & \text{on } \gamma_1, \end{cases}$$

which can be rewritten as

$$\begin{cases} a_1(\tilde{u}_1^{n+1},v)\leq(F(u_{1h}^n)-F(u_1^n)+F(u_1^n),v) & \forall v\in\mathring{\mathbb{V}}_1, v\geq 0 \\ \tilde{u}_1^{n+1}\leq\pi_h(u_{2h}^n)\ -\pi_h(u_2^n)\ +\pi_h(u_2^n) & \text{on } \gamma_1. \end{cases}$$

Since $F(.)$ is Lipschitz continuous, $\gamma_1 \subset \Omega_2$, this implies that

$$\begin{cases} a_1(\tilde{u}_1^{n+1}, v) \leq (F(u_1^n) + (2n)Ch^2 \left|\ln h\right|, v) & \forall v \in \mathring{\mathbb{V}}_1, v \geq 0 \\ \tilde{u}_1^{n+1} \leq \pi_h(u_2^n) + (2n+1)Ch^2 \left|\ln h\right| & \text{on } \gamma_1. \end{cases}$$

This means that $\tilde{u}_1^{n+1}$ is a subsolution for the problem with source term $F(u_1^n) + (2n)Ch^2 \left|\ln h\right|$ and boundary term $\pi_h(u_2^n) + (2n+1)Ch^2 \left|\ln h\right|$. Let $\tilde{U}_1^{n+1}$ be the solution of such a problem. That is,

$$\tilde{U}_1^{n+1} = \partial\left(F(u_1^n) + (2n)Ch^2 \left|\ln h\right| \;,\; \pi_h(u_2^n) + (2n+1)Ch^2 \left|\ln h\right|\right).$$

Then, making use of lemma 2.2, we have

$$\tilde{u}_1^{n+1} \leq \tilde{U}_1^{n+1} = \partial(F(u_1^n) + (2n)Ch^2 \left|\ln h\right| \;,\; \pi_h(u_2^n) + (2n+1)Ch^2 \left|\ln h\right|)$$

But

$$u_1^{n+1} = \partial(F(u_1^n) \;,\; u_2^n),$$

then, making use of proposition 2.1, we have

$$\left\|\tilde{U}_1^{n+1} - u_1^{n+1}\right\|_1 \leq (2n+1)Ch^2 \left|\ln h\right|,$$

and due to lemma 2.2,

$$\tilde{u}_1^{n+1} \leq \tilde{U}_1^{n+1} \leq u_1^{n+1} + (2n+1)Ch^2 \left|\ln h\right|.$$

Putting

$$\alpha_1^{n+1} = \tilde{u}_1^{n+1} - (2n+1)Ch^2 \left|\ln h\right|, \tag{4.38}$$

yields

$$\alpha_1^{n+1} \leq u_1^{n+1}.$$

And, using (4.5), we get

$$\left\|\alpha_1^{n+1} - u_{1h}^{n+1}\right\|_1 \leq (2n+1)Ch^2 \left|\ln h\right|. \tag{4.39}$$

The solution of (4.3) is also a subsolution:

$$\begin{cases} a_1(\tilde{u}_{1h}^{n+1}, \phi_s) \leq (F(u_1^n), \phi_s) & \forall \phi_s \geq 0, \forall s \\ \tilde{u}_{1h}^{n+1} \leq \pi_h(u_2^n) & \text{on } \gamma_1, \end{cases}$$

which, in turn, can be rewritten as

$$\begin{cases} a_1(\tilde{u}_{1h}^{n+1}, \phi_s) \leq (F(u_1^n) - F(u_{1h}^n) + F(u_{1h}^n), \phi_s) & \forall \phi_s \geq 0, \forall s \\ \tilde{u}_{1h}^{n+1} \leq \pi_h(u_2^n) - \pi_h(u_{2h}^n) + \pi_h(u_{2h}^n) & \text{on } \gamma_1. \end{cases}$$

Since $F(.)$ is Lipschitz continuous, $\gamma_1 \subset \Omega_2$, this implies

$$\begin{cases} a_1(\tilde{u}_{1h}^{n+1}, \phi_s) \leq (F(u_{1h}^n) + (2n)Ch^2 \left|\ln h\right|, \phi_s) \; \forall \phi_s \geq 0, \forall s \\ \tilde{u}_{1h}^{n+1} \leq \pi_h(u_{2h}^n) + (2n+1)Ch^2 \left|\ln h\right| & \text{on } \gamma_1. \end{cases}$$

In other words, $\tilde{u}_{1h}^{n+1}$ is a subsolution for the problem with data $F(u_{1h}^n) + (2n)Ch^2 |\ln h|$ and $\pi_h(u_{2h}^n) + (2n+1)Ch^2 |\ln h|$. Let $\tilde{U}_{1h}^{n+1}$ be the solution of such a problem. That is,

$$\tilde{U}_{1h}^{n+1} = \partial_h(F(u_{1h}^n) + (2n)Ch^2 |\ln h|\,,\ \pi_h(u_{2h}^n) + (2n+1)Ch^2 |\ln h|).$$

But, as

$$u_{1h}^{n+1} = \partial_h(F(u_{1h}^n)\ ,\ \pi_h(u_{2h}^n)),$$

making of proposition 2.2, we get

$$\left\|\tilde{U}_{1h}^{n+1} - u_{1h}^{n+1}\right\|_1 \leq (2n+1)Ch^2 |\ln h|\,.$$

And so, due to lemma 2.4, we have

$$\tilde{u}_{1h}^{n+1} \leq \tilde{U}_{1h}^{n+1} \leq u_{1h}^{n+1} + (2n+1)Ch^2 |\ln h|\,.$$

Now putting

$$\alpha_{1h}^{n+1} = \tilde{u}_{1h}^{n+1} - (2n+1)Ch^2 |\ln h|\,, \tag{4.40}$$

and using (4.6), we obtain

$$\left\|\alpha_{1h}^{n+1} - u_1^{n+1}\right\|_1 \leq 2(n+1)Ch^2 |\ln h|\,. \tag{4.41}$$

Hence, similarly to above, combining (4.38), (4.39), (4.40) and (4.41), we obtain

$$\left\|u_1^{n+1} - u_{1h}^{n+1}\right\|_1 \leq 2(n+1)Ch^2 |\ln h|\,. \tag{4.42}$$

Which is the desired result in Ω_1.

Likewise, the estimate for the iterate $n+1$ in Ω_2 can be proved using similar arguments as above, which yields

$$\left\|u_2^{n+1} - u_{2h}^{n+1}\right\|_1 \leq (2n+3)Ch^2 |\ln h|\,.$$

□

Corollary 4.1. *$\forall n \geq 1$ fixed, we have*

$$\lim_{h\to 0} \|u_i^n - u_{ih}^n\|_i = 0,\ \ i = 1, 2.$$

Proof. The proof is straightforward. For fixed $n \geq 1$, passing to the limit as $h \to 0$ to both (4.8) and (4.9), the corollary follows on both subdomains.

□

Now, we are in position to prove the following convergence result:

Corollary 4.2. *There exists $h_n > 0$ with $h_n \to 0$, such that*

$$\lim_{n\to\infty} \left\|u_i - u_{ih_n}^n\right\|_i = 0; i = 1, 2. \tag{4.43}$$

Proof. Let us give the proof of (4.43) on Ω_1, the one on Ω_2 is similar. We know that

$$\|u_1 - u_{1h}^n\|_1 \leq \|u_1 - u_1^n\|_1 + \|u_1^n - u_{1h}^n\|_1$$

Letting $\epsilon > 0$, Theorem 3.1 implies that there exists $N \in \mathbb{N}$ such that

$$\|u_1 - u_1^n\|_1 \leq \frac{\epsilon}{2} \quad \forall n > N$$

Hence, due to (4.8), we have

$$\|u_1^n - u_{1h}^n\|_1 \leq (2n)Ch^2 \left|\ln h\right|,$$

Thus, the convergence result follows by choosing $h_n > 0$ such that

$$h_n^2 \left|\ln h_n\right| \leq \frac{\epsilon}{4Cn} \quad \forall n > N.$$

□

5. Numerical Experiments

In this section, we conduct numerical experiments on two model problems to validate the theory. The first model is chosen so that it does not have an exact solution, while we know an exact solution for the second one. For both models, we adopt the following notations:

- h_i; $i = 1, 2$, are the mesh sizes of the triangulations in Ω_i.
- δ is the size of overlap between both subdomains.
- $ERROR_{h_i} = \left\|u - u_{ih_i}^n\right\|_i$ is the maximum error between the exact solution u and the discrete Schwarz iterate on each subdomain.

We shall conduct the two tests to investigate the behavior of $ERROR_{h_i}$ as follows:

(1) We fix the mesh sizes h_i and vary the number of Schwarz iterations n,
(2) We fix the number of iterations n and vary the mesh sizes h_i,
(3) We consider the sequence of mesh sizes $h_{i,n}$ as n varies.

For all the experiments, we consider $\Omega = [0, 1] \times [0, 1]$. The "FreeFEM++" software, see [19], is adapted to obtain the numerical results for both models.

5.1. **First example:** In this example, we consider the boundary value problem

$$\begin{cases} -\Delta u = \dfrac{-\sigma u}{1 + au + bu^2} & \text{in } \Omega \\ \\ u = x + y & \text{on } \partial\Omega \end{cases} \tag{5.1}$$

where $\sigma = 1$, $a = 0$ and $b = 0.25$. This problem describes the enzyme kinetics model with inhibition. The value of the constant c is defined by determining suitable lower and upper solutions to the problem

(5.1) satisfying Definitions (3.2) and (3.3), respectively. The sector $\langle \breve{u}, \hat{u} \rangle = \langle 0, 12 \rangle$ is taken and c is evaluated by [4]:

$$c = \max \left\{ -\frac{\partial f}{\partial u}; x \in \bar{\Omega}, \breve{u} \leq u \leq \hat{u} \right\}. \tag{5.2}$$

For $c = 1$, $f(u)$ satisfies the one-sided Lipshitz condition

$$f(u_1) - f(u_2) \geq -(u_1 - u_2) \quad \text{for} \quad \breve{u} \leq u_2 \leq u_1 \leq \hat{u}. \tag{5.3}$$

A unique positive solution for the above model is also ensured in the sector $\langle 0, 2 \rangle$, (see [4]). Since it is difficult to obtain the exact solution for the problem, we use a P_2- finite element approximation of the exact solution of the problem on Ω, instead. Figure 2 represents the solution u of the above problem using a uniform fixed mesh size of $\frac{1}{30}$.

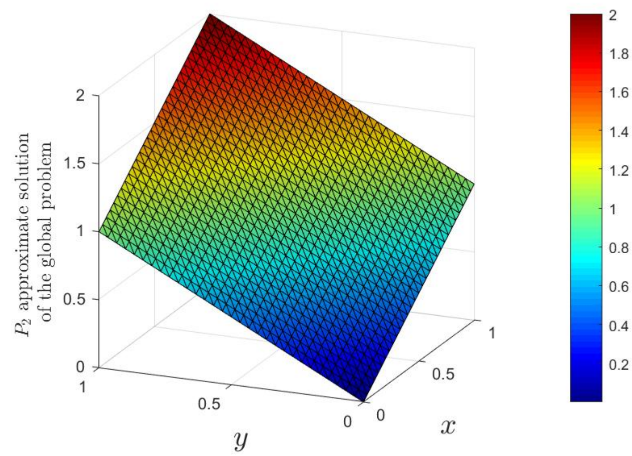


Figure 2. P_2-Approximate solution

We divide Ω into two overlapping non-matching subdomains Ω_1 and Ω_2 such that each subdomain is independently discretized into a quasi-uniform mesh with P_1 triangular elements and different mesh size $h_i; i = 1, 2$. In order for the maximum principle to be satisfied here, we construct a triangulation with acute angles for every $K \in \mathcal{T}^{h_i}; i = 1, 2$, using a Delaunay triangulation algorithm.

When both mesh sizes are fixed to be $h_1 = \frac{1}{32}$ and $h_2 = \frac{1}{24}$ with two sizes of the overlap $\delta = \frac{1}{8}$ and $\delta = \frac{1}{4}$, the approximated solution of the $35th$ iterate are represented in Figure 3, respectively. This shows the convergence of Schwarz sequences $u^n_{ih_i}$ to u. The same result can be easily shown for different mesh refinements.

Furthermore, Tables 1 and 2 represent the approximate solution values at some points in the domain with a stopping criterion $\varepsilon = e^{-5}$ for both subdomains. The obtained information of the tables show the monotone convergence of the Schwarz sequences $u^n_{ih_i}$, where u_{1h_1} and u_{2h_2} are the P_2-approximation of the nonlinear PDE problem on Ω_1 and Ω_2, respectively. It is also seen that the number of Schwarz iterations decreases as the overlap size increases.

Table 1. Approximate solution values at some points when $\delta = \frac{1}{8}$

n	$u_{1h_1}(\frac{1}{4}, \frac{1}{2})$	$u_{1h_1}(\frac{1}{2}, \frac{1}{2})$	$u_{2h_2}(\frac{2}{3}, \frac{1}{4})$	$u_{2h_2}(\frac{3}{4}, \frac{2}{3})$
0	0	0	0	0
1	0.345889	0.0817873	0.642487	1.1308
2	0.572373	0.536428	0.814077	1.32854
3	0.689899	0.809101	0.890129	1.40256
4	0.745107	0.943227	0.926039	1.43526
5	0.770443	1.00566	0.942779	1.44996

Table 2. Approximate solution values at some points when $\delta = \frac{1}{4}$

n	$u_{1h_1}(\frac{1}{4}, \frac{1}{2})$	$u_{1h_1}(\frac{1}{2}, \frac{1}{2})$	$u_{2h_2}(\frac{2}{3}, \frac{1}{4})$	$u_{2h_2}(\frac{3}{4}, \frac{2}{3})$
0	0	0	0	0
1	0.390914	0.16399	0.741398	1.23655
2	0.677421	0.800115	0.902063	1.41263
3	0.762939	0.992884	0.943188	1.45016
4	0.784635	1.04217	0.953611	1.45921

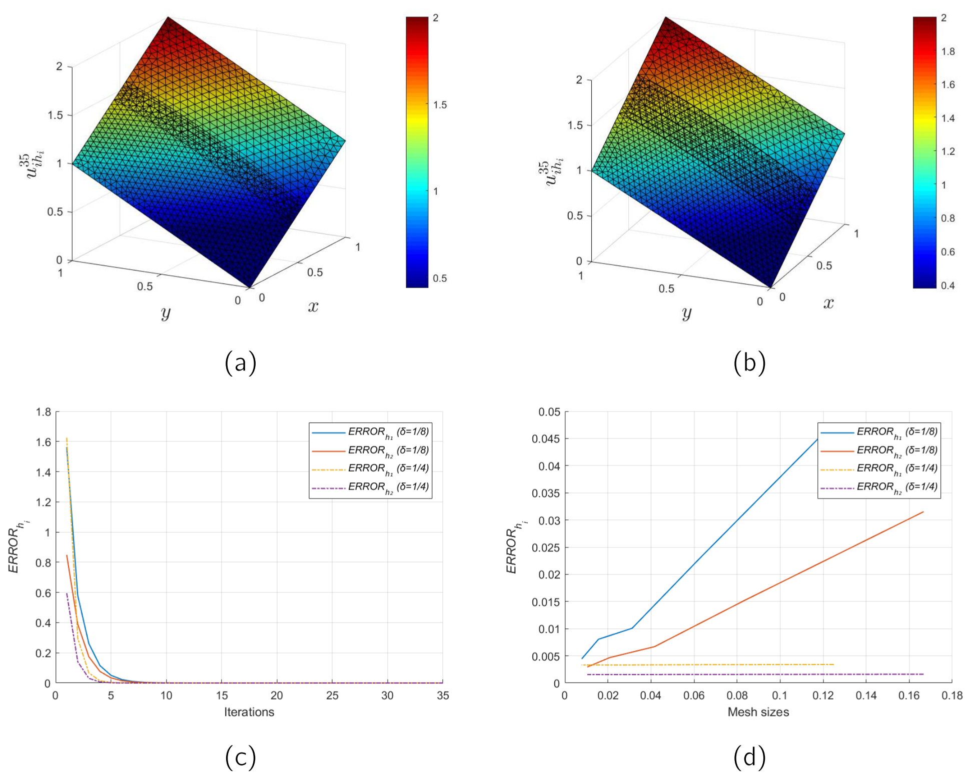


Figure 3. Iterative process of the first example on both Ω_1 and Ω_2. Approximate solution at 35th iteration when (A)$\delta = \frac{1}{8}$ and (B) $\delta = \frac{1}{4}$. Maximum errors versus number of iterations (C) and versus meshsizes (D).

In the first place, when we put $h_1 = \frac{1}{32}$, $h_2 = \frac{1}{24}$ with both $\delta = \frac{1}{8}$ and $\delta = \frac{1}{4}$ as before, and vary the number of Schwarz iterations over $1 \leq n \leq 35$, one can observe how the maximum errors decrease as the number of Schwarz iterations and the overlap size increase.

Next, we fix the number of iterations to be $n = 8$ when $\delta = \frac{1}{8}$, $n = 5$ when $\delta = \frac{1}{4}$ and vary the mesh sizes to be $h_1 = \frac{1}{4\times 2^N}$, $h_2 = \frac{1}{3\times 2^N}$ when $1 \leq N \leq 5$, instead. One can notice that the maximum errors decrease as the mesh sizes get smaller. Also, the bigger overlap size is, the smaller errors and the closer the curves are. Figure 3 shows both plots of $ERROR_{h_i}$.

5.2. **Second example:** We consider the following problem

$$\begin{cases} -\Delta u = \sigma u^p & \text{in } \Omega \\ \\ u = \frac{12}{(x+y+1)^2} & \text{on } \partial\Omega \end{cases} \tag{5.4}$$

where $\sigma = -1$ and $p = 2$. This problem describes the concentration of free atoms in the dissociation process. One can verify that the function $f(u)$ satisfies the one-sided Lipschitz condition (5.3). Also, the value of c satisfying (5.2) is 24. Hence, there exists a positive solution for the model in the sector $\langle 0, 12 \rangle$. Furthermore, one can verify that the exact solution of the model is given by

$$u = \frac{12}{(x+y+1)^2}$$

The exact solution u is represented in Figure 4 using a uniform fixed mesh size of $\frac{1}{30}$.

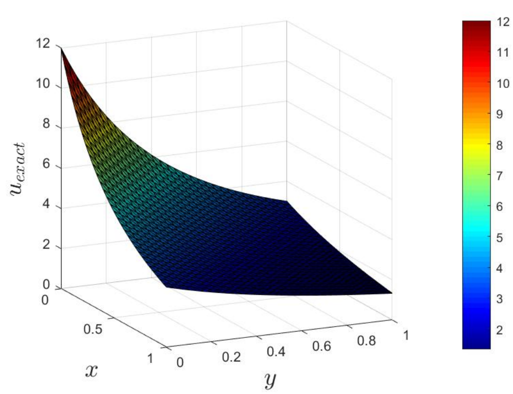


Figure 4. Exact solution

In this example, we build the same triangulation as in the first experiment in order to satisfy the maximum principle. We also examine the performance of the iterative approach for different values of the number of iterations, with only overlap size of $\delta = \frac{1}{8}$, by doing a similar analysis to the one made in the first example. The numerical results are shown in Figure 5, where the first and second figures represent the approximate solution at the initial and 35th iteration, while the third and fourth ones display the relationship of maximum errors with number of iterations and mesh sizes, respectively.

Moreover, the approximated solution values at some certain points in the domain with the same stopping criterion as in the first example for both subdomains are represented in Table 3. We notice the monotone convergence of the Schwarz sequences $u_{ih_i}^n$.

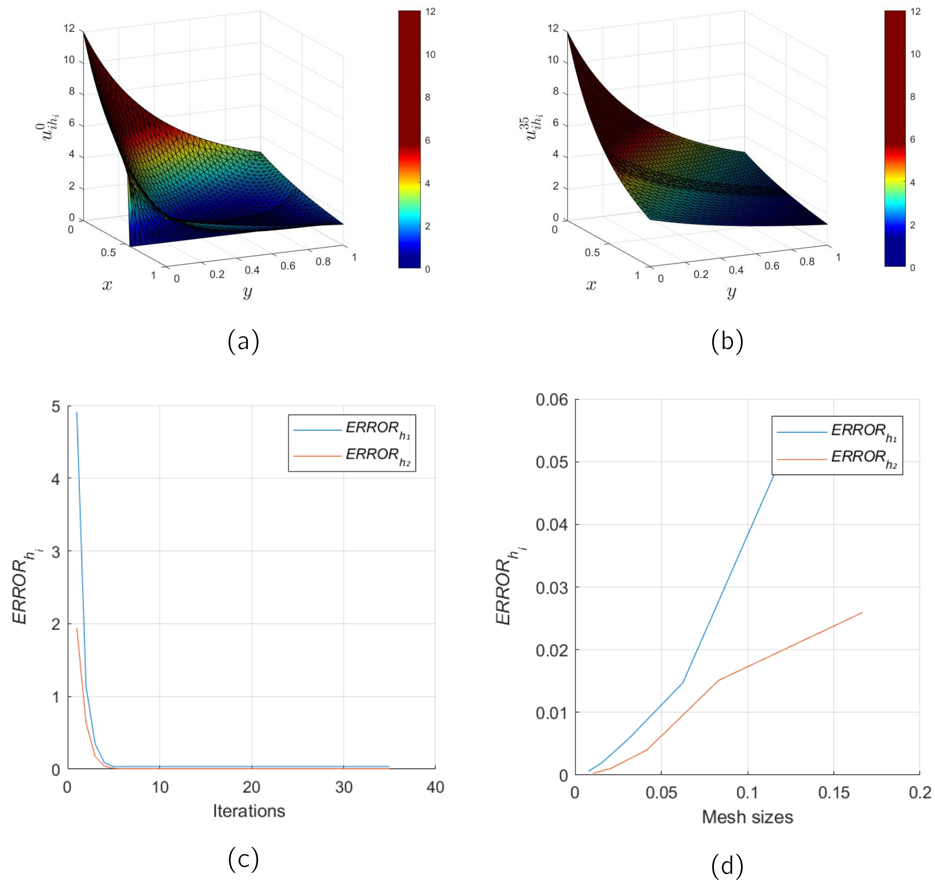


Figure 5. Iterative process of the second example on both Ω_1 and Ω_2. (A) Approximate solution at first iteration. (B) Approximate solution at 35th iteration. (C) Maximum errors versus number of iterations with fixed mesh sizes. (D)Maximum errors versus meshsizes with fixed number of iterations.

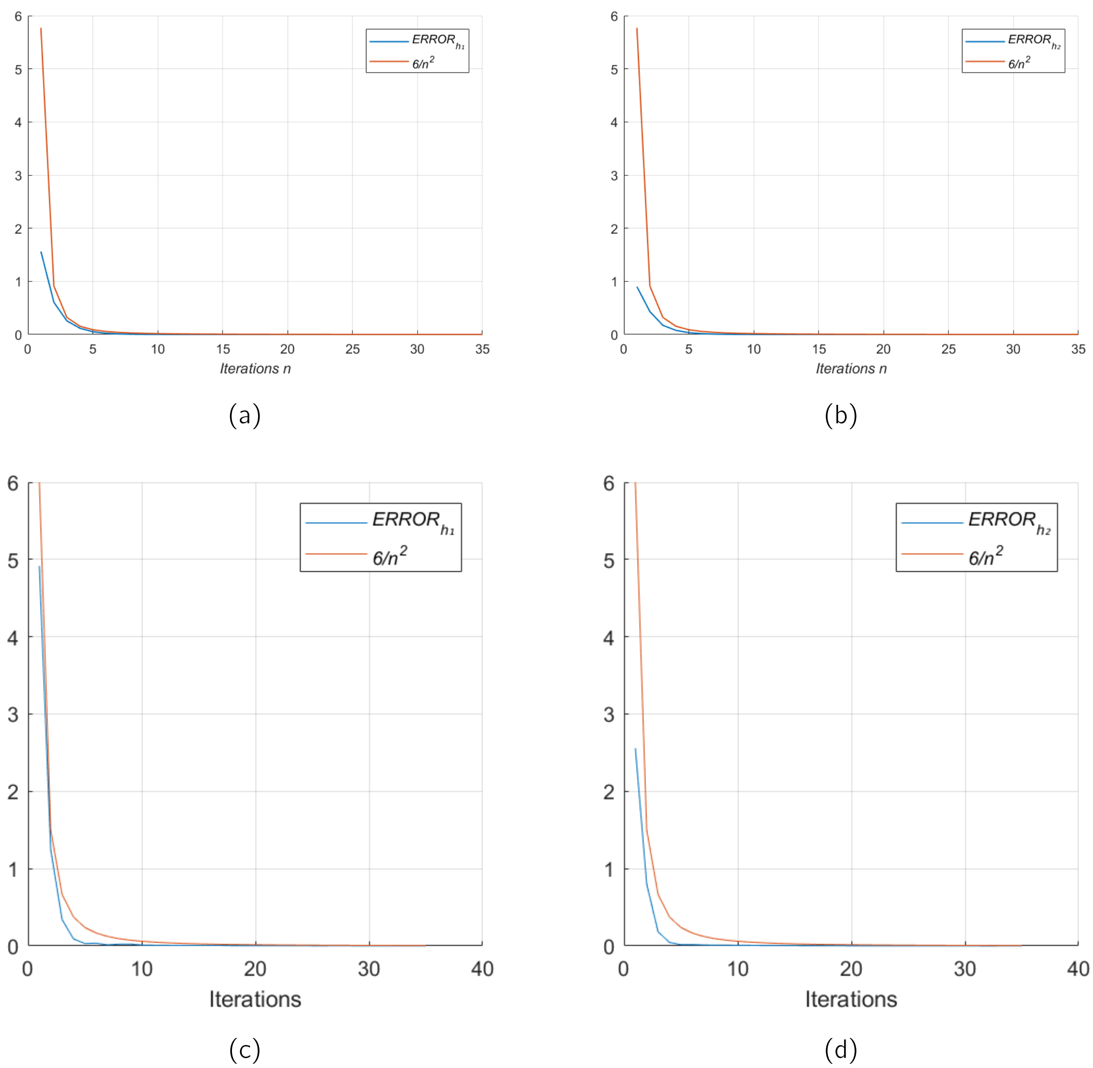
Table 3. Approximate solution values at some points when $\delta = \frac{1}{8}$

n	$u_{1h_1}(\frac{1}{4}, \frac{1}{2})$	$u_{1h_1}(\frac{1}{2}, \frac{1}{2})$	$u_{2h_2}(\frac{2}{3}, \frac{1}{4})$	$u_{2h_2}(\frac{3}{4}, \frac{2}{3})$
0	0	0	0	0
1	3.31901	0.709984	2.71408	1.64465
2	3.685	2.14238	3.13503	1.93498
3	3.86836	2.7503	3.23898	2.02499
4	3.90591	2.9386	3.26161	2.04795
5	3.9165	2.98647	3.2672	2.05378
6	3.91889	2.99829	3.26853	2.0552
7	3.9195	3.00117	3.26885	2.05555

We conclude this section by validating the convergence result (Corollary 4.2). Applying the context of it to both examples with mesh sizes of $h_{1,n} = \frac{1}{2n+1}$ and $h_{2,n} = \frac{1}{3n+1}$ on both subdomains when $1 \leq n \leq 35$, we see in Figure 6 that

$$\left\| u_i - u^n_{ih_{i,n}} \right\|_i \leq \frac{6}{n^2}; i = 1, 2, \forall n$$

In particular, the asymptotic behavior of our iterative approach is indicated to be at least $O(\frac{1}{n^2})$. This proves that our numerical results are in agreement with our theory.



(a) (b) (c) (d)

Figure 6. Plots of the maximum errors for both examples by considering the meshsize sequences $h_{i,n}$ for $1 \leq n \leq 35$. (A) First maximum error for Example 1. (B) Second maximum error for Example 1. (C) First maximum error for Example 2. (D) Second maximum error for Example 2.

6. CONCLUSION

In this paper, we have proved the convergence of the standard finite element approximation of monotone linear Schwarz alternating procedure for a class of semilinear elliptic PDEs, in the context of non-matching grids. In order to prove the main result, we used the concept of subsolutions to estimate, at each Schwarz iteration, the gap between the continuous and approximated Schwarz sequences. We have also conducted numerical experiments to show the agreement with the theory. We believe that the availability of a rate of convergence of the Schwarz procedure will help to derive an error estimate between the discrete Schwarz sequence and the exact solution of the semilinear PDE on each subdomain.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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