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Regularity and Paracompactness in Fuzzy Topological Spaces

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ABSTRACT. In this paper we obtain two characterizations of regular fuzzy topological spaces using Luo's and Abd El-Monsef and others' paracompact fuzzy topological spaces.

1. Introduction

In this paper we obtain two characterizations of fuzzy regularity as a fuzzy covering property. Indeed, we show that one can characterize fuzzy regularity as a paracompact-type fuzzy property in Luo's and Abd El-Monsef and others' both senses.

2. Definitions and Main Results

Definition 1 [1] Let μ be a set in a fts (X,τ) and let $r\in(0,1]$, $s\in[0,1)$; we define

$$\mu_{[r]}=\chi_{\{x\in X:\mu(x)\geqq r\}}$$

$$\mu_{(s)}=\chi_{\{x\in X:\mu(x)>s\}}$$

$$\mu_{<r>}=r\mu_{[r]}$$

Definition 2 [1] Let $\mathscr{A}$ be a family of sets and μ be a set in a fts (X,τ). We say that $\mathscr{A}$ is locally finite (resp. $*$-locally finite) in μ for each point e in μ, there exists $\nu\in Q(e)$ such that ν

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is quasi-coincident (resp. intersects) with at most a finite number of sets of $\mathscr{A}$; we often omit the word "in μ " when $\mu = X$.

Definition 3 [1] A family of sets $\mathscr{A}$ is called a Q-cover of a set μ if for each $x \in supp(\mu)$, there exist a $\nu \in \mathscr{A}$ such that ν and μ are quasi-coincident at x. Let $r \in (0,1]$. $\mathscr{A}$ is called a $r - Q$ cover of μ if $\mathscr{A}$ is a Q-cover of $\mu_{\langle r \rangle}$.

Definition 4 [1] Let $r \in (0,1]$, μ be a set in a fts (X,τ). We say that μ is r-paracompact (resp. r^*-paracompact) if for each r-open Q-cover of μ there exists an open refinement of it which is both locally finite (resp.∗-locally finite) in μ and a r-Q-cover of μ. The fuzzy set μ is called S-paracompact (resp. S*-paracompact) if for every $r \in (0,1]$, μ is r-paracompact (resp. r^* -paracompact).

Definition 5 [2] A family of fuzzy sets $\mathscr{U}$ is called an L-cover of a fuzzy set μ if $\bigvee_{\upsilon \in \mathscr{U}} \upsilon \geqq \mu.$

Definition 6 [2] Let μ be a fuzzy set in a fts (X,τ). We say that μ is fuzzy paracompact (resp. ∗-fuzzy paracompact) if for each open L-cover $\mathscr{B}$ of μ and for each $\xi \in (0,1]$, there exists an open refinement $\mathscr{B}^*$ of $\mathscr{B}$ which is both locally finite (resp. ∗-locally finite) in μ and L-cover of $\mu - \xi$. We say that a fts (X,τ) is fuzzy paracompact (resp. ∗-fuzzy paracompact) if each constant set in X is fuzzy paracompact (resp. ∗-fuzzy paracompact).

Theorem 1 Let (X,τ) a fuzzy Hausdorff fts (in any of Wuyts and Lowen's definitions that are good extensions of Hausdorffness). Then (X,τ) is fuzzy regular if and only if for each $r \in (0,1]$, for each r-open Q-cover of (X,τ) and for each fuzzy point x_λ of X there exists an open refinement of it which is both ∗-locally finite in x_λ and a r-Q-cover of (X,τ).

Proof (⇒) For each $r \in (0,1]$, let $\mathscr{U}$ be a r-open Q-cover of (X,τ), and x_λ be a fuzzy point of X . Then, we have that the family of crisp sets $\{U_{(1-r)} | U \in \mathscr{U}\}$, is an open cover of $(X,[\tau])$, which is Hausdorff and regular ([3], [4]). Then ([5], [6], [7]), it has an open refinement $\mathscr{V}_x \subset [\tau]$

which is a cover of X, and is locally finite in x. For each $V \in \mathscr{V}_x$ we have an $U_V \in \mathscr{U}$ with $V \subset (U_V)_{(1-r)}$.

Let $\mathscr{W}_x = \{\chi_V \wedge U_V \mid V \in \mathscr{V}_x\}$. Then, $\mathscr{W}_x \subset \tau$, is both an open refinement of $\mathscr{U}$ and a $r-Q$ -cover of (X,τ), and also is $*$-locally finite in x_λ, indeed, because $\mathscr{V}_x$ is locally finite in x, we have an open neigborhood G of x that G intersects with only finite number of members of $\mathscr{V}_x$. Then $\chi_G \in Q(x_\lambda)$ intersects with only a finite number of members of $\mathscr{W}_x$.

($\Leftarrow$) Let $\mathscr{U} \subset [\tau]$ be an open cover of $(X,[\tau])$; then $\{\chi_U \mid U \in \mathscr{U}\}$ is an open Q-cover of 1_X, and, for each $x \in X$, it has an open refinement $\mathscr{V}_{x_r}$ which is a Q-cover of 1_X and also locally finite in x_{1-r}. Let $\mathscr{W} = \{V_{(1-r)} | V \in \mathscr{V}_{x_r}\}$; then $\mathscr{W} \subset [\tau]$ is both a refinement of $\mathscr{U}$ and a cover of $(X,[\tau])$. Also, $\mathscr{W}$ is locally finite in x. Indeed: we take $O_1 \in Q(\mathrm{x}_{1-r})$ such that O_1, is quasi-coincident with only a finite number of members $V_1,\ldots,V_n$, of $\mathscr{V}_{x_r}$. Let $\mathscr{O} = (O_1)_{(r)}$, then $x \in O \in [\tau]$. For each $\mathrm{V} \in \mathscr{V}_{x_r}$, if $O \wedge V_{(1-r)} \neq \varnothing$, we have a crisp point $y \in X$, such that $O_1(y) > r,\ V(y) > 1-r,\ O_1(y) + V(y) > 1,$ then $O_1 q V$ and $V \in \{V_1,\ldots,V_n\}$. Hence the neighborhood O of x intersects with only a finite number of members $(V_1)_{(1-r)},\ldots,(V_n)_{(1-r)} \in \mathscr{W}$.

Theorem 2. Let (X,τ) be a fuzzy Hausdorff fts (in any of Wuyts and Lowen's definitions that are good extensions of Hausdorffness [3]). Then (X,τ) is fuzzy regular if and only if for each $r \in I$, and for each open L-cover $\mathscr{B}$ of r, for each $\xi \in (0,1]$, and for each fuzzy point x_λ of X, there exists an open refinement $\mathscr{B}^*$ of it which is both $*$-locally finite in x_λ and L-cover of $r-\xi$.

Proof ($\Rightarrow$) For each $r \in I$, and for each open L-cover $\mathscr{B}$ of r, for each $\xi \in (0,1]$, and for each fuzzy point x_λ of X, we have that the family of crisp sets $\mathscr{U} = \{G^{-1}((r-\xi,1]) \mid G \in \mathscr{B}\} \subset [\tau]$ is an open cover of $(X,[\tau])$ which is Hausdorff and regular ([3, 4]). Then ([5],[6],[7]), it has an open refinement $\mathscr{U}_x^* \subset [\tau]$ which is a cover of X and is locally finite in x. For each $V \in \mathscr{U}_x^*$, there exists $G_V^{-1}((r-\xi,1])\in\mathscr{U}$, such that $V\subset G_V^{-1}((r-\xi,1])$. So $\mathscr{B}^*=\{\chi_V\wedge G_V|V\in\mathscr{U}_x^*\}\subset\tau$ is refinement of $\mathscr{B}$. Then, there exists $V\in\mathscr{U}_x^*$ such that $x\in V$ and $G_V(x)>r-\xi.$ So, $(\chi_V\wedge G_V)(x)\geq r-\xi,$ and $\bigvee\{\chi_V\wedge G_V|V\in\mathscr{U}_x^*\}\geq r-\xi.$ Since $\mathscr{U}_x^*$ is locally finite in x, there exists $A\in[\tau]$ with $x\in A$, such that intersects with at most a finite number of members of $\mathscr{U}_x^*$. Then, there exists $\chi_A\in\tau$ such that $x_\lambda q\chi_A$ and χ_A intersects with a finite number of fuzzy sets of $\mathscr{B}^*$.

(⇐) Let $\mathscr{U}\subset[\tau]$ be an open cover of X and $x\in X$, then $\mathscr{B}=\{\chi_U|U\in\mathscr{U}\}$ is an open L-cover of (X,τ) and for each, $r\in I$ is $\bigvee\{\chi_U|U\in\mathscr{U}\}\geq r$. For each $\xi\in(0,1],$ there exists an open refinement $\mathscr{B}^*$ of $\mathscr{B}$ which is both locally finite in x_λ and L-cover of $r-\xi$. This implies that $\bigvee\{G|\,G\in\mathscr{B}^*\}\geq r-\xi_1$ for all $\xi_1>\xi.$ Let $\mathscr{U}^*=\{G^{-1}((r-\xi,1])\,|\,G\in\mathscr{B}^*\}$, then $\mathscr{U}^*\subset[\tau]$ is an open refinement of $\mathscr{U}$ (indeed, for each $G^{-1}((r-\xi,1])\in\mathscr{U}^*$ there exists $V_G\in\mathscr{U}$ such that $G^{-1}((r-\xi,1])\subset V_G$). Since $\bigvee\{G\,|\,G\in\mathscr{B}^*\}\geq r-\xi_1$, then $\mathscr{U}^*$ is an open refinement of $\mathscr{U}$. And, since $x\in X$ there exists $A\in Q(x_\lambda)$ which intersects with only $G_1,..,G_n\in\mathscr{B}^*.$ Since $A(x)+\lambda>1,$ we have $A(x)\ >1-\lambda$, then $x\in A^{-1}((1-\lambda,1])\in[\tau].$

If $A^{-1}((1-\lambda,1])\cap G^{-1}((r-\xi,1])\neq\varnothing$, there exists some point z, such that $A(z)\ >1-\lambda$ and $G(z)>r-\xi$, so $A\wedge G\neq\varnothing$. Then, if the neigborhood $A^{-1}((1-\lambda,1])$ of x intersects with infinite members of $\mathscr{U}^*$, A intersects with infinite members of $\mathscr{B}^*$.Thus $\mathscr{U}^*$ is locally finite in x.

This yields that the Hausdorff topological space $(X,[\tau])$ is regular ([5], [6], [7]) and (X,τ) is fuzzy regular ([4]).

3. Discussion

In this paper, fuzzy regularity is characterized as a fuzzy covering property. Future research could obtain characterization of other fuzzy separation properties as fuzzy covering properties.

Conflicts of Interest: The author declares that there are no conflicts of interest regarding the publication of this paper.

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