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On $\omega_{\tilde{\theta}}$-μ-Open Sets in Generalized Topological Spaces

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Abstract. In this paper analogous to [1], we introduce a new class of sets called $\omega_{\tilde{\theta}}$-μ-open sets in generalized topological spaces which lies strictly between the class of $\tilde{\theta}_\mu$-open sets and the class of ω-μ-open sets. We prove that the collection of $\omega_{\tilde{\theta}}$-μ-open sets forms a generalized topology. Finally, several characterizations and properties of this class have been given.

1. Introduction

One notion that has received much attention lately is the so-called ω-open sets in a topological space (X, τ) was introduced by Hdeib [12], which forms a topology finer than τ. Recently, many topological concepts and several interesting results related to this notion have obtained by many authors such as [3], [10], [9], [2]. A collection μ of subsets of a nonempty set X is a generalized topology (GT) if $\emptyset \in \mu$ and μ is closed under arbitrary unions, this notion was introduced by Császár in the sense of [5]. We call the pair (X, μ) a generalized topological space (briefly GTS) on X. The elements of μ are called μ-open sets and their complements are called μ-closed sets, see [7], the union of all elements of μ will be denoted by $\mathcal{M}_\mu$ and a GTS (X, μ) is said to be strong [7] if $X \in \mu$. If A is a subset of a GTS (X, μ), then the μ-closure of A, $c_\mu(A)$, is the intersection of all μ-closed sets containing A and the μ-interior of A, $i_\mu(A)$, is the union of all μ-open sets contained in A (see [5, 7]). It is easy to observe that operators i_μ and c_μ are idempotent and monotonic A subset A of a GTS (X, μ) is μ-open if and only if $A = i_\mu(A)$, and and $i_\mu(A) = X \setminus c_\mu(X \setminus A)$. Evidently, A is μ-closed if and only if $A = c_\mu(A)$, $c_\mu(A)$ is the smallest μ-closed set containing A, $i_\mu(A)$ is the largest μ-open set contained in A. Over recent years several authors have been working in formulate many topological

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concepts to establish new concepts in the structure of GTS, see [4], [8], [6] [11], [17], [15], [13] and others. Then motivated by the notion of ω-open set in a topological space (X, τ), Al Ghour and Wafa Zareer (2016) [1] defined the notions of ω-μ-closed sets and ω-μ-open sets in the structure of GTS as follows : A subset A of GTS (X, μ) is called ω-μ-closed if it contains all its condensation points. The complement of an ω-μ-closed set is called ω-μ-open. The family of all ω-μ-open subsets of X forms a GT on X, denoted by ω_μ.

Let us now recall some notions defined in [14]. A subset A of GTS (X, τ) is said to be $\tilde{\theta_\mu}$-open if and only if for each $x \in A$, there exists $U \in \mu$ such that $x \in U \subseteq c_\mu(U) \cap \mathcal{M}_\mu \subseteq A$ and the collection of all $\tilde{\theta_\mu}$-open subsets of a GTS (X, μ) is denoted by $\tilde{\theta}_\mu$. Then $\tilde{\theta}_\mu$ is also a GT included in μ. Analogous to [1] and by using the notion of $\tilde{\theta_\mu}$-open, we introduce the relatively new notions of $\omega_{\tilde{\theta}}$-μ-open as a new class of sets . We present several characterizations, properties, and examples related to the new concepts.

In section 2, we use the the notion of $\tilde{\theta_\mu}$-open to introduce $\omega_{\tilde{\theta}}$-μ-open sets in GTS as a new class of sets and we prove that this class lies strictly between the class of $\tilde{\theta_\mu}$-open sets and the class of ω-μ-open sets. Moreover, we give some sufficient conditions for the equivalence between the class of $\omega_{\tilde{\theta}}$-μ-open sets and the class of ω-μ-open sets.

In section 3, several interesting properties of $\omega_{\tilde{\theta}}$-μ-open subsets are discussed via the operations of $\omega_{\tilde{\theta}}$-interior and $\omega_{\tilde{\theta}}$-closure.

Definition 1.1. *[16] A GTS (X, μ) is said to be μ-locally indiscrete if every μ-open set in (X, μ) is μ-closed.*

Definition 1.2. *[1] A GTS (X, μ) is called μ-locally countable if $\mathcal{M}_\mu$ is nonempty and for every point $x \in \mathcal{M}_\mu$, there exists a $U \in \mu$ such that $x \in U$ and U is countable.*

Definition 1.3. *[14] Let (X, μ) be a GTS , $A \subseteq X$ and $\gamma_{\tilde{\theta}} : P(X) \to P(X)$ be an operation defined as the following:*

$$\gamma_{\tilde{\theta}_\mu}(A) = \{x \in X : c_\mu(U) \cap \mathcal{M}_\mu \cap A \neq \emptyset \ for \ \ all \ \ U \in \mu, x \in U\}.$$

Theorem 1.1. *[1] Let (X, μ) be a GTS. Then $\mathcal{M}_\mu = \mathcal{M}_{\omega_\mu}$.*

Theorem 1.2. *[1] If (X, μ) is a μ-locally countable GTS, then $\omega_\mu$is the discrete topology on $\mathcal{M}_\mu$.*

2. $\omega_{\tilde{\theta}}$-μ-open sets

We begin this section by introducing the following definition.

Definition 2.1. *Let (X, μ) be a GTS and $A \subseteq X$. Consider an operation $\Gamma_{\omega_{\tilde{\theta}}} : P(X) \to P(X)$ defined as the following:*
$\Gamma_{\omega_{\tilde{\theta}}}(A) = \{x \in X : U \cap A \ \ is \ uncountable \ for \ all \ U \in \tilde{\theta}_\mu \ and \ x \in U\}$. *A point $x \in X$ is called a*

$\tilde{\theta}_\mu$-condensation point of A if for all $U \in \tilde{\theta}_\mu$ such that $x \in U$ and $U \cap A$ is uncountable. The set of all $\tilde{\theta}_\mu$-condensation points of A is denoted by $\Gamma_{\omega_{\tilde{\theta}}}(A)$.

Lemma 2.1. *Let (X, μ) be a GTS. The operation $\Gamma_{\omega_{\tilde{\theta}}} : P(X) \to P(X)$ has the following properties:*
(1) *if $A \subseteq B \subset X$, then $\Gamma_{\omega_{\tilde{\theta}}}(A) \subseteq \Gamma_{\omega_{\tilde{\theta}}}(B)$ (monotonic property);*
(2) *$\Gamma_{\omega_{\tilde{\theta}}}(\Gamma_{\omega_{\tilde{\theta}}}(A)) \subseteq \Gamma_{\omega_{\tilde{\theta}}}(A)$ for any $A \subseteq X$ (restricting property);*
(3) *if A is any countable subset of X, then $\Gamma_{\omega_{\tilde{\theta}}}(A) = \emptyset$.*

Proof. (1) Let $A \subseteq B \subset X$ and $x \in \Gamma_{\omega_{\tilde{\theta}}}(A)$. Then $U \cap A$ is uncountable for each $U \in \tilde{\theta}_\mu$ and $x \in U$. Since $A \subseteq B$, then $U \cap B$ is uncountable. Thus $x \in \Gamma_{\omega_{\tilde{\theta}}}(B)$ and hence $\Gamma_{\omega_{\tilde{\theta}}}(A) \subseteq \Gamma_{\omega_{\tilde{\theta}}}(B)$.
(2) Let $x \in \Gamma_{\omega_{\tilde{\theta}}}(\Gamma_{\omega_{\tilde{\theta}}}(A))$. Then $U \cap \Gamma_{\omega_{\tilde{\theta}}}(A)$ is an uncountable for all $U \in \tilde{\theta}_\mu$ and $x \in U$. Let $y \in U \bigcap \Gamma_{\omega_{\tilde{\theta}}}(A)$. Then $y \in U$ and $y \in \Gamma_{\omega_{\tilde{\theta}}}(A)$ which implies that $U \cap A$ is an uncountable set. Hence $x \in \Gamma_{\omega_{\tilde{\theta}}}(A)$ and therefore $\Gamma_{\omega_{\tilde{\theta}}}(\Gamma_{\omega_{\tilde{\theta}}}(A)) \subseteq \Gamma_{\omega_{\tilde{\theta}}}(A)$.
(3) The proof is obvious by Definition 2.1. □

Definition 2.2. *Let (X, μ) be a GTS and $A \subseteq X$. Then A is said to be $\omega_{\tilde{\theta}}$-μ-closed if $\Gamma_{\omega_{\tilde{\theta}}}(A) \subseteq A$. The complement of an $\omega_{\tilde{\theta}}$-μ-closed set is said to be $\omega_{\tilde{\theta}}$-μ-open.*

The family of all $\omega_{\tilde{\theta}}$-μ-open subsets of (X, μ) is denoted by $\omega_{\tilde{\theta}}$, where $\omega_{\tilde{\theta}} = \{W \subseteq X : \Gamma_{\omega_{\tilde{\theta}}}(X \backslash W) \subseteq X \setminus W\}$. The following theorem and lemma give a necessary and sufficient condition for $\omega_{\tilde{\theta}}$-μ-open sets.

Theorem 2.1. *Let (X, μ) be a GTS and $W \subseteq X$. Then the following statements are equivalent:*
(1) *W is $\omega_{\tilde{\theta}}$-μ-open;*
(2) *if for every $x \in W$ there exists a $U \in \tilde{\theta}_\mu$ such that $x \in U$ and $U \setminus W$ is a countable set.*

Proof. (1) $\Rightarrow$ (2): Suppose W is $\omega_{\tilde{\theta}}$-μ-open. Since $X \backslash W$ is $\omega_{\tilde{\theta}}$-μ-closed set, then $\Gamma_{\omega_{\tilde{\theta}}}(X \backslash W) \subseteq X \backslash W$. This means that for every $x \in W$, $x \notin \Gamma_{\omega_{\tilde{\theta}}}(X \setminus W)$ and hence there exists a $U \in \tilde{\theta}_\mu$ such that $x \in U$ and $U \cap (X \setminus W) = U \setminus W$ is countable.
(2) $\Rightarrow$ (1): Let $x \in W$. Then by assumption there exists a $U \in \tilde{\theta}_\mu$ such that $x \in U$ and $U \cap (X \setminus W)$ is countable. Which implies that $x \notin \Gamma_{\omega_{\tilde{\theta}}}(X \setminus W)$, $\Gamma_{\omega_{\tilde{\theta}}}(X \setminus W) \subseteq X \setminus W$ and hence $X \setminus W$ is $\omega_{\tilde{\theta}}$-μ-closed. Therefore W is $\omega_{\tilde{\theta}}$-μ-open set. □

Lemma 2.2. *A subset W of a GTS (X, μ) is $\omega_{\tilde{\theta}}$-μ-open if and only if for every $x \in W$ there exists a $U \in \tilde{\theta}_\mu$ and a countable $C \subseteq \mathcal{M}_\mu$ such that $x \in U \setminus C \subseteq W$.*

Proof. Necessity. Let W be $\omega_{\tilde{\theta}}$-μ-open and $x \in W$. By Theorem 2.1, there exists $U \in \tilde{\theta}_\mu$ such that $x \in U$ and $U \setminus W$ is countable. Let $C = U \setminus W$. Then C is countable, $C \subseteq \mathcal{M}_\mu$ and $x \in U \cap (X \setminus C) = U \cap \big(X \setminus (U \cap X \setminus W)\big) = U \cap W \subseteq W$ and hence $x \in U \setminus C \subseteq W$.
Sufficiency. Let $x \in W$. From assumption there exists $U \in \tilde{\theta}_\mu$ and a countable set $C \subseteq \mathcal{M}_\mu$ such that $x \in U \setminus C \subseteq W$. Therefore, $U \setminus W \subseteq C$ and $U \setminus W$ is a countable set and this completes the proof. □

Theorem 2.2. *Let (X, μ) be a GTS and $C \subseteq X$. If C is $\omega_{\tilde{\theta}}$-μ-closed, then $C \subseteq F \cup B$ for some $\omega_{\tilde{\theta}}$-μ-closed set F and a countable subset B.*

Proof. Let C be any $\omega_{\tilde{\theta}}$-μ-closed set in (X, μ). Then $X \setminus C$ is $\omega_{\tilde{\theta}}$-μ-open. By Lemma 2.2, for each $x \in X \setminus C$, there exist a $\tilde{\theta_\mu}$-open set U containing x and a countable subset $B \subseteq \mathcal{M}_\mu$ such that $x \in U \setminus B \subseteq X \setminus C$. Thus $C \subseteq X \setminus (U \setminus B) = X \setminus (U \cap (X \setminus B)) = (X \setminus U) \cup B$. Let $F = X \setminus U$. Then F is $\omega_{\tilde{\theta}}$-μ-closed such that $C \subseteq F \cup B$. □

Theorem 2.3. *Let (X, μ) be a GTS. Then the collection $\omega_{\tilde{\theta}}$ forms a generalized topology on X.*

Proof. It is clear that $\emptyset \in \omega_{\tilde{\theta}}$. Let $\{W_\lambda : \lambda \in \Delta\}$ be a collection of $\omega_{\tilde{\theta}}$-μ-open subsets of (X, μ) and $x \in \bigcup\limits_{\lambda \in \Delta} W_\lambda$. There exists an $\lambda_0 \in \Delta$ such that $x \in W_{\lambda_0}$. Since W_{λ_0} is $\omega_{\tilde{\theta}}$-open set, then by Lemma 2.2, there exist $U \in \tilde{\theta}_\mu$ and a countable set $C \subseteq M_\mu$ such that $x \in U \setminus C \subseteq W_{\lambda_0} \subseteq \bigcup\limits_{\lambda \in \Delta} W_\lambda$. By Lemma 2.2, it follows that $\bigcup\limits_{\lambda \in \Delta} W_\lambda$ is $\omega_{\tilde{\theta}}$-μ-open. Hence the collection $\omega_{\tilde{\theta}}$ is generalized topology on X. □

The next theorem obtains that the new class of $\omega_{\tilde{\theta}}$-μ-open sets lies strictly between the class of $\tilde{\theta}$-μ-open sets and the class of ω-μ-open sets.

Theorem 2.4. *Let (X, μ) be a GTS. Then $\tilde{\theta}_\mu \subseteq \omega_{\tilde{\theta}} \subseteq \omega_\mu$.*

Proof. To show that $\tilde{\theta}_\mu \subseteq \omega_{\tilde{\theta}}$, let $W \in \tilde{\theta}_\mu$ and $x \in W$. Take $U = W$ and $C = \emptyset$. Then $U \in \tilde{\theta}_\mu$, $C \subseteq \mathcal{M}_\mu$ such that $x \in U \setminus C \subseteq W$. Therefore, by Lemma 2.2, it follows that $W \in \omega_{\tilde{\theta}}$.
To show that $\omega_{\tilde{\theta}} \subseteq \omega_\mu$, Let $W \in \omega_{\tilde{\theta}}$. By Theorem 2.1, for each $x \in W$ there exists a $U \in \tilde{\theta}_\mu$ such that $x \in U$ and $U \setminus W$ is countable. Since $\tilde{\theta_\mu} \subseteq \mu$, then $U \in \mu$ and hence W is ω-μ-open. Therefore $W \in \omega_\mu$. □

The following diagram follows immediately from the definitions and Theorem 2.4.

$$\begin{array}{ccc} \tilde{\theta_\mu} - open & \Longrightarrow & \omega_{\tilde{\theta}} - \mu - open \\ \Downarrow & & \Downarrow \\ \mu - open & \Longrightarrow & \omega - \mu - open \end{array}$$

The converse of these implications need not be true in general as shown by the following examples.

Example 2.1. *Consider $X = \mathbb{R}$, $A = \{4n : n \in \mathbb{N}\}$ and $\mu = \{\emptyset, [0, 2], [1, 3] \cup A, [0, 3] \cup A\}$. Then (X, μ) is a generalized topological space and the family of all $\tilde{\theta}_\mu$-open sets is $\tilde{\theta}_\mu = \{\emptyset, [0, 3] \cup A\}$. Then $[1, 3] \in \omega_\mu \setminus \omega_{\tilde{\theta}}$, i.e. $[1, 3]$ is ω-μ-open but it is not $\omega_{\tilde{\theta}}$-μ-open. Also, it is easy to check that $\Gamma_{\omega_{\tilde{\theta}}}(\mathbb{R} \setminus [0, 3]) \subseteq \mathbb{R} \setminus [0, 3]$. Thus $[0, 3] \in \omega_{\tilde{\theta}} \setminus \tilde{\theta}_\mu$, i.e. $[0, 3]$ is $\omega_{\tilde{\theta}}$-μ-open but it is not $\tilde{\theta_\mu}$-open*

Example 2.2. *Let $X = \{a, b, c, d\}$ with GT $\mu = \{\emptyset, \{a, b\}, \{a, c\}, \{a, b, c\}\}$. Then $\{a, c\} \in \omega_{\tilde{\theta}} \setminus \tilde{\theta}_\mu$, i.e. the set $\{a, c\}$ is $\omega_{\tilde{\theta}}$-μ-open but it is not $\tilde{\theta_\mu}$-open.*

Note that the previous examples show that $\tilde{\theta}_\mu \neq \omega_{\tilde{\theta}} \neq \omega_\mu$ in general.

Remark 2.1. *The notions of μ-open and $\omega_{\tilde{\theta}}$-μ-open sets are independent of each other. For more clarity in Example 2.1, the set $[0, 3]$ is $\omega_{\tilde{\theta}}$-μ-open but it is not μ-open and the set $[1, 3] \cup A$ is μ-open but it is not $\omega_{\tilde{\theta}}$-μ-open.*

Theorem 2.5. *If a GTS (X, μ) is a μ-locally indiscrete, then $\mu \subseteq \omega_{\tilde{\theta}}$.*

Proof. To show that $\mu \subseteq \omega_{\tilde{\theta}}$, let $A \in \mu$ and $x \in A$. Take $U = A$. Since (X, μ) is μ-locally indiscrete, then $c_\mu(U) = U$ and we have $x \in U \subseteq c_\mu(U) \cap \mathcal{M}_\mu \subseteq A$. Thus $A \in \tilde{\theta}_\mu$ and by Theorem 2.4, $\tilde{\theta}_\mu \subseteq \omega_{\tilde{\theta}}$. Therefore $A \in \omega_{\tilde{\theta}}$. □

Lemma 2.3. *Let (X, μ) be a GTS. Then $\mathcal{M}_\mu \in \tilde{\theta}_\mu$.*

Proof. Let $A = \mathcal{M}_\mu$ and $x \in A$. Then there exists $U_x \in \mu$ such that $x \in U_x$. Since $U_x \subseteq c_\mu(U_x) \bigcap \mathcal{M}_\mu \subseteq A$, then $A = \mathcal{M}_\mu \in \tilde{\theta}_\mu$. □

For a GT μ on a nonempty set X, let $\mathcal{M}_{\omega_{\tilde{\theta}}} = \bigcup\{U \subseteq X : U \in \omega_{\tilde{\theta}}\}$. Thus we have the following theorem.

Theorem 2.6. *Let (X, μ) be a GTS. Then $\mathcal{M}_\mu = \mathcal{M}_{\omega_{\tilde{\theta}}}$*

Proof. By Lemma 2.3, $\mathcal{M}_\mu \in \tilde{\theta}_\mu$ and form Theorem 2.4, $\tilde{\theta}_\mu \subseteq \omega_{\tilde{\theta}}$ and hence $\mathcal{M}_\mu \subseteq \mathcal{M}_{\omega_{\tilde{\theta}}}$. On the other hand, let $x \in \mathcal{M}_{\omega_{\tilde{\theta}}}$. Since, $\mathcal{M}_{\omega_{\tilde{\theta}}} \in \omega_{\tilde{\theta}}$, then by Lemma 2.2, there exists a $U \in \tilde{\theta}_\mu$ and a countable set $C \subseteq \mathcal{M}_\mu$ such that $x \in U \setminus C \subseteq \mathcal{M}_{\omega_{\tilde{\theta}}}$. Since $U \subseteq \mathcal{M}_\mu$ and U is μ-open, it follows that $x \in \mathcal{M}_\mu$ and hence $\mathcal{M}_{\omega_{\tilde{\theta}}} \subseteq \mathcal{M}_\mu$. Therefore $\mathcal{M}_\mu = \mathcal{M}_{\omega_{\tilde{\theta}}}$. □

By Theorem 1.1 and Theorem 2.6, we obtain the following corollary

Corollary 2.1. *Let (X, μ) be a GTS. Then $\mathcal{M}_\mu = \mathcal{M}_{\omega_{\tilde{\theta}}} = \mathcal{M}_{\omega_\mu}$*

We will denote by $(\tau_{coc})_X$, the cocountable topology on a nonempty set X.

Theorem 2.7. *Let (X, μ) be a GTS. Then $(\tau_{coc})_U \subseteq \omega_{\tilde{\theta}}$ for all $U \in \tilde{\theta}_\mu \setminus \{\emptyset\}$.*

Proof. Let $U \in \tilde{\theta}_\mu \setminus \{\emptyset\}$, $W \in (\tau_{coc})_U$ and $x \in W$. Since $W \subseteq U$, we have $x \in U$ and $U \setminus W = U \setminus (U \cap V)$ for some $V \in \tau_{coc}$. Now, $U \setminus W = U \setminus (U \cap V) = U \setminus V$. Thus $U \setminus W$ is countable set and by Theorem 2.1, it follows that $W \in \omega_{\tilde{\theta}}$. This shows that $(\tau_{coc})_U \subseteq \omega_{\tilde{\theta}}$. □

Theorem 2.8. *For any GTS (X, μ), the following statements are equivalent.*
(1) $\tilde{\theta}_\mu = \omega_{\tilde{\theta}}$.
(2) $(\tau_{coc})_U \subseteq \tilde{\theta}_\mu$ *for all* $U \in \tilde{\theta}_\mu \setminus \{\emptyset\}$.

Proof. (1) $\Longrightarrow$ (2): Assume that $\tilde{\theta}_\mu = \omega_{\tilde{\theta}}$ and $U \in \tilde{\theta}_\mu \setminus \{\emptyset\}$. Then by Theorem 2.7, $(\tau_{coc})_U \subseteq \omega_{\tilde{\theta}} = \tilde{\theta}_\mu$.
(2) $\Longrightarrow$ (1): Suppose that $(\tau_{coc})_U \subseteq \tilde{\theta}_\mu$ for all $U \in \tilde{\theta}_\mu \setminus \{\emptyset\}$. It is enough to show that $\omega_{\tilde{\theta}} \subseteq \tilde{\theta}_\mu$. Let

$W \in \omega_{\tilde{\theta}}$ and $x \in W$. By Lemma 2.2, there exists $U_x \in \tilde{\theta}_\mu$ and a countable set $C_x \subseteq \mathcal{M}_\mu$ such that $x \in U_x \setminus C_x \subseteq W$. Thus $U_x \cap X \setminus C_x \in (\tau_{coc})_{U_x}$, where $X \setminus C_x \in \tau_{coc}$. From assumption $U_x \setminus C_x \in (\tau_{coc})_{U_x} \subseteq \tilde{\theta}_\mu$ for all $x \in W$, and so $U_x \setminus C_x \in \tilde{\theta}_\mu$. It follows that $W = \bigcup\{U_x \setminus C_x : x \in W\} \in \tilde{\theta}_\mu$, and hence $\tilde{\theta}_\mu = \omega_{\tilde{\theta}}$. □

Proposition 2.1. *Let (X, μ) be a GTS. If $\tilde{\theta}_\mu$ is a topology on X, then $\omega_{\tilde{\theta}}$ is a topology.*

Proof. Suppose that $\tilde{\theta}_\mu$ is a topology. By Theorem 2.3, $\omega_{\tilde{\theta}}$ is generalized topology. It is enough to show that the collection $\omega_{\tilde{\theta}}$ is closed under finite intersection. Let W, G be $\omega_{\tilde{\theta}}$-μ-open sets and $x \in W \cap G$. Then by Theorem 2.1, there exist $U, V \in \tilde{\theta}_\mu$ containing x such that $U \setminus W$ and $V \setminus G$ are countable sets. Since $\tilde{\theta}_\mu$ is a topology, we have $x \in U \cap V \in \tilde{\theta}_\mu$. Furthermore, $(U \cap V) \setminus (W \cap G) = (U \cap V) \cap \left[X \setminus W \cup X \setminus G\right] = [(U \cap V) \setminus W)] \cup [(U \cap V) \setminus G)] \subset (U \setminus W) \cup (V \setminus G)$. Therefore, $(U \cap V) \setminus (W \cap G)$ is a countable set and hence $W \cap G$ is $\omega_{\tilde{\theta}}$-μ-open.

□

Definition 2.3. *Let (X, μ) be a GTS . Then (X, μ) is said to be $\tilde{\theta}_\mu$-locally countable if $\mathcal{M}_\mu$ is nonempty and for every point $x \in \mathcal{M}_\mu$, there exists a $U \in \tilde{\theta}_\mu$ such that $x \in U$ and U is countable.*

The following corollary is a direct result from Definition 2.3 and Definition 1.2.

Corollary 2.2. *Let (X, μ) be a GTS. If (X, μ) is $\tilde{\theta}_\mu$-locally countable, then (X, μ) is μ-locally countable.*

Theorem 2.9. *If (X, μ) is a $\tilde{\theta}_\mu$-locally countable GTS, then $\omega_{\tilde{\theta}}$ is the discrete topology on $\mathcal{M}_\mu$.*

Proof. It is enough to show that every singleton subset of $\mathcal{M}_\mu$ is $\omega_{\tilde{\theta}}$-μ-open. Since (X, μ) is $\tilde{\theta}_\mu$-locally countable, then for each $x \in \mathcal{M}_\mu$, there exists a $U \in \tilde{\theta}_\mu$ such that $x \in U$ and U is countable. By Theorem 2.7, we have $(\tau_{coc})_U \subseteq \omega_{\tilde{\theta}}$. Therefore $U \setminus (U \setminus \{x\}) = \{x\} \in \omega_{\tilde{\theta}}$. □

The following corollary is a direct result of Theorem 2.9.

Corollary 2.3. *Let (X, μ) be a strong GTS. If (X, μ) is a $\tilde{\theta}_\mu$-locally countable, then $\omega_{\tilde{\theta}}$ is the discrete topology on X.*

Proposition 2.2. *If (X, μ) is a $\tilde{\theta}_\mu$-locally countable GTS, then $\omega_{\tilde{\theta}} = \omega_\mu$.*

Proof. Since (X, μ) is $\tilde{\theta}_\mu$-locally countable, then by Theorem 2.9, $\omega_{\tilde{\theta}}$ is the the discrete topology on $\mathcal{M}_\mu$. From Corollary 2.2 and Theorem 1.2, we get $\omega_{\tilde{\theta}} = \omega_\mu$. □

Corollary 2.4. *Let (X, μ) be a GTS. If $\mathcal{M}_\mu$ is a countable nonempty set, then $\omega_{\tilde{\theta}}$ is the discrete topology on $\mathcal{M}_\mu$.*

Proof. Since $\mathcal{M}_\mu$ is countable nonempty set, then for $x \in \mathcal{M}_\mu$, there exists $U \in \tilde{\theta}_\mu$ such that U is countable set. Thus (X, μ) is $\tilde{\theta}_\mu$-locally countable. From Theorem 2.9, we get $\omega_{\tilde{\theta}}$ is the discrete topology on $\mathcal{M}_\mu$. □

3. Further properties of $\omega_{\tilde{\theta}}$-μ-open sets

Definition 3.1. *Let (X, μ) be a GTS and $A \subseteq X$. A point $x \in X$ is called an $\omega_{\tilde{\theta}}$-closure point of A if and only if $U \cap A \neq \emptyset$ for all $U \in \omega_{\tilde{\theta}}$ and $x \in U$. Consider the following operations are defined as follows:*
(1) $\gamma_{\omega_{\tilde{\theta}}}(A) = \{x \in X : U \cap A \neq \emptyset,\ for\ all\ U \in \omega_{\tilde{\theta}}\ \ and\ \ x \in U\}$;
(2) $c_{\omega_{\tilde{\theta}}}(A) = \cap\{F : A \subseteq F, F\ \ is\ \ \omega_{\tilde{\theta}}\text{-}\mu\text{-}closed\ \ in\ \ X\}$.

Lemma 3.1. *Let (X, μ) be a GTS. Then $c_{\omega_{\tilde{\theta}}}(A) = \gamma_{\omega_{\tilde{\theta}}}(A)$ for any $A \subseteq X$.*

Proof. It is enough to show that $\gamma_{\omega_{\tilde{\theta}}}(A)$ is the smallest $\omega_{\tilde{\theta}}$-μ-closed set containing A. Clearly $A \subseteq \gamma_{\omega_{\tilde{\theta}}}(A)$. Further $\gamma_{\omega_{\tilde{\theta}}}(A)$ is $\omega_{\tilde{\theta}}$-μ-closed, that is $X \setminus \gamma_{\omega_{\tilde{\theta}}}(A)$ is $\omega_{\tilde{\theta}}$-μ-open because for each $x \in X \setminus \gamma_{\omega_{\tilde{\theta}}}(A)$ there is $U_x \in \omega_{\tilde{\theta}}$ such that $x \in U_x$ and $U_x \cap A = \emptyset$. Now, for any $y \in U_x$ implies $y \in X \setminus \gamma_{\omega_{\tilde{\theta}}}(A)$ so that $X \setminus \gamma_{\omega_{\tilde{\theta}}}(A) = \bigcup_{x \in X \setminus \gamma_{\omega_{\tilde{\theta}}}(A)} U_x \in \omega_{\tilde{\theta}}$.

Finally if $A \subseteq F$ and F is any $\omega_{\tilde{\theta}}$-μ-closed, then $X \setminus F$ is $\omega_{\tilde{\theta}}$-μ-open and $(X \setminus F) \cap A = \emptyset$ so that $X \setminus F \subseteq X \setminus \gamma_{\omega_{\tilde{\theta}}}(A)$ and hence $\gamma_{\omega_{\tilde{\theta}}}(A) \subseteq F$. Therefore $\gamma_{\omega_{\tilde{\theta}}}(A)$ is the smallest $\omega_{\tilde{\theta}}$-μ-closed set containing A, and by Definition 3.1(2), $\gamma_{\omega\tilde{\theta}}(A) = c_{\omega\tilde{\theta}}(A)$. □

The proof of the following theorem is straightforward and thus omitted.

Theorem 3.1. *For subsets A, B of GTS (X, μ), the following properties hold:*
(1) *if $A \subseteq B \subset X$, then $c_{\omega_{\tilde{\theta}}}(A) \subseteq c_{\omega_{\tilde{\theta}}}(B)$;*
(2) *$A \subseteq c_{\omega_{\tilde{\theta}}}(A)$ for $A \subseteq X$;*
(3) *$c_{\omega_{\tilde{\theta}}}(c_{\omega_{\tilde{\theta}}}(A)) = c_{\omega_{\tilde{\theta}}}(A)$ for $A \subseteq X$;*
(4) *A is $\omega_{\tilde{\theta}}$-μ-closed if and only if $c_{\omega_{\tilde{\theta}}}(A) = A$.*

Definition 3.2. *Let (X, μ) be a GTS and $A \subseteq X$. Then we define the following notions:*
(1) $c_{\tilde{\theta}_\mu}(A) = \cap\{F : A \subseteq F, F\ \ is\ \ \tilde{\theta}_\mu\text{-}closed\ \ in\ \ X\}$;
(2) $c_{\omega_\mu}(A) = \cap\{F : A \subseteq F, F\ \ is\ \ \omega\text{-}\mu\text{-}closed\ \ in\ \ X\}$.

The proof of the following corollary is straightforward and thus omitted.

Corollary 3.1. *For a subset A of a GTS (X, μ), the following properties hold:*
(1) *A is $\tilde{\theta}_\mu$-closed if and only if $c_{\tilde{\theta}_\mu}(A) = A$;*
(2) *A is ω-μ-closed if and only if $c_{\omega_\mu}(A) = A$.*

Lemma 3.2. *Let (X, μ) be a GTS. Then $\gamma_{\tilde{\theta}_\mu}(A) \subseteq c_{\tilde{\theta}_\mu}(A)$ for any $A \subseteq X$.*

Proof. Let $x \notin c_{\tilde{\theta}_\mu}(A)$. Then $x \in X \setminus c_{\tilde{\theta}_\mu}(A)$ so that there is $U \in \tilde{\theta}_\mu$ satisfying $x \in U$ and $U \cap A = \emptyset$. Since $U \in \tilde{\theta}_\mu$, then there is $V \in \mu$ such that $x \in V \subseteq c_\mu(V) \cap \mathcal{M}_\mu \subseteq U$ and $c_\mu(V) \cap \mathcal{M}_\mu \cap A = \emptyset$, consequently $x \notin \gamma_{\tilde{\theta}}(A)$. Thus we have $\gamma_{\tilde{\theta}}(A) \subseteq c_{\tilde{\theta}}(A)$. □

Theorem 3.2. *Let (X, μ) be a GTS and $A \subseteq X$. Then the following properties hold:*
(1) $c_{\omega_\mu}(A) \subseteq c_{\omega_{\tilde{\theta}}}(A) \subseteq c_{\tilde{\theta}_\mu}(A)$;
(2) *If A is $\tilde{\theta_\mu}$-closed, then A is $\omega_{\tilde{\theta}}$-μ-closed;*
(3) *If A is $\omega_{\tilde{\theta}}$-μ-closed, then A is ω-μ-closed.*

Proof. (1) To show that $c_{\omega_\mu}(A) \subseteq c_{\omega_{\tilde{\theta}}}(A)$, let $x \notin c_{\omega_{\tilde{\theta}}}(A)$ and so there is a $U \in \omega_{\tilde{\theta}}$ containing x such that $U \cap A = \emptyset$. From Theorem 2.4, we have $\omega_{\tilde{\theta}} \subseteq \omega_\mu$, $U \in \omega_\mu$, and hence $x \notin c_{\omega_\mu}(A)$. To show that $c_{\omega_{\tilde{\theta}}}(A) \subseteq c_{\tilde{\theta}_\mu}(A)$, let $x \notin c_{\tilde{\theta}_\mu}(A)$ and so there is a $U \in \tilde{\theta}_\mu$ containing x such that $U \cap A = \emptyset$. From Theorem 2.4, we have $\tilde{\theta}_\mu \subseteq \omega_{\tilde{\theta}}$, $U \in \omega_{\tilde{\theta}}$, and hence $x \notin c_{\omega_{\tilde{\theta}}}(A)$.
(2) Suppose that A is $\tilde{\theta_\mu}$-closed. Then by Corollary 3.1(1), $c_{\tilde{\theta}_\mu}(A) = A$. Thus by (1), $c_{\omega_{\tilde{\theta}}}(A) = A$ and hence A is $\omega_{\tilde{\theta}}$-μ-closed.
(2) Suppose that A is $\omega_{\tilde{\theta}}$-μ-closed. Then by Theorem 3.1(4), $c_{\omega_{\tilde{\theta}}}(A) = A$. Thus by (1), $c_{\omega_\mu}(A) = A$ and hence A is ω-μ-closed. □

Proposition 3.1. *Let (X, μ) be a $\tilde{\theta}_\mu$-locally countable GTS and $A \subseteq X$. Then $c_{\omega_\mu}(A) = c_{\omega_{\tilde{\theta}}}(A)$*

Proof. By Theorem 3.2(1), $c_{\omega_\mu}(A) \subseteq c_{\omega_{\tilde{\theta}}}(A)$. Let $x \in c_{\omega_{\tilde{\theta}}}(A)$. Then $U \cap A \neq \emptyset$ for all $U \in \omega_{\tilde{\theta}}$ and $x \in U$. Since (X, μ) is a $\tilde{\theta}_\mu$-locally countable, then by Theorem 2.9, $\omega_{\tilde{\theta}}$ is the discrete topology on $\mathcal{M}_\mu$ and hence $\omega_\mu = \omega_{\tilde{\theta}}$. Which implies that $x \in c_{\omega_\mu}(A)$ and $c_{\omega_{\tilde{\theta}}}(A) \subseteq c_{\omega_\mu}(A)$. Hence $c_{\omega_\mu}(A) = c_{\omega_{\tilde{\theta}}}(A)$. □

Theorem 3.3. *Let (X, μ) be a μ-locally indiscrete GTS and let $A \subseteq X$. Then the following properties hold.*
(1) $c_\mu(A) = c_{\tilde{\theta}_\mu}(A)$;
(2) $c_{\omega_{\tilde{\theta}}}(A) \subseteq c_\mu(A)$;
(3) *If A is μ-closed in (X, μ), then A is $\tilde{\theta_\mu}$-closed in (X, μ).*
(4) *If A is μ-closed in (X, μ), then A is $\omega_{\tilde{\theta}}$-μ-closed in (X, μ).*

Proof. (1) Clearly $c_\mu(A) \subseteq c_{\tilde{\theta}_\mu}(A)$. To show that $c_{\tilde{\theta}_\mu}(A) \subseteq c_\mu(A)$, let $x \notin c_\mu(A)$. Then there exists $U \in \mu$ such that $x \in U$ and $U \cap A = \emptyset$. Since (X, μ) is a μ-locally indiscrete, $c_\mu(U) = U$. It follows that $U \subseteq c_\mu(U) \cap \mathcal{M}_\mu \subseteq U$ and hence $U \in \tilde{\theta}_\mu$. Thus $x \notin c_{\tilde{\theta}_\mu}(A)$.
(2) Since (X, μ) is μ-locally indiscrete. then by Theorem 2.5, $\mu \subseteq \omega_{\tilde{\theta}}$ and hence $c_{\omega_{\tilde{\theta}}}(A) \subseteq c_\mu(A)$.
(3) Suppose that A is μ-closed in (X, μ), then $c_\mu(A) = A$. Thus by (1), $A = c_{\tilde{\theta}_\mu}(A)$ and hence A is $\tilde{\theta_\mu}$-closed in (X, μ).
(4) Suppose that A is μ-closed in (X, μ), then $c_\mu(A) = A$. Thus by (2), $A = c_{\omega_{\tilde{\theta}}}(A)$ and hence A is $\omega_{\tilde{\theta}}$-μ-closed in (X, μ). □

Definition 3.3. *A GTS (X, μ) is said to be $\omega_{\tilde{\theta}}$-anti-locally countable if the intersection of any two $\omega_{\tilde{\theta}}$-μ-open sets is either empty or uncountable.*

The following lemma is used to prove the theorem which is stated below.

Lemma 3.3. *Let (X,μ) be $\omega_{\tilde{\theta}}$-anti-locally countable and $A \subseteq X$. If $A \in \omega_{\tilde{\theta}}$, then $c_{\tilde{\theta}_\mu}(A) = c_{\omega_{\tilde{\theta}}}(A)$.*

Proof. Suppose that $\emptyset \neq A \subseteq X$ and $A \in \omega_{\tilde{\theta}}$. By Theorem 3.2(1), $c_{\omega_{\tilde{\theta}}}(A) \subseteq c_{\tilde{\theta}_\mu}(A)$. To Show that $c_{\tilde{\theta}_\mu}(A) \subseteq c_{\omega_{\tilde{\theta}}}(A)$, let $x \in c_{\tilde{\theta}_\mu}(A)$ and $W \in \omega_{\tilde{\theta}}$ such that $x \in W$. Then by Lemma 2.2, there exists $U \in \tilde{\theta}_\mu$ and a countable set $C \subseteq \mathcal{M}_\mu$ such that $x \in U \setminus C \subseteq W$. Since $x \in U \cap c_{\tilde{\theta}_\mu}(A)$, $U \cap A \neq \emptyset$. Choose $y \in U \cap A$. Since $A \in \omega_{\tilde{\theta}}$, there exists $V \in \tilde{\theta}_\mu$ and a countable set $D \subseteq \mathcal{M}_\mu$ such that $y \in V \setminus D \subseteq A$. Since $y \in U \cap V$ and (X,μ) is $\omega_{\tilde{\theta}}$-anti-locally countable, then $U \cap V$ is uncountable. Thus, $(U \setminus C) \cap (V \setminus D) \neq \emptyset$ and hence $A \cap W \neq \emptyset$. Therefore, $x \in c_{\omega_{\tilde{\theta}}}(A)$. □

A subset A of GTS (X,μ) is said to be $\tilde{\theta_\mu}$-clopen(resp. $\omega_{\tilde{\theta}}$-μ-clopen) if it is both $\tilde{\theta_\mu}$-open and $\tilde{\theta_\mu}$-closed (resp. $\omega_{\tilde{\theta}}$-μ-open and $\omega_{\tilde{\theta}}$-μ-closed).

In the following, by using Lemma 3.3, we prove the main result in this section.

Theorem 3.4. *Let (X,μ) be $\omega_{\tilde{\theta}}$-anti-locally countable and $A \subseteq X$. Then, A is $\tilde{\theta_\mu}$-clopen if and only if A is $\omega_{\tilde{\theta}}$-μ-clopen.*

Proof. $\Rightarrow$) Suppose that A is $\tilde{\theta_\mu}$-clopen, then A and $X \setminus A$ are $\tilde{\theta_\mu}$-open. Since $\tilde{\theta}_\mu \subseteq \omega_{\tilde{\theta}}$, then A and $X \setminus A$ are $\omega_{\tilde{\theta}}$-μ-open, and hence A is $\omega_{\tilde{\theta}}$-μ-clopen.
$\Leftarrow$) Suppose that A is $\omega_{\tilde{\theta}}$-μ-clopen. Since A and $X \setminus A$ are $\omega_{\tilde{\theta}}$-μ-open, the by Lemma 3.3,

$$c_{\tilde{\theta}_\mu}(A) = c_{\omega_{\tilde{\theta}}}(A) \text{ and } c_{\tilde{\theta}_\mu}(x \setminus A) = c_{\omega_{\tilde{\theta}}}(X \setminus A).$$

Since A is $\omega_{\tilde{\theta}}$-μ-clopen., then

$$c_{\tilde{\theta}_\mu}(A) = c_{\omega_{\tilde{\theta}}}(A) = A \text{ and } c_{\omega_{\tilde{\theta}}}(X \setminus A) = X \setminus A.$$

Therefore,

$$c_{\tilde{\theta}_\mu}(A) = A \text{ and } c_{\tilde{\theta}_\mu}(X \setminus A) = X \setminus A$$

and hence A and $X \setminus A$ are $\tilde{\theta_\mu}$-closed sets. This means that A is $\tilde{\theta_\mu}$-clopen. □

Definition 3.4. *Let (X,μ) be a GTS and $A \subseteq X$. Then, we define the following notions:*
(1) $i_{\omega_{\tilde{\theta}}}(A) = \cup\{U \subseteq X : U \subseteq A,\ U \text{ is } \omega_{\tilde{\theta}}\text{-}\mu\text{-open}\}$;
(2) $i_{\tilde{\theta}}(A) = \cup\{U \subseteq X : U \subseteq A,\ U \text{ is } \tilde{\theta_\mu}\text{-open}\}$;
(3) $i_{\omega_\mu}(A) = \cup\{U \subseteq X : U \subseteq A,\ U \text{ is } \omega\text{-}\mu\text{-open}\}$.

Theorem 3.5. *For subsets A, B of GTS (X,μ), the following properties hold:*
(1) *if $A \subseteq B \subset X$, then $i_{\omega_{\tilde{\theta}}}(A) \subseteq i_{\omega_{\tilde{\theta}}}(B)$;*
(2) *for $A \subseteq X$, then $i_{\omega_{\tilde{\theta}}}(A) \subseteq A$;*
(3) *$i_{\omega_{\tilde{\theta}}}(i_{\omega_{\tilde{\theta}}}(A)) = i_{\omega_{\tilde{\theta}}}(A)$ for $A \subseteq X$;*
(4) *A is $\omega_{\tilde{\theta}}$-μ-open if and only if $i_{\omega_{\tilde{\theta}}}(A) = A$.*

Proof. The proof is obvious □

Corollary 3.2. *Let (X,μ) be a GTS and $A \subseteq X$. Then $i_{\tilde{\theta}_\mu}(A) \subseteq i_{\omega_{\tilde{\theta}}}(A) \subseteq i_{\omega_\mu}(A)$.*

Proof. To show that $i_{\tilde{\theta}_\mu}(A) \subseteq i_{\omega_{\tilde{\theta}}}(A)$, let $x \in i_{\tilde{\theta}_\mu}(A)$. Then there is $U \in \tilde{\theta}_\mu$ such that $x \in U \subseteq A$. By Theorem 2.4, U is $\omega_{\tilde{\theta}}$-μ-open. Thus $x \in i_{\omega_{\tilde{\theta}}}(A)$. To show that $i_{\omega_{\tilde{\theta}}}(A) \subseteq i_{\omega_\mu}(A)$, let $x \in i_{\omega_{\tilde{\theta}}}(A)$. Then there is $U \in \omega_{\tilde{\theta}}$ such that $x \in U \subseteq A$. Then by Theorem 2.4, U is ω-μ-open and hence $x \in i_{\omega_\mu}(A)$ □

Theorem 3.6. *Let (X,μ) be a GTS and $A \subseteq X$. Then the following properties hold:*
(1) $c_{\omega_{\tilde{\theta}}}(X \setminus A) = X \setminus i_{\omega_{\tilde{\theta}}}(A)$;
(2) $i_{\omega_{\tilde{\theta}}}(X \setminus A) = X \setminus c_{\omega_{\tilde{\theta}}}(A)$.

Proof. (1) Let $x \in c_{\omega_{\tilde{\theta}}}(X \setminus A)$ and $U \in \omega_{\tilde{\theta}}$ with $x \in U$. Since $x \in c_{\omega_{\tilde{\theta}}}(X \setminus A)$, $U \cap (X \setminus A) \neq \emptyset$. This implies that $x \notin i_{\omega_{\tilde{\theta}}}(A)$ and hence $x \in X \setminus i_{\omega_{\tilde{\theta}}}(A)$.
Conversely, for $x \in X \setminus i_{\omega_{\tilde{\theta}}}(A)$, $x \notin i_{\omega_{\tilde{\theta}}}(A)$, and then $U \cap (X \setminus A) \neq \emptyset$ for all $U \in \omega_{\tilde{\theta}}$ and $x \in U$ which implies $x \in c_{\omega_{\tilde{\theta}}}(X \setminus A)$.
(2) Let $x \in X \setminus c_{\omega_{\tilde{\theta}}}(A)$ if and only if $x \notin c_{\omega_{\tilde{\theta}}}(A)$ if and only if there is $U \in \omega_{\tilde{\theta}}$ with $x \in U$ such that $U \cap A = \emptyset$ if and only if $x \in i_{\omega_{\tilde{\theta}}}(X \setminus A)$. □

4. Conclusion

In this paper, we introduced the notion of $\omega_{\tilde{\theta}}$-μ-open sets in the sense of generalized topology given in [5]. We have proved that the collection of $\omega_{\tilde{\theta}}$-μ-open sets forms a generalized topology on X that lies between the class of $\tilde{\theta}_\mu$-open sets and the class of ω-μ-open sets. The relationships of $\omega_{\tilde{\theta}}$-μ-open and other well-known generalized open sets are given. Several properties of $\omega_{\tilde{\theta}}$-μ-open sets which enable us to prove certain of our results are studied and verified. In the upcoming work, we plan to : (1) introduce some concepts in GTS using $\omega_{\tilde{\theta}}$-μ-open sets such as connectedness, compactness and Lindelöfness; (2) introduce continuity and decomposition of continuity via $\omega_{\tilde{\theta}}$-μ-open sets.

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