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Unusual Nonpolynomial Van der Pol Oscillator Equations With Exact Harmonic and Isochronous Solutions

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Abstract. We do not know Van der Pol-type equations with nonlinear restoring force having explicitly an exact periodic solution. We present, for the first time, nonpolynomial Van der Pol oscillator equations that do not satisfy the classical existence theorems. We exhibit their exact harmonic and isochronous solutions and prove the existence of limit cycles by using averaging theory. We also present first integrals and exact solutions of polynomial Van der Pol-Duffing equations to show that they do not have any limit cycle. Additionally, we prove that the damped Duffing-type equations are equivalent to the conservative Duffing equations exhibiting nonoscillatory solutions.

1. Introduction

The Lienard equation:

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1.1}$$

where overdot is the derivative with respect to time and $f(x)$ and $g(x)$ are functions of x, is one of the most important autonomous second-order differential equations. This importance results in the fact that it often occurs in mathematical modelling of physical and engineering systems. Consequently, Equation (1.1) has been widely investigated in the literature. A celebrated equation of this type is the Van der Pol oscillator [1–3]:

$$\ddot{x} + \beta(x^2 - 1)\dot{x} + x = 0, \tag{1.2}$$

where $\beta \succ 0$, mentioned having a unique limit cycle only in the light of qualitative theory of differential equations and existence theorems [1–3] since it has no known exact and general solutions. Thus,

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the result known currently under the name of the Lienard-Levinson-Smith theorem [1–3] has been intensively applied to investigate the existence of limit cycles for the equations of type (1.1). In this way the Van der Pol-Duffing equation [4] is:

$$\ddot{x} + \beta(x^2 - 1)\dot{x} + \gamma x + \alpha x^3 = 0, \tag{1.3}$$

where α, β and γ are constants and has been considered for a long time as a self-excited oscillator. However, similar to the Van der Pol equation, Equation (1.3) is not integrable in general. Recently, Udwadia and Cho [5] succeeded in showing that equation (1.3) in the form:

$$\ddot{x} + \beta(x^2 - 1)\dot{x} - 3\alpha\left(1 + \frac{3\alpha}{\beta^3}\right)x + \alpha x^3 = 0, \tag{1.4}$$

does not have any limit cycle by explicitly calculating its general solution. In [6], the author established a first integral of equations of type (1.3) but in terms of special functions, namely, in terms of hypergeometric functions. Monsia et al. [4], for the first time, successfully derived the time-independent first integral of a more general form of equation (1.4) in terms of elementary functions such that it became possible to obtain the Lagrangian of equations of type (1.3). The above explicitly shows that the Van der Pol equation with polynomial nonlinear restoring force can have no limit cycle. Moreover, it is worth mentioning that the exact and general solution depicted by Udwadia and Cho [5] is not periodic. Recently, Akande et al. [7] successfully showed that the Lienard equation:

$$\ddot{x} - \frac{q_1 x}{\sqrt{q_2^2 - x^2}}\dot{x} = 0, \tag{1.5}$$

admits exact harmonic and isochronous periodic solutions, while this equation does not satisfy the classical existence theorems. This fact is also observed for the Lienard equations:

$$\ddot{x} \pm q_3\sqrt{q_4 q_5 - q_6 q_7 x^2}\dot{x} + q_4 q_7 x + q_4 q_7 x^2 - q_3 q_4 q_5 = 0, \tag{1.6}$$

studied by Akplogan et al. [8], where the parameters q_i, $i = 1, ..., 7$ are arbitrary constants. In this situation, let us consider the following equation with Van der Pol damping called Van der Pol-type equation:

$$\ddot{x} + \beta(x^2 - a)\dot{x} + \gamma x + \lambda(c_1 - c_2 x^2)^p = 0, \tag{1.7}$$

where $\beta \succ 0$, $\gamma \succeq 0$, $a \succ 0$, λ, c_1, c_2, and p are constants. When $\lambda = 0$, Equation (1.7) reduces to the Van der Pol equation (1.2) for $\gamma = 1$. Equation (1.7) generalizes the equation given in [[1] p.146] as an exercise for $c_1 = 0$, and $p = 1$. If $c_1 = 0$ and $p = \frac{3}{2}$, Equation (1.7) becomes the Van der Pol-Duffing equation (1.3) mentioned above. For $c_2 = 0$, Equation (1.7) takes the form of the biased Van der Pol equation, which appears in [[9] p.287]. To the best of our knowledge, such a Van der Pol equation with nonpolynomial restoring force has not been previously investigated in the literature. From this perspective, the question is to ask if Equation (1.7) can exhibit harmonic and limit cycle oscillations. Thus, the objective is to show the nonexistence of limit cycles for Van der Pol equations with polynomial restoring force and to prove the existence of the harmonic and limit cycle solutions of Equation (1.7). To that end, we first present some results for the Van der Pol equations with polynomial restoring force (Section 2) and second, explicitly prove the existence of the exact harmonic and limit cycle solutions of Equation (1.7) (Section 3). Finally, the results are compared to numerical solutions using the fourth-order Runge-Kutta algorithm, and we give a conclusion for the work.

2. Van der Pol equations with polynomial restoring force

In this part, we present time-independent first integrals allowing the determination of Lagrangian and exact solutions of Van der Pol-type equations with polynomial restoring forces.

2.1. **Van der Pol-Duffing equations.** Consider the equation:

$$\ddot{x} - (k_1 x^2 + 3k_2)\dot{x} + 2k_2^2 x - 2k_1^2 x^5 = 0, \tag{2.1}$$

where k_1 and k_2 are arbitrary parameters. A first integral of this equation can be written as follows:

$$\dot{x} = k_2 x + k_1 x^3. \tag{2.2}$$

Putting $k_1 = -\mu$ and $3k_2 = k$, one can, from Equation (2.1), arrive at:

$$\ddot{x} + (\mu x^2 - k)\dot{x} + \frac{2k^2}{9}x - 2\mu^2 x^5 = 0, \tag{2.3}$$

which is a quintic Duffing-Van der Pol equation.

Consider the equation:

$$\ddot{x} + (3k_1 x^2 - 2k_2)\dot{x} + k_2^2 x - k_1 k_2 x^3 = 0, \tag{2.4}$$

which denotes the cubic Duffing-Van der Pol equation. The first integral of Equation (2.4) has the form:

$$\dot{x} = k_2 x - k_1 x^3. \tag{2.5}$$

The cubic-quintic Duffing-Van der Pol equation:

$$\ddot{x} + k_1(x^2 - 1 - 2k_1)\dot{x} + 2k_1^3 x + 2k_1^2(1 - k_1)x^3 - 2k_1^2 x^5 = 0, \tag{2.6}$$

has the time-independent first integral:

$$k_1 x^5 = k_1 x^3 - x^2 \dot{x}. \tag{2.7}$$

The first integral of the equation:

$$\ddot{x} - k_2(1 - x^2)\dot{x} - k_1(k_1 + k_2)x + \frac{k_1 k_2}{3}x^3 - k_1 k_3 = 0, \tag{2.8}$$

can be written as follows:

$$\dot{x} = (k_1 + k_2)x - \frac{k_2}{3}x^3 + k_3, \tag{2.9}$$

where k_3 is an arbitrary parameter.

Consider the equation:

$$\ddot{x} - k_1(1-x^2)\dot{x} + k_2(k_1-k_2)x - \frac{k_1k_2}{3}x^3 = 0. \tag{2.10}$$

Its first integral can be read:

$$\dot{x} = (k_1-k_2)x - \frac{k_1}{3}x^3. \tag{2.11}$$

2.2. **Generalized Van der Pol-type equation.** Now, consider the more general Van der Pol-type equation with polynomial restoring force:

$$\ddot{x} + \left(\frac{k_2}{k_3}\ell + \frac{k_1}{k_3}x^{1-\ell}\right)\dot{x} + \frac{(\ell-1)k_1^2}{k_3^2}x^{3-2\ell} + \frac{2(\ell-1)k_1k_2}{k_3^2}x^{2-\ell} + \frac{(\ell-1)k_2^2}{k_3^2}x = 0. \tag{2.12}$$

One can verify that Equation (2.12) has the time-independent first integral:

$$\dot{x} = -\frac{1}{k_3}\left(k_1x^{2-\ell} + k_2x\right), \tag{2.13}$$

where ℓ is an arbitrary parameter. It suffices to put $\ell = (2-n)$ into Equation (2.12) to recover the general form considered in [6]. The author derived the first integral of this type of Equation (2.12) in terms of hypergeometric functions. Equation (2.12) with $\ell = (2-n)$ has also been investigated by Chandrasekar et al. [10]. The first integrals derived by these authors [10] using the so-called generalized extended Prelle-Singer method are functions of time. The general solution of Equation (2.12), as can be verified, using the first integral (2.13), becomes:

$$x(t) = \left[\frac{1}{k_2}\left(-k_1 + e^{-\frac{k_2}{k_3}(\ell-1)(t+K)}\right)\right]^{\frac{1}{\ell-1}}, \tag{2.14}$$

where $\ell \neq 1$, and K is an integration constant. An interesting case of Equation (2.12) consists of putting $\ell = -1$ to obtain:

$$\ddot{x} + \left(\frac{k_1}{k_3}x^2 - \frac{k_2}{k_3}\right)\dot{x} - \frac{2k_1^2}{k_3^2}x^5 - \frac{4k_1k_2}{k_3^2}x^3 - \frac{2k_2^2}{k_3^2}x = 0. \tag{2.15}$$

The Van der Pol-Duffing equation (2.15) is equivalent to the conservative cubic-quintic Duffing equation:

$$\ddot{x} - \frac{3k_1^2}{k_3^2}x^5 - \frac{4k_1k_2}{k_3^2}x^3 - \frac{k_2^2}{k_3^2}x = 0, \tag{2.16}$$

obtained by using the first integral (2.13), where $\ell = -1$. In this regard, the general solution of Equation (2.15) or (2.16), taking into consideration Equation (2.14), can read:

$$x(t) = \frac{(k_2)^{\frac{1}{2}}}{\left[e^{\frac{2k_2}{k_3}(t+K)} - k_1\right]^{\frac{1}{2}}}. \tag{2.17}$$

This result shows that the widely studied cubic-quintic Duffing equation is (2.12) in fact a pseudo-oscillator. From Equation (2.13), the more general Van der Pol-type equation with polynomial nonlinear restoring force becomes equivalent to the generalized Duffing equation:

$$\ddot{x}-\frac{k_1k_2}{k_3^2}(3-\ell)x^{2-\ell}-\frac{k_1^2}{k_3^2}(2-\ell)x^{3-2\ell}-\frac{k_2^2}{k_3^2}x=0. \tag{2.18}$$

Conservative equation (2.18) also has the general solution (2.14). It is worth noting that the conservative cubic-quintic Duffing equation (2.16) is equivalent by using the first integral (2.13) to the dissipative Lienard equation:

$$\ddot{x}+\frac{4k_2}{k_3}\dot{x}-\frac{3k_1^2}{k_3^2}x^5+\frac{3k_2^2}{k_3^2}x=0, \tag{2.19}$$

with the general solution (2.17). Equation (2.19) is also known as the damped-quintic Duffing equation.The general solution (2.17) of Equation (2.19) contradicts the results of existence of the bounded periodic solutions exhibited in [11]. The above shows that the Van der Pol-type equations with polynomial restoring force do not have any limit cycle. That being so, we can establish the exact general harmonic solution and limit cycle of Equation (1.7).

3. Exact harmonic and limit cycle solutions

3.1. **Exact harmonic and isochronous solutions.** In this part, we look for Equation (1.7), an exact harmonic solution:

$$x(t)=Acos(wt+\varphi), \tag{3.1}$$

where A, w and φ are arbitrary parameters. Thus, substituting Equation (3.1) into Equation (1.7) yields, after a few algebraic calculations, the results are $c_1=a=A^2$, $\gamma=w^2$, $c_2=1$, $\lambda=\pm\beta w$, and $p=\frac{3}{2}$. The arbitrary constant φ can be determined by using initial conditions. From these integrability conditions, the desired Van der Pol-type equation (1.7) with nonlinear restoring force takes the form:

$$\ddot{x}+\beta(x^2-A^2)\dot{x}+w^2x\pm\beta w(A^2-x^2)^{\frac{3}{2}}=0. \tag{3.2}$$

Comparing Equation (3.2) with Lienard equation (1.1) yields $f(x)=\beta(x^2-A^2)$ and $g(x)=w^2x\pm\beta w(A^2-x^2)^{\frac{3}{2}}$. Hence, for $g(x)=w^2x-\beta w(A^2-x^2)^{\frac{3}{2}}$, Equation (3.2) is written:

$$\ddot{x}+\beta(x^2-A^2)\dot{x}+w^2x-\beta w(A^2-x^2)^{\frac{3}{2}}=0, \tag{3.3}$$

which admits the exact and general solution (3.1). For $g(x)=w^2x+\beta w(A^2-x^2)^{\frac{3}{2}}$, Equation (3.2) becomes:

$$\ddot{x}+\beta(x^2-A^2)\dot{x}+w^2x+\beta w(A^2-x^2)^{\frac{3}{2}}=0, \tag{3.4}$$

and has the exact and general solution:

$$x(t)=Asin(wt+\varphi_1), \tag{3.5}$$

where φ_1 is an arbitrary constant that can be calculated by the application of initial conditions.

3.2. **Existence theorem analysis.** As seen, the functions $f(x)$ and $g(x)$ do not satisfy the conditions required by usual theorems for the existence of a centre at the origin [1–3,12]. For example, according to Theorem 11.3 of [[1] p.390], the origin is a centre for the Lienard equation (1.1) when $f(x)$ and $g(x)$ are odd, and $g(0) = 0$. However, the previous expression of $f(x)$ is not odd but rather even. The previous formulas of $g(x)$ are not odd, and $g(0) \neq 0$ while the general solutions (3.1) and (3.5) show that the origin is an isochronous centre for Equations (3.3) and (3.4).

3.3. **Phase plane analysis.** Consider Equation (3.3). Then, the equivalent dynamical system can be written as follows:

$$\begin{cases} \dot{x} = y \\ \\ \dot{y} = -\beta(x^2 - A^2)y - w^2 x + \beta w(A^2 - x^2)^{\frac{3}{2}}. \end{cases} \tag{3.6}$$

The equilibrium points are given by $y = 0$, and:

$$\beta w(A^2 - x^2)^{\frac{3}{2}} - w^2 x = 0. \tag{3.7}$$

One can easily observe that $x = 0$ does not satisfy Equation (3.7). Therefore, for the qualitative theory of differential equations, Equation (3.3) cannot have the origin as an equilibrium point, which contradicts the exact and general harmonic solutions (3.1). According to Equation (3.7), $x = 0$, when:

$$\beta w A^3 = 0. \tag{3.8}$$

Equation (3.8) holds only, as $w \neq 0$, and $A \neq 0$, when $\beta = 0$. In this case, Equation (3.3) reduces to the linear harmonic oscillator equation:

$$\ddot{x} + w^2 x = 0. \tag{3.9}$$

The previous analysis also holds for Equation (3.4). The above shows that the classical theorems for the existence of isochronous centres clearly exclude a number of Lienard equations, as seen in several previous papers [7,8].

3.4. **Application of averaging method for equation (3.3).** In this part, we investigate the existence of a limit cycles for Equation (3.3) using the averaging method. Equation (3.3) can be written as follows:

$$\ddot{x} + w^2 x + \beta\left[(x^2 - A^2)\dot{x} - w(A^2 - x^2)^{\frac{3}{2}}\right] = 0. \tag{3.10}$$

The equation (3.10) has the form:

$$\ddot{x} + w^2 x = \varepsilon F(\dot{x}, x), \tag{3.11}$$

where $F(\dot{x}, x) = -(x^2 - A^2)\dot{x} + w(A^2 - x^2)^{\frac{3}{2}}$ and $\beta = \varepsilon$, such that ε is small parameter, that is, $0 < \varepsilon \ll 1$, as required in the application of the averaging method. Now, we seek for Equation (3.11) the solution of the form:

$$x(t) = r(t)cos\Big(wt + \phi(t)\Big), \tag{3.12}$$

under the initial conditions:

$$x(0) = A, \qquad\qquad \dot{x}(0) = 0, \tag{3.13}$$

such that:

$$\dot{x}(t) = -wr(t)sin\Big(wt + \phi(t)\Big), \tag{3.14}$$

and

$$\dot{r}(t)cos\Big(wt + \phi(t)\Big) - r(t)\dot{\phi}(t)sin\Big(wt + \phi(t)\Big) = 0. \tag{3.15}$$

Hence, knowing that:

$$(A^2 - x^2)^{\frac{3}{2}} = (A^2 - \frac{r^2(t)}{2})\Big[1 - \frac{3r^2(t)}{4(A^2 - \frac{r^2(t)}{2})}cos(2wt + 2\phi(t))\Big], \tag{3.16}$$

one can, after a little algebraic manipulation, obtain:

$$\begin{aligned} F(\dot{x}, x) = w\Big[A^2 - \frac{r^2(t)}{2}\Big]^{\frac{3}{2}} - \frac{3}{4}w\Big[A^2 - \frac{r^2(t)}{2}\Big]^{\frac{1}{2}} r^2(t)cos(2wt + 2\phi(t)) + \\ \frac{1}{4}wr^3(t)sin(3wt + 3\phi(t)) + wr(t)\Big[\frac{r^2(t)}{4} - A^2\Big]sin(wt + \phi(t)). \end{aligned} \tag{3.17}$$

From:

$$\dot{r}(t) = -\frac{\varepsilon}{2\pi w}\int_0^{2\pi} F\Big[r(t)cos(wt + \phi(t)), -wr(t)sin(wt + \phi(t))\Big]sin(wt + \phi(t))d(wt + \phi(t)), \tag{3.18}$$

and

$$\dot{\phi}(t) = -\frac{\varepsilon}{2\pi wr(t)}\int_0^{2\pi} F\Big[r(t)cos(wt+\phi(t)), -wr(t)sin(wt+\phi(t))\Big]cos(wt+\phi(t))d(wt+\phi(t)). \tag{3.19}$$

After a few algebra, it results in:

$$\dot{r}(t) = \frac{\varepsilon r(t)}{8}\Big[4A^2 - r^2(t)\Big] \tag{3.20}$$

Integrating, after separation of variables, yields:

$$r(t) = \frac{2A}{\Big[1 + e^{-\varepsilon A^2(t+K)}\Big]^{\frac{1}{2}}}, \tag{3.21}$$

where K is a constant of integration. Using the initial conditions (3.13), the following is obtained:

$$r(t) = \frac{2A}{\left[1 + 3e^{-\varepsilon A^2 t}\right]^{\frac{1}{2}}}, \tag{3.22}$$

where $e^{-\varepsilon A^2 K} = 3$. Now, from Equation (3.19):

$$\dot{\phi}(t) = 0,$$

that is:

$$\phi(t) = \varphi_0, \tag{3.23}$$

where φ_0 is a constant. In this context, the desired solution (3.12) takes the form:

$$x(t) = \frac{2A}{\left[1 + 3e^{-\varepsilon A^2 t}\right]^{\frac{1}{2}}} cos(wt + \varphi_0). \tag{3.24}$$

When $A = w = 1$, and $\varphi_0 = 0$, the solution (3.23) becomes:

$$x(t) = \frac{2}{\sqrt{1 + 3e^{-\varepsilon t}}} cos(t). \tag{3.25}$$

It is worth mentioning that the formula (3.25) is the solution obtained for the Van der Pol oscillator equation:

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0, \tag{3.26}$$

by Strogatz [[9], p.225] using the averaging method.

4. Numerical applications

To do so, it is first necessary to determine the constants φ and φ_1 from the initial conditions.

4.1. Solution (3.1) in terms of x_0 and v_0.

From the general initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$, one can obtain, using the general solution (3.1), the system of algebraic equations:

$$x_0 = Acos\varphi, \qquad v_0 = -wAsin\varphi, \tag{4.1}$$

which yields the constant:

$$\varphi = arctan\left(-\frac{v_0}{wx_0}\right). \tag{4.2}$$

Using this expression, the general solution (3.1) takes the form:

$$x(t) = Acos\left[wt - arctan\left(\frac{v_0}{wx_0}\right)\right]. \tag{4.3}$$

Figure 1 shows the graphical comparison of the result (4.3) in the circles line with the solution obtained by numerical integration of Equation (3.3) in solid line where $A = 1$, $w = 1$, $\beta = 0.01$, $x_0 = 1$, and $v_0 = 0.001$.

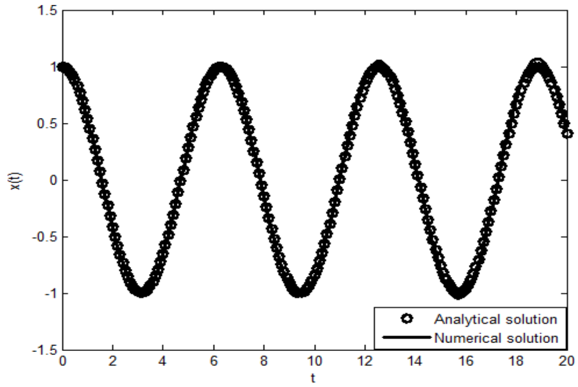

Figure 1. Comparison of solution (4.3) to the numerical solution of Equation (3.3). Typical values are $A = 1$, $w = 1$, $\beta = 0.01$, $x_0 = 1$, and $\vartheta = 0.001$.

4.2. **Solution (3.12) in terms of x_0 and v_0.** Under the general initial conditions that $x(0) = x_0$ and $\dot{x}(0) = v_0$, solution (3.5) leads to the system of algebraic equations:

$$x_0 = Asin\varphi_1, \qquad\qquad v_0 = wAcos\varphi_1, \tag{4.4}$$

such that the constant φ_1 is defined as:

$$\varphi_1 = arccotan\left(\frac{v_0}{wx_0}\right). \tag{4.5}$$

From this, the general solution (3.5) can be rewritten in the form:

$$x(t) = Asin\left[wt + arccot\left(\frac{v_0}{wx_0}\right)\right]. \tag{4.6}$$

The graphical comparison of this solution (4.6) in the circles line with the result obtained by numerical integration of Equation (3.4) is depicted in Figure 2, where $A = 1$, $w = 1$, $\beta = 0.01$, $x_0 = 1$, and $v_0 = 0.001$.

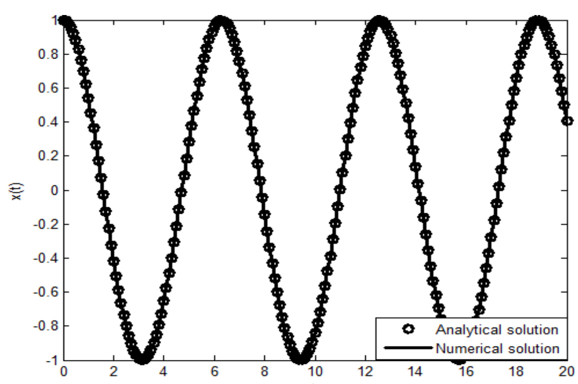

Figure 2. Comparison of solution (4.6) to the numerical solution of Equation (3.4). Typical values are $A = 1$, $w = 1$, $\beta = 0.01$, $x_0 = 1$, and $\vartheta = 0.001$.

5. Conclusion

We have presented exceptional nonpolynomial Van der Pol oscillator equations in this paper. We have exhibited their exact harmonic and limit cycle solutions while they do not satisfy classical theorems for the existence of at least one periodic solution. We have proven that the Van der Pol-Duffing-type equations are equivalent to the conservative Duffing equations so that they could not admit limit cycle oscillations. Additionally, we have shown that the damped quintic Duffing equations are equivalent to the conservative cubic-quintic Duffing equations. It was for the first time such results have been obtained in the literature.

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