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Mappings and Finite Product of Pairwise Expandable Spaces

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Abstract. In this work, we intend to study certain mappings in bitopological spaces such as pairwise perfect and pairwise countably perfect mappings that possess the property (E). Some properties of such mappings are provided which have helped us to obtain some finite product theorems concerning with pairwise expandable and almost pairwise expandable spaces. Two illustrative examples are given to demonstrate the effectiveness of some proposed results.

1. Introduction

A bitopological space (X, τ_1, τ_2) is a non-empty set X with two arbitrary topologies τ_1, τ_2. The idea of the bitopological space was induced by topologies generalized by the following two sets:

$$B_{\rho_\epsilon} = \{y \in X \mid \rho(x, y) \leq \epsilon\}$$

and

$$B_{\delta_\epsilon} = \{y \in X \mid \delta(x, y) \leq \epsilon\},$$

where ρ and δ are quasi-metric spaces of X with $\rho(x, y) = \delta(y, x)$. From 1963, when Kellay introduced the concept of bitopological space, several topological properties, which are already included in a single topology, are generalized into bitopological spaces [1]. Some of these properties are compactness,

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paracompactness, separation axioms, connectedness, some special types of functions and many others [1–4]. Many authors studied and investigated these bitopological spaces after Kelly, like Fletcher et al. [5], Birsan [6], Reilly [7], Datta [8], Hdeib and Fora [9], Bose et al. [10], Killiman and Salleh [11], Abushaheen et al. [12], and Qoqazeh et al. [13]. For further exploration in this field, this work aims to present some properties of the pairwise perfect and the pairwise countably perfect mappings for the purpose of using them to obtain some novel finite product theorems concerning with pairwise expandable and almost pairwise expandable spaces.

To go forward in this paper and for more simplification, we state some notations and notions which will be used later on. In particular, the closure and interior of the set A will be denoted respectively by $CL(A)$ and $Int(A)$ with noting that (X, τ) is a topological space. Besides, if A is a subset of a bitopological space $X = (X, \tau_1, \tau_2)$, then the relative topology, which is a subspace topology of X, on the set A inherited by τ will be denoted by τ_A. The cardinality of the set Δ will be denoted by $|\Delta|$. The sets $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{N}$, and $\mathbb{Z}$ will denote the sets of real numbers, rational numbers, natural numbers and integer numbers, respectively. On the other hand, ω_0 and ω_1 will denote respectively the first two uncountable ordinals, and m will denote generally for an infinite cardinal. Finally, the terms τ_u, τ_{dis}, τ_{cof} and τ_{coc} will denote the usual, discrete, cofinite and the co-countable topologies, respectively.

2. Basic Definitions

In the following content, we state certain definitions and preliminaries associated with the bitopological space for completeness.

Definition 2.1. *[14] A collection subset $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ of a bitopological space (X, τ_1, τ_2) is said to be pairwise locally finite if for each $x \in X$ there exist an τ_1–open set U containing x such that U intersects only finitely many members of $\tilde{F}$, or there exist τ_2– open set V containing x such that V intersects only finitely many members of $\tilde{F}$.*

Definition 2.2. *[15] A P–open cover $\tilde{V}$ of a bitopological space (X, τ_1, τ_2) is called parallel refinement of a P–open cover $\tilde{U}$ of X if each τ_i–open set of $\tilde{V}$ is contained in some τ_i–open set of $\tilde{U}$, where $i = 1, 2$.*

Definition 2.3. *[8] A bitopological space X is called P–m–paracompact, if every P–open cover $\tilde{U}$ of X, so that $\left|\tilde{U}\right| \leq m$, has a pairwise locally finite open parallel refinement. If $m = \omega_0$, then the space X is called P–countably paracompact. If the space X is P–m–paracompact for every m, then X is called P–paracompact.*

Definition 2.4. *[14] Let m be an infinite cardinal, then the bitopological space (X, τ_1, τ_2) is called τ_ι–m–expandable space with respect to τ_j if for every τ_ι–locally finite $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ with $|\triangle| \leq m$, there exist τ_j–locally finite collection $\tilde{G} = \{G_\alpha : \alpha \in \Delta\}$ of open subsets of X such that $F_\alpha \subset G_\alpha$ for all $\alpha \in \Delta$ and for $i \neq j$, where $i, j = 1, 2$.*

Definition 2.5. *[14] A bitopological space (X, τ_1, τ_2) is called τ_i–expandable with respect to τ_j, if it is an τ_i–m–expandable for every cardinal m and $i \neq j$ where $i, j = 1, 2$.*

Definition 2.6. *[14] A bitopological space $(X, \tau_{1,}\tau_2)$ is called a pairwise expandable (or simply P–expandable), if it is P–T_2–space and it is τ_1–expandable with respect to τ_2 and τ_2– expandable with respect to τ_1.*

Definition 2.7. *Let $X = (X, \tau_1, \tau_2)$, $Y = (Y, \sigma_1, \sigma_2)$ be two bitopological spaces and $f : X \longrightarrow Y$ be a given map. Then f is called pairwise closed (P–closed) map if it maps a τ_i–closed subset of X onto a σ_i–closed subset of Y, for each $i = 1, 2$.*

Definition 2.8. *let $f : X \longrightarrow Y$ be a P–closed, P–continuous map from a bitopological space $X = (X, \tau_1, \tau_2)$ onto a bitopological space $Y = (Y, \sigma_1, \sigma_2)$ such that $f^{-1}(y)$ is compact for each $y \in Y$. Then the map f is called pairwise perfect (P–perfect) map.*

Definition 2.9. *According to definition 2.2 above, if the preimage $f^{-1}(Y)$ is P–countably compact for each $y \in Y$ then f is called a pairwise countably perfect (P–countably perfect) map.*

Definition 2.10. *The topologies τ_i and τ_j on a nonempty set X are said to have the property (E) if $A \in \lambda C(X, \tau_i)$ and $B \in \lambda C(X, \tau_j)$ imply $A \cap B \in \lambda C(X, \tau_i)$.*

3. Main Theoretical Results

In this part, we intend to state and derive some novel significant theoretical results in light of the aforesaid definitions. At the beginning, one can observe that it is easy to prove the following two lemmas:

Lemma 3.1. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a P–countably perfect map. Then we have:*

- *If $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ be a P–locally finite collection of subsets of Y, then $f^{-1}(\tilde{F}) = \{f^{-1}(F_\alpha) : \alpha \in \Delta\}$ is a P–locally finite collection of X.*
- *If $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ be a P–locally finite collection of subsets of X, then $f(\tilde{F}) = \{f(F_\alpha) : \alpha \in \Delta\}$ is also P–locally finite collection of subsets of Y.*

Lemma 3.2. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a P–closed map, if $\tilde{G} = \{G_\alpha : \alpha \in \Delta\}$ be a P–locally finite of subsets of X and $\tilde{V} = \{V_\alpha : \alpha \in \Delta\}$ be a σ_i–open in Y for $i = 1, 2$ such that $V_\alpha \subseteq f(G_\alpha)$ for $\alpha \in \Delta$, then the family $\{f(G_\alpha) : \alpha \in \Delta\}$ is P–locally finite collection of subsets of Y.*

Next, based on the two lemmas provided above, we state and prove the following important theorem that concerns with the *P–m*-expandable space.

Theorem 3.1. *Let f be a P–countably perfect map from a bitopological space $X = (X, \tau_1, \tau_2)$ onto a bitopological space $Y = (Y, \sigma_1, \sigma_2)$. Then X is P–m–expandable if and only if Y is P–m–expandable.*

Proof. $\Rightarrow$) Suppose $X = (X, \tau_1, \tau_2)$ is P–m–expandable, and $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ is a P–locally finite collection of subsets of Y with $|\Delta| \leq m$. Then, by Lemma 3.2, the following mapping:

$$f^{-1}(\tilde{F}) = \{f^{-1}(F_\alpha) : \alpha \in \Delta\}$$

will be a P–locally finite collection of subsets of X with $|\Delta| \leq m$. Now, since X is P–m–expandable, then there exists a P–locally finite collection $\tilde{G} = \{G_\alpha : \alpha \in \Delta\}$ of open subsets of X such that $f^{-1}(F_\alpha) \subseteq G_\alpha$, for each $\alpha \in \Delta$. Set

$$V_\alpha = Y - f(X - G_\alpha), \ \alpha \in \Delta.$$

Consequently, we have the following claim:
Claim $F_\alpha \subseteq G_\alpha$.
To prove this claim, we have clearly:

$$X - G_\alpha \subseteq X - f^{-1}(F_\alpha),$$

and hence $f(X - G_\alpha) \subseteq Y - F_\alpha$. Again, we have:

$$F_\alpha \subseteq Y - f(X - G_\alpha).$$

This consequently implies that $F_\alpha \subseteq V_\alpha$, for each $\alpha \in \Delta$.
Now, it remains to show that $\tilde{V} = \{V_\alpha : \alpha \in \Delta\}$ is a P–locally finite collection of open subsets of Y. For this purpose, we should note that V_α is σ_i–open in Y, for $i = 1, 2$. Now, since f is P–closed map, $V_\alpha \subseteq f(G_\alpha)$, and by Lemma 3.2, the family $\{f(G_\alpha) : \alpha \in \Delta\}$ will be a P–locally finite collection of subsets of Y. Therefore, we conclude that $\tilde{V}$ is locally finite collection of subsets of Y, which implies that Y is indeed a P–m–expandable.
$\Leftarrow$) In order to show the converse inclusion, we assume that Y is P–m–expandable and $\tilde{F} = \{F_\alpha : \alpha \in \Delta\}$ is a P–locally finite collection of subsets of X such that $|\Delta| \leq m$. By Lemma 3.1, we can deduce that:

$$f(\tilde{F}) = \{f(F_\alpha) : \alpha \in \Delta\}$$

represents a P–locally finite collection of subsets of Y. Hence, there exists P–locally finite collection of subsets of σ_i–open subsets of Y, for $i = 1, 2$, say $\tilde{G} = \{G_\alpha : \alpha \in \Delta\}$, such that $F_\alpha \subseteq G_\alpha$ for each $\alpha \in \Delta$. Then:

$$F_\alpha \subseteq f^{-1}\left(f(F_\alpha)\right) \subseteq f^{-1}(G_\alpha).$$

Since f is P–continuous, we get $f^{-1}(G_\alpha)$ is τ_i–open in X for $i = 1, 2$. Now, by Lemma 3.1, the following collection

$$f^{-1}(\tilde{G}) = \{f^{-1}(G_\alpha) : \alpha \in \Delta\}$$

will be a τ_i–open locally finite collection for $i = 1, 2$. Hence, X is P–m–expandable. □

In the same context, we should notice that the P–closed continuous image of the P–expandable space need not be P–expandable. However, The following example aims to illustrate this assertion.

Example 3.1. *Let $W = (W, \tau, \tau)$ where $W - [0, \omega_1] \times [0, \omega_1)$ and τ be an order topology define on W and $X = W \times N$ where $N = (N, \tau_{dis}, \tau_{dis})$ and τ_{dis} is a discrete topology. Clearly, W is P–countably compact and hence X is p–expandable. Now, let*

$$f_1 : (X, \tau \times \tau_{dis}) \longrightarrow (Y, \sigma_1)$$

be a P–closed onto continuous map. We have illustrated that Y is not ω_o–expandable. Hence,

$$f_2 : (X, \tau \times \tau_{dis}, \tau \times \tau_{dis}) \longrightarrow (Y, \sigma_1, \sigma_2)$$

is a P–closed onto continuous map and then Y is not P–ω_o–expandable.

On the other hand, in order to turn into the other theoretical results, we will recall bellow the following definition that relates with the P–expandable space.

Definition 3.1. *Let $X = (X, \tau_1, \tau_2)$ be a bitopological space. A τ_1–subset F is called P–expandable relative to X if and only if every P–open, P–A_σ–cover of F in X has a P–open, P–locally finite refinement in X.*

Thus, we are now ready to introduce the following important results that deal with the P–expandable space.

Lemma 3.3. *Let M be a τ_i–closed expandable subset of a bitopological space $X = (X, \tau_1, \tau_2)$. If F is τ_i–closed in $Int(M)$, then F is P–expandable relative to X.*

Proof. Let $\tilde{U}$ be a P–open, P–A_σ–cover of F in X. Then:

$$\tilde{B} = \{U \cap M : U \in \tilde{U}\} \cup (M - F)$$

will be an P–open, P–A_σ–cover of M. Since M is τ_i–expandable, then there exists a P–open, P–locally finite refinement of $\tilde{B}$, say $\tilde{V}$, such that for each $V \in \tilde{V}$, there exists $B \in \tilde{B}$ such that $V \subseteq B$. Now, let

$$\tilde{W} = \{V \cap Int(M) : v \in \tilde{V}\},$$

then $\tilde{W}$ will be a P–open locally finite refinement of $\tilde{U}$ in X. Therefore, F is indeed a P–expandable relative to X. □

Theorem 3.2. *Let $i = 1, 2$ and $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a P–continuous map from a bitopological space X onto a bitopological space Y. Then f is P–closed if and only if for each $y \in Y$ and each τ_i–open set U in X such that $f^{-1}(y) \subseteq U$, there exists a σ_i–open set O_y in Y containing y and $f^{-1}(O_y) \subseteq U$.*

Proof. $\Rightarrow$) Let $i = 1, 2$ and suppose f is P–closed. Let $y \in Y$ and let U be any τ_i–open set such that $f^{-1}(y) \subseteq U$. If one lets $A = X - U$, then $f(A) = f(X - U)$ will be σ_i–closed in Y. Moreover, letting

$$O_y = Y - f(X - U)$$

yields O_y to be σ_i–open in Y and $y \in O_y$, which consequently implies that $f^{-1}(O_y) \subseteq U$. Hence, if we let $t \in f^{-1}(O_y)$, then $f(t) \in O_y$ and $f(t) \notin f(X-U)$, which implies $t \notin X-U$, and therefore $t \in U$.

$\Leftarrow$) Conversely, let A be any τ_i–closed subset of X and $y \in Y - f(A)$. Since f is onto, then there exists $x \in X$ such that $y = f(x)$. Now, we have the following claim:

Claim $f^{-1}(y) \cap A = \phi$.

To prove this claim, suppose that $f^{-1}(y) \cap A \neq \phi$. Then, there exists $z \in f^{-1}(y)$ and $z \in A$, which implies that $f(z) = y$. But $f(z) = Y - f(A)$, which gives that $f(z) \notin f(A)$. Hence, $z \notin A$. This yields a contradiction.

Thus, $f^{-1}(y) \cap A = \phi$ or $f^{-1}(y) \subseteq X - A$. Consequently, due to $X - A$ is τ_i–open, then by assumption there exists a σ_i–open set O_y in Y such that $f^{-1}(O_y) \subseteq X - A$. Now, we have:

$$y \in O_y \subseteq f(X - A) = Y - f(A).$$

This implies that $Y - f(A)$ is a σ_i–open subset of Y, which means that $f(A)$ is σ_i–closed. Hence, we have f is P–closed. □

Corollary 3.1. *Let X be a P–compact space and $Y = (Y, \sigma_1, \sigma_2)$ be any bitopological space. Then, the projection map $\pi : X \times Y \longrightarrow Y$ is P–closed map.*

Proof. The proof follows directly from Theorem 3.2. □

Theorem 3.3. *Let $X = (X, \tau_1, \tau_2)$ be a P–expandable space and $Y = (Y, \sigma_1, \sigma_2)$ be a P–compact space. Then, $X \times Y$ is P–expandable.*

Proof. The projection map $\pi : X \times Y \longrightarrow X$ is a P–closed map by Corollary 3.1 and because

$$\pi^{-1}(x) = \{x\} \times Y \cong Y$$

is P–compact, for each $x \in X$. Therefore, Y is P–countably compact, which implies that π is a P–countably perfect map. Hence, by Theorem 3.2, we conclude that $X \times Y$ is P–expandable. □

It is worth mentioning that if X and Y are two P–expandable bitopological spaces, then $X \times Y$ need not be in general P–expandable. However, this assertion can be illustrated by the following example.

Example 3.2. *Suppose $X = (X, \tau_s, \tau_s)$ so that τ_s represents the Sorgenfrey topology. Due to X is P–Lindelof and P–T_3–space, then X is P–paracompact and thus P– expandable. But $X \times X$ is not P–expandable since the collection $\tilde{F} = \{(x, -x) : x \in X\}$ is P–locally finite. However, one can show, by a category argument, that there is no P–locally finite collection of open sets $\{G_x : x \in X\}$ such that $(x, -x) \in G_x$, for each $x \in X$.*

Theorem 3.4. *Let $X = (X, \tau_s, \tau_s)$ be a P–regular, P–subparacompact and P–expandable bitopological space and $Y = (Y, \sigma_1, \sigma_2)$ be a P–expandable bitopological space. Then $X \times Y$ is P–expandable* *if and only if for each $x \in X$ there is a τ_i–open set $U(x)$ of X such that $CL(U(x)$ is P–expandable, for each $i = 1, 2$.*

Proof. $\Rightarrow$) Suppose $X \times Y$ is P–expandable. Let $x \in X$ and take X to be the τ_i–open set containing x. This implies that $CL(X) \times Y = X \times Y$ is P–expandable.
$\Leftarrow$) Conversely, suppose that for each $x \in X$ there exist a τ_i–open set of x, $U(x)$, such that $CL(U(x)) \times Y$ is P–expandable. It suffices to show that the following projection map:

$$\pi : X \times Y \longrightarrow X$$

has property (E). Let $\tilde{U}$ be a P–open, P–A_σ–cover of $X \times Y$. Now, by P–regularity, and due to there is a τ_i–open set of x, $U(x)$, such that $x \in U(x)$, for each $x \in X$ and for $i = 1, 2$, then there is a τ_i–open set $V(x)$ of x such that:

$$x \in V(x) \subseteq CL(V(x)) \subseteq U(x) \subseteq CL(U(x)).$$

Then

$$V(x) \times Y \subseteq CL(V(x)) \times Y \subseteq CL(U(x)) \times Y.$$

By Lemma 3.3, $CL(V(x)) \times Y$ is P–expandable relative to $X \times Y$, which consequently implies that $\tilde{U}$ has a P–open, P–locally finite refinement in $X \times Y$, say $\tilde{A}(x)$ such that:

$$CL(V(x)) \times Y \subseteq \cup\{\tilde{A}(x) : x \in X\}.$$

Now, we have:

$$\tilde{A} = \cup\{\tilde{A}(x) : x \in X\}$$

is P–open, P–refinement of $\tilde{U}$. Further, $\tilde{V} = \{V(x) : x \in X\}$ is a P–open cover of X such that:

$$\pi^{-1}(V(x)) = V(x) \times Y \subseteq \bigcup_{x \in X} \tilde{A}(x),$$

where $\tilde{A}(x)$ is a P–locally finite subfamily of $\tilde{A}$. Since X is P–subparacompact, then $\tilde{V}$ is a P–open, $P - A_\sigma$–cover of X. Therefore, we have for each P–open, P–A_σ–cover $\tilde{U}$ of $X \times Y$, there is a P–open, P–A_σ–cover $\tilde{V}$ of X and P–open refinement $\tilde{A}$ of $\tilde{U}$ such that $\pi^{-1}(V(x))$ is contained in the union of P–locally finite subfamily of $\tilde{A}$, for each $x \in X$ and $V(x) \in \tilde{V}$. Hence, π has property (E), and $X \times Y$ is P–expandable space. $\square$

Corollary 3.2. *Let $X = (X, \tau_s, \tau_s)$ be a P–paracompact, P–locally compact bitopological space and $Y = (Y, \sigma_1, \sigma_2)$ be a P–expandable bitopological space. Then $X \times Y$ is P–expandable.*

Proof. The proof follows immediately from Theorem 3.3. $\square$

4. Conclusions

This paper has studied and explored some certain mappings in bitopological spaces like the pairwise perfect and pairwise countably perfect mappings. In order to gain some finite product theorems concerning with pairwise expandable and almost pairwise expandable spaces, some significant properties of such mappings have been stated and derived well. Some illustrative examples have been provided for completeness.

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